NEW METHODS OF COMPUTING THE PROJECTIVE POLYNOMIAL RESULTANT BASED ON DIXON, JOUANOLOU AND JACOBIAN MATRICES

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NEW METHODS OF COMPUTING THE PROJECTIVE POLYNOMIAL RESULTANT BASED ON DIXON, JOUANOLOU AND JACOBIAN MATRICES

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To my beloved mother late Fatima Abdulkarim, my respected father Mallam Sulaiman Abdullahi and my lovely children Abdullahi Surajo, Fatima Surajo and Zainab Surajo.
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In elimination theory, particularly when using the matrix method to compute multivariate resultant, the ultimate goal is to derive or construct techniques that give a resultant matrix that is of considerable size with simple entries. At the same time, the method should be able to produce no or less superfluous factors. In this thesis, three different techniques for computing the resultant matrix are presented, namely the Jouanolou-Jacobian method, the Dixon-Jouanolou methods for bivariate polynomials, and their generalizations to the multivariate case. The Dixon-Jouanolou method is proposed based on the existing Jouanolou matrix method which is subjected to bivariate systems. To further extend this method to multivariate systems, the entry formula for computing the Dixon resultant matrix is first generalized. This extended application of the loose entry formula leads to the possibility of generalizing the Dixon-Jouanolou method for the bivariate systems of three polynomials to systems of \( n + 1 \) polynomials with \( n \) variables. In order to implement the Dixon-Jouanolou method on systems of polynomials over the affine and projective space, respectively, the concept of pseudo-homogenization is introduced. Each space is subjected to its respective conditions; thus, pseudo-homogenization serves as a bridge between them by introducing an artificial variable. From the computing time analysis of the generalized loose entry formula used in the computation of the Dixon matrix entries, it is shown that the method of computing the Dixon matrix using this approach is efficient even without the application of parallel computations. These results show that the cost of computing the Dixon matrix can be reduced based on the number of additions and multiplications involved when applying the loose entry formula. These improvements can be more pronounced when parallel computations are applied. Further analyzing the results of the hybrid Dixon-Jouanolou construction and implementation, it is found that the Dixon-Jouanolou method had performed with less computational cost with cubic running time in comparison with the running time of the standard Dixon method which is quartic. Another independent construction produced in this thesis is the Jouanolou-Jacobian method which is an improvement of the existing Jacobian method since it avoids multi-polynomial divisions. The Jouanolou-Jacobian method is also able to produce a considerably smaller resultant matrix compared to the existing Jacobian method and is therefore less computationally expensive. Lastly all the proposed methods have considered a systematic way of detecting and removing extraneous factors during the computation of the resultant matrix whose determinant gives the polynomial resultant.
ABSTRAK

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<td>BKK</td>
<td>Bernstein Kouchnirenko Khovanskii</td>
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<tr>
<td>CAS</td>
<td>Computer Algebra System</td>
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<tr>
<td>CPU</td>
<td>Central Processing Unit</td>
</tr>
<tr>
<td>GCD</td>
<td>Greatest Common Divisor</td>
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<tr>
<td>GPS</td>
<td>Global Positioning System</td>
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<tr>
<td>LCM</td>
<td>Lowest Common Multiple</td>
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<td>RSC</td>
<td>Rank Submatrix Computation</td>
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<td>UFD</td>
<td>Unique Factorization Domain</td>
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**LIST OF SYMBOLS**

\[ V(F) \] - Affine varieties of the system \( F \)

\[ \text{Det}(M) \] - Determinant of the square matrix \( M \)

\[ \Delta_{i,j} \] - Differentials

\[ \oplus \] - Direct sum

\[ \theta(f_1, f_2, \ldots, f_{n+1}) \] - Dixon polynomial of the system \( f_1, f_2, \ldots, f_{n+1} \)

