

The Method of Lines solution of the Forced Korteweg-de Vries-Burgers Equation (FKdVB)

NazatulsyimaMohdYazid^a, Kim GaikTay^b, Wei King Tiong^c, Yan Yee Choy^d,
AzilaMdSudin^e & CheeTiongOng^f

^a Faculty of Science, Technology and Human Development,
UniversitiTun Hussein Onn Malaysia, 86400 Parit raja, BatuPahat, Johor, Malaysia
nazatulsyima91@yahoo.com

^b Department of Communication Engineering, Faculty of Electric and Electronic, UniversitiTun
Hussein Onn Malaysia, 86400 Parit raja, BatuPahat, Johor, Malaysia
tay@uthm.edu.my

^c Department of Computational Science and Mathematics, Faculty of Computer Science and Information Technology
Universiti Malaysia Sarawak, 94300 Kota Samarahan, Malaysia
wktiong@fit.unimas.my

^{d, e} Department of Mathematics and Statistics, Faculty of Science, Technology and Human Development,
UniversitiTeknologi Malaysia, 81310 Johor Bahru, Malaysia
yychoy@uthm.edu.my, azzila@uthm.edu.my

^f Department of Mathematical Sciences, Faculty of Science
UniversitiTeknologi Malaysia, 81310 Johor Bahru, Malaysia
octiong@utm.my

Abstract. In this paper, the application of the method of lines (MOL) to the Forced Korteweg-de Vries-Burgers equation with variable coefficient (FKdVB) is presented. The MOL is a powerful technique for solving partial differential equations by typically using finite-difference approximations for the spatial derivatives and ordinary differential equations (ODEs) for the time derivative. The MOL approach of the FKdVB equation led to a system of ODEs. Solution of the system of ODEs was obtained by applying the Fourth-OrderRunge-Kutta (RK4) method. In order to show the accuracy of the presented method, the numerical solutions obtained were compared with its progressive wave solution in terms of maximum absolute error at certain times. It was found that the maximum absolute errors are in the order of 10^{-6} .

Keywords: FKdVB Equation; The Method of Lines; System of Differential Equation; RungeKutta.

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INTRODUCTION

There are many physical phenomena in engineering and physics which can be expressed by some nonlinear partial differential equations (Demiray, 2003). However, most of them do not have exact analytical solutions. Therefore, these nonlinear equations should be solved using approximate methods (Zedan & Adrous, 2012).

In literature, weakly nonlinear wave propagation in a prestressed fluid-filled stenosed elastic tube filled with a Newtonian fluid with variable viscosity fluid has been studied by (Tay, 2007) by applying the reductive perturbation method and long wave approximation. By using the stretched coordinate of initial-value type and extending the field quantities into the asymptotic series of order ε where ε is a small parameter, the governing equations are reduced to the forced Korteweg-de Vries-Burgers (FKdVB) equation with variable coefficients

$$u_{\tau} + \mu_1 u u_{\xi} - \mu_2 u_{\xi\xi} + \mu_3 u_{\xi\xi\xi} + \mu_4(\tau) u_{\xi} = \mu(\tau) \quad (1)$$

where ξ is a spatial variable, τ is a temporal variable, $\mu_1, \mu_2, \mu_3, \mu_4(\tau)$ and $\mu(\tau)$ are the coefficients of nonlinearity, dissipation, dispersion, variable coefficient and forcing term respectively. The presence of forcing term $\mu(\tau)$, and variable coefficient term, $\mu_4(\tau) u_{\xi}$ show the presence of stenosis. The dissipative term $-\mu_2 u_{\xi\xi}$ in FKdVB equation is caused by the effect of variable viscosity. The coefficients of $\mu_1, \mu_2, \mu_3, \mu_4(\tau)$ and $\mu(\tau)$ are defined by (Tay, 2007) as

$$\mu_1 = \frac{5}{2\lambda_\theta} + \frac{\beta_2}{\beta_1}, \quad \mu_2 = \frac{\nu}{2c}, \quad \mu_3 = \frac{m}{4\lambda_z} + \frac{\lambda_\theta^2}{16} - \frac{\beta_0}{2\beta_1},$$

$$\mu_4(\tau) = \frac{\lambda_\theta \gamma_2}{\beta_1} G(\tau) - \left[\frac{\beta_2}{\beta_1} + \frac{1}{2\lambda_\theta} \right] g(\tau), \quad \mu(\tau) = \frac{1}{2} g'(\tau) - \frac{\lambda_\theta \gamma_1}{2\beta_1} G'(\tau). \quad (2) \text{ where}$$

$$\gamma_0 = \frac{1}{\lambda_\theta \lambda_z} \left(\lambda_\theta - \frac{1}{\lambda_\theta^3 \lambda_z^2} \right) F(\lambda_\theta, \lambda_z),$$

$$\gamma_1 = \frac{1}{\lambda_\theta \lambda_z} \left[\left(1 + \frac{3}{\lambda_\theta^4 \lambda_z^2} \right) + 2\alpha \left(\lambda_\theta - \frac{1}{\lambda_\theta^3 \lambda_z^2} \right)^2 \right] F(\lambda_\theta, \lambda_z),$$

$$\gamma_2 = \frac{1}{2\lambda_\theta \lambda_z} \left[-\frac{12}{\lambda_\theta^5 \lambda_z^2} + 6\alpha \left(\lambda_\theta - \frac{1}{\lambda_\theta^3 \lambda_z^2} \right) \left(1 + \frac{3}{\lambda_\theta^4 \lambda_z^2} \right) + 4\alpha^2 \left(\lambda_\theta - \frac{1}{\lambda_\theta^3 \lambda_z^2} \right)^3 \right] F(\lambda_\theta, \lambda_z),$$

$$\beta_0 = \frac{1}{\lambda_\theta} \left(\lambda_z - \frac{1}{\lambda_\theta^3 \lambda_z^3} \right) F(\lambda_\theta, \lambda_z), \quad \beta_1 = \gamma_1 - \frac{\gamma_0}{\lambda_\theta}, \quad \beta_2 = \gamma_2 - \frac{\beta_1}{\lambda_\theta}, \quad (3)$$

given that $F(\lambda_\theta, \lambda_z) = \exp \left[\alpha \left(\lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2 \lambda_z^2} - 3 \right) \right]$, $\alpha = 1.948$, $\lambda_\theta = \lambda_z = 1