\[ \Theta \] - Dixon resultant matrix

\[ R_r(F) \] - Dixon-Jouanolou resultant matrix

\[ D(f_1, f_2, \ldots, f_{n+1}) \] - Dixon resultant matrix of the system \( f_1, f_2, \ldots, f_{n+1} \)

\[ \xi_{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n} \] - Element of the loose entry formula

\[ D_{a,b,c,d} \] - Entries of the Dixon matrix for bivariate system

\[ D_{m_1-1, \ldots, m_n-1; n_{m_1-1}, \ldots, m_n-1} \] - Entries of the generalize Dixon matrix of the system \( F \)

\[ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \] - Exponent vectors of the monomial \( x^\alpha \)

\[ J_r(F) \] - Jouanolou resultant matrix

\[ \mu (\mathbb{Q}_1, \mathbb{Q}_2, \ldots, \mathbb{Q}_n) \] - Mixed volume of the polytopes \( \mathbb{Q}_1, \mathbb{Q}_2, \ldots, \mathbb{Q}_n \)

\[ x^\alpha \] - Monomial \( x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \) where \( \alpha \in \mathbb{N}^n \)

\[ N(F) \] - Newton polytopes of the system \( F \)

\[ \mathbb{P}^n \] - Projective space

\[ P(F) \] - Projective varieties of the system \( F \)

\[ H_r(f_1, f_2, \ldots, f_n) \] - Resultant matrix of the Dixon-Jouanolou method

\[ \mathbb{C}[x_1 x_1^{-1}, \ldots, x_n x_n^{-1}] \] - Ring of Laurent polynomial over a field \( \mathbb{C} \)

\[ K[x_1, x_2, \ldots, x_n] \] - Ring of polynomial over a field \( K \)

\[ \mathbb{C} \] - Set of complex numbers
\( \mathbb{Z} \) - Set of integers
\( \mathbb{N} \) - Set of natural numbers
\( \mathbb{R} \) - Set of real numbers
\( S(H_r(f_1, f_2, \ldots, f_n)) \) - Size of the Generalized Dixon-Jouanolou matrix
\( S(R_r) \) - Size of the Dixon-Jouanolou matrix
\( A = (A_1, A_2, \ldots, A_{n+1}) \) - Support of the system \( F \)
\( Syl(f_1, f_2, \ldots, f_n) \) - Sylvester resultant matrix of the system \( f_1, f_2, \ldots, f_n \)
\( F = \{f_1, f_2, \ldots, f_{n+1}\} \) - System of polynomials
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CHAPTER 1

INTRODUCTION

1.1 Preface

The role played by a system of polynomial equations in scientific research has a variety of applications in real life situations. For example, in modelling the components in computer-aided design represented by the Bezier Bernstein splines [1], detecting whether a moving robot will collide with an obstacle or not [2], designing curves and surfaces [3], differential elimination [4] and application of Global Positioning System (GPS) in geodesy and geoinformatics [5]. Another important application is the modelling of geometric and kinematics constraints where a well-constrained system of polynomials equations are used to represent the motion of a camera.

Dealing with the above-mentioned applications requires a technique of variables elimination. There are three powerful elimination techniques; Grobner basis [6, 7], set characteristics or Ritt-Wu method [8] and the resultant matrix method [9, 10]. Some of the disadvantages of both Grobner basis and Ritt-Wu as reported in [11, 12] are:

1. These methods require large storage capacity during the computations.
2. High computational complexity.
The matrix method for computing the resultant is a popular tool used in eliminating variables which reduces the system of polynomial equations into simpler forms. The resultant of a system of polynomial equations can be obtained from the determinant of the resultant matrix. The determinant of the resultant matrix is also referred to as the projection operator. Exact resultant can be achieved if the projection operator exactly equals the resultant. Otherwise, the projection operator consists of a product of polynomials which are multiples of the resultant. These other factors of the projection operator besides the resultant polynomial are called extraneous factors. The presence of extraneous factors in the resultant formulation gives rise to the problem of extracting the resultant polynomial from the determinant.

Much of the concern in researches related to multivariate polynomials resultant is to determine a method that can give exact resultant. In most cases, exact resultant only exist on certain classes of the generic system of polynomial equations and these conditions are determined and proven to give exact resultant. Besides finding methods that can produce a determinantal formula which can give exact resultant, a method that can reduce the presence of extraneous factors in the resultant matrix formulation reduces the complexity of the problem. It becomes the aim of this thesis to find new methods that can reduce the complexity of computing the resultant matrix and resultant polynomial.

The rest of this chapter is as follows. Section 1.2 gives the research background leading to the problem statement in Section 1.3. The objectives of the study are given in Section 1.4 followed by the scope of the study, the significance of the study and thesis organization.

1.2 Research Background

When dealing with systems of polynomials in more than one variable, there are basically two matrix base constructions which depend on the nature of the resultant matrix [13]. If each entry of the matrix is either the coefficient of one of the polynomials or
zero, the matrix is regarded as Sylvester type [14]. Sometimes the entries of the resultant matrix are polynomials in terms of the coefficients of the given system of polynomial equations, such type is referred to as Cayley/Dixon type. Methods such as Macaulay, Jouanolou (Generalized Macaulay), Newton sparse, incremental and Salmon Jacobian which is also referred as Sturmfel resultant [5] are considered to be Sylvester type while Dixon is regarded as Cayley/Dixon type [15–17]. All Cayley/Dixon resultant matrix have complicated entries, but with relatively small matrix [18–23]. On the other hand, Sylvester type resultant matrix have simple entries with large size matrix [24, 25].

In a situation where the resultant matrix is constructed based on the two types of the constructions, such formulation is referred to as the hybrid resultant matrix [26]. The foundation work for hybrid resultant was first introduced in [27], derived for certain class of the multivariate polynomials of multi-graded type. Independently, in 1999 Chionh et al. in [28] had proposed another hybrid construction which possibly is the first construction that can be applied to a more general class of system of polynomials.

Apart from the classical hybrid resultant matrix, the sparse hybrid formulation was constructed, due to the frequent appearance of such systems in many engineering applications [29]. However, it is not clear whether or not the constructions can generate exact resultant. Another construction was given by [30] and unlike the work of [29], Khetan presents his formulation and computes the hybrid resultant matrix based on certain examples. His construction also only considers systems of polynomials with unmixed support and the size of the matrix can still be very large [30]. A complete implementation of the Sylvester-Bezout construction is given by Ahmad in [13] giving conditions that can give optimal resultant matrix and describes some limitations in the implementations.

Apart from the matrix method for computing resultant, the second most commonly used algorithmic method is the Ritt-Wu’s approach introduced by Ritt in [31] and further improved by the Chinese mathematician Wu Wen-Tsün. The method has two important steps namely reduction to triangular form and successive pseudo-division [32, 33].
A triangular set of polynomials with almost the same set of common solutions as the original system of equations is defined as the characteristic set of a set of polynomials [8, 34]. Ritt presents the first algorithm to compute the characteristic set that was resurrected by Wu and Ritt respectively in [8, 31]. Characteristic sets are typically computed by eliminating variables sequentially in some predetermined order using successive pseudo-division of polynomials.

Ritt-Wu’s method requires a large storage capacity during the computation. For example, Heymann’s question can be resolved using the matrix method within 300 seconds, compared to almost 19 hours using characteristics set method [11, 18]. The implementation by Gao and Wang in [11] is carried out using SUN 4/470.

The Grobner basis of a polynomial ideal is a basis with many useful properties and provides answers to most of the theoretic questions about the ideals, such as ideal description and membership problem. The notion generalizes three well-known algorithms namely: Gaussian elimination algorithm, particularly reduced row echelon form for linear systems, the Euclidean algorithm for computing the greatest common divisor of both univariate and multivariate polynomials and lastly, the simplex algorithm for minimizing or maximizing linear and non-linear functions.

Buchberger’s algorithms resolved the issue of the ideal membership using S−polynomial of \( f_1, f_2, \ldots, f_n \in k[x_1, \ldots, x_n] \) which is defined to be \( S(f_1, f_2) = \frac{x^\alpha}{\text{LT}(f_1)} f_1 - \frac{x^\alpha}{\text{LT}(f_2)} f_2 \), where LT is the leading term of \( f_i \) and \( x^\alpha \) is the least common multiple (LCM) of the leading monomials \( \text{LM} \) of \( f_1 \) and \( f_2 \) \( (x^\alpha = \text{lcm}(\text{LM}(f_1), \text{LM}(f_2))) \) [32, 35].

The first algorithm to compute the Grobner basis of an ideal is given by [7, 36] and since then, many efficient variations have been proposed. Along with other resultant methods, Grobner basis can be considered as an effective tool for solving a polynomial equation which also include finding the solutions of the system of polynomial equations, variables elimination and ideal membership problem. The approach of Grobner basis provides a criterion for which a polynomial must satisfy in order to be a member of a certain ideal.
The Grobner basis method can also be used in a variety of applications such as solving polynomials systems and implicitization of curves and surfaces. This method computes the exact resultant [18, 37]. However, the Grobner basis approach is not as simple as the matrix method and run out of time when the total degree is very large since it requires large storage capacity during the computations.

The Grobner basis method also is less effective, when computing the resultant of a polynomial system, for example, deriving the implicit equation of a bi cubic surface takes only 50 seconds using the matrix method, compared to almost 10,000 seconds using Grobner basis. In an implementation using SUN 4/470, sometimes the system runs out of memory before the computation ends [11, 18].

Another setback of the Grobner basis method reported by Zheng et al. [12] is that the approach fails to generate the implicit equation of some parametric equations with base points as given in Equation (1.1). On the other hand, the matrix method of computing the resultant is able to compute the implicit equation despite having these base points. For rational parametric equations defined as

\[
x = \frac{x(s, t)}{w(s, t)}, \quad y = \frac{y(s, t)}{w(s, t)} \quad \text{and} \quad z = \frac{z(s, t)}{w(s, t)},
\]

a base point is a value \((s, t)\) for which \(x(s, t) = y(s, t) = z(s, t) = w(s, t) = 0\). At this point the values \(x, y\) and \(z\) are not defined. Another implication of the base point is that, no matter what values the coefficients of the rational curves or surfaces will be, there is always a common solution at infinity.

\[
F = \begin{cases} 
  x(s, t) = 2t^3 + 4t^2 + 2t + 4st + s^2t + 2 + 3s + s^2 \\
  y(s, t) = -2st^2 - 2t - st + 2 + s - 2s^2 - s^3 \\
  z(s, t) = 2t^2 - 3st^2 - 2t - 3st - 2s^2t - 2s - 3s^2 - s^3 \\
  w(s, t) = t^3 + t^2 - t + s^4 - 1 - s + s^2 + s^3 
\end{cases} \quad (1.1)
\]
In the implementation of the Grobner basis method, some of the reasons for large storage requirement and the CPU time is the swell of intermediate system of equations encountered during the computation of the basis. These intermediate polynomials do not satisfy the requirement of the basis, thus, are not included in the resulting Grobner basis [38, 39].

![Diagram](image)

**Figure 1.1** Usual routines when computing resultant

In an attempt to improve the effectiveness of the Grobner basis, several algorithms were introduced by different scholars such as signature base algorithm [40, 41], F4, F5, F5C
algorithms among others [42–44]. Until today reducing the cost of computing Grobner bases remain an open area of research. Figure 1.1 shows the different techniques of computing resultant and how they are related.

Since both Ritt-Wu and Grobner basis techniques require large storage capacity and huge CPU time while computing the resultant polynomial, this work focuses on the matrix approach of computing the resultant. Existing methods are revisited giving emphasis on the method of construction, complexity, size of the matrix, nature of the entries, size of the unwanted factors and space requirement in the implementation of these methods.

1.3 Statement of the Problem

The resultant techniques for solving multivariate polynomial equations have received lots of attention with emphasis on eliminating or at least reducing the terms of the extraneous factors in the projection operator. This is because the presence of extraneous factors constitute to one of the biggest problem common to all matrix methods. These factors do not provide any information on the solutions of the polynomials; thus can be misleading and the process of identifying them is time-consuming.

Recent research on the resultant matrix methods focus on the hybrid resultant formulations. However, the existing hybrid resultant matrix methods either produce a large resultant matrix size or extraneous factors embedded in the projection operator [13, 29]. On the other hand, there exist hybrid resultant matrix that gives exact resultant [30], but the method is confined to certain class of polynomials. The Sylvester Bezout type resultant matrix is implemented by [13] and proven to produce an exact result, but under certain conditions, the method had failed to generate the desired Bezout block of the matrix.

Generally, for any given system of multivariate polynomials, none of the existing resultant
matrix methods can give exact resultant. However, in some special cases, almost all existing method can produce exact resultant [45] which is due to the special structure of the Newton polytopes corresponding to the system. Among the factors that contribute to the effectiveness and the efficiency of the resultant matrix method is the nature of the matrix elements and the large matrix size. If the entries of the matrix are polynomials, the symbolic computation of the determinant will be more complex then if the entries are numerical values [46]. Therefore the nature of the matrix entries as well as the size of the matrix determine the efficiency of computing the resultant polynomial.

Several formulations have been given with notable improvements. Yet the problem of reducing the size of the resultant matrix and reducing or eliminating extraneous factors is still an open problem in the study of resultant. Thus, deriving or constructing a new hybrid resultant matrix with considerable size, that can eliminate, or at least reduce, the number of extraneous factors remains an important problem of research, which when solved adequately will produce positive dividends.

1.4 Objectives of the Study

Based on the formulated problem, the following research objectives are outlined:


2. To generalize the loose entry formula for computing the entries of the Dixon matrix and generalize the construction of Dixon-Jouanolou method to multivariate systems of \( n + 1 \) polynomials with \( n \) variables, applying the generalize loose entry formula to compute the entries of the Dixon-Jouanolou matrix.

3. To determine the computational complexity of computing the Dixon-Jouanolou and Jouanolou-Jacobian matrices and compare with the complexity of computing the
Dixon and Jouanolou matrices respectively.

4. To determine the possible causes for the existence of extraneous factors and provide a suitable approach of eliminating them.

1.5 Scope of the Study

The research focuses on the construction of the hybrid resultant matrix methods for computing the resultant of a system of multivariate polynomial equation. The methods involved elimination theory, an area under algebraic geometry. The polynomials under consideration are assumed to be unmixed, generic and symbolic. Although, the new hybrid methods can handle $n$ system of polynomials with $n$ or $n - 1$ variables, depending on the requirements of the method, the examples given in this thesis only include system of polynomials with at most four variables. Basic tools of algebraic geometry are applied in solving some problems encountered throughout this research. The computer algebra system Maple version 2015 is used to evaluate the resultant matrices.

1.6 Significance of the Study

So far most of the matrix-based elimination techniques fail to produce an exact resultant. Instead, these methods generate a polynomial called a projection operator which is a multiple of the resultant containing some unwanted factors which looks like an integral part of the resultant. For lower dimensional cases the approach of computing and extracting the resultant is well understood [47], but for higher dimensional cases the problem is still subjected to further research. The contribution of this work is to be able to produce new resultant matrix method that can eliminate or minimize the difficulties faced when extracting the resultant from the projection operator. This study will be beneficial to many industrial applications, in areas like computer-aided design, robot design and control, modeling of geometric object and many other applications within the scope of
algebraic geometry.

1.7 Thesis Organization

Chapter 1 introduces the concept of polynomial resultant which begins with preface, research background, statement of the problem, objectives of the study, scope of the study and finally the significance of the study. This chapter provides the introduction to the research area and highlights some of the existing problems. This chapter served as introductory part of this research work.

Chapter 2 serves as the review of the existing literatures. Referring to Figure 1.2, this chapter contains eight sections which include introduction, preliminary definitions and theorems and the matrix methods for computing resultant. Others are Dixon resultant, Macaulay resultant, Jouanolou resultant and the hybrid resultants. This chapter highlights major setbacks of the existing classical and hybrid techniques of computing resultant. Based on these limitations, the research problem have been identified. Hence the new constructions presented in Chapter 4, 5 and 6 are designed to reduce the size of extraneous factors, space requirements and cost of computations. The eighth section concludes the chapter.

Chapter 3 presents the methodology of this research work. As described in Figure 1.2, this chapter contains five sections which include introduction, research assumptions, research framework and computational tools. Details of the three constructions are provided with explanation. The chapter describes how these methods are designed to produce relatively smaller resultant matrix. Finally, the fifth section concludes the methodology.

Chapter 4 presents the Jouanolou-Jacobian constructions, To provide a clear presentation, this chapter contains four sections. The first three sections are introduction, Jacobian block and construction of Jouanolou-Jacobian method. The fourth section presents
the complexity analysis of the Jouanolou-Jacobian method. This complexity analysis provides a yardstick for comparison with the existing Jouanolou method to determine whether the Jouanolou-Jacobian technique is computationally expensive or not. Referring to Figure 1.2, the fifth section concludes the chapter.

Chapter 5 presents the Dixon-Jouanolou constructions of type 1 and 2. This chapter contains four sections which include introduction, pseudo-homogenization and Dixon-Jouanolou formulations for bivariate systems. The fourth section concludes the chapter. The concept of pseudo-homogenization allows the constructions to switch from a projective space to affine space using an artificial variable.

Chapter 6 presents the generalization of the Dixon-Jouanolou method, from the bivariate system to the system of \( n + 1 \) equations with \( n \) variables. The loose entry formula for computing the Dixon resultant matrix is generalized to the system of \( n + 1 \) equations with \( n \) variables. This allows the generalization of the Dixon-Jouanolou method. Figure 1.2 shows that this chapter contained five sections which include introduction, generalized Dixon resultant matrix, generalized entry formula and the generalized Dixon-Jouanolou method is presented followed by conclusion.

Chapter 7 presents the summary of the thesis and highlights how each of the objectives are achieved. This chapter also provide the direction for further research. These suggestions are derived from the conclusions of this chapter.
NEW METHODS OF COMPUTING THE PROJECTIVE POLYNOMIAL RESULTANT BASED ON DIXON, JOUANOLOU AND JACOBIAN MATRICES

CHAPTER ONE: Introduction

Preface  Research Background  Statement of the Problem  Objective of the Study  Scope of the Study  Significance of the Study

CHAPTER TWO: Literature review

Introduction  Preliminary Definition and Theorems  Matrix Method For Computing Resultant  Dixon Resultant  Macaulay Resultant  Jouanolou Resultant  Hybrid Resultant  Conclusion

CHAPTER THREE: Methodology

Introduction  Research Assumptions  Research Framework  Computational Tools  Conclusion

CHAPTER FOUR: Jouanolou-Jacobian Formulation

Introduction  Jacobian Block  Construction of Jouanolou-Jacobian Method  Complexity Analysis of the Jouanolou-Jacobian Method  Conclusion

CHAPTER FIVE: Two Dixon-Jouanolou Resultant Formulation

Introduction  Pseudo Homogenization  Dixon-Jouanolou Method  Conclusion

CHAPTER SIX: Generalized Dixon-Jouanolou Formulation

Introduction  Generalized Dixon Resultant Matrix  Generalized Entry Formula  Generalized Dixon-Jouanolou Method  Conclusion

CHAPTER SEVEN: Conclusion

Introduction  Summary of the Thesis  Conclusion  Future Directions

Figure 1.2 Thesis organization
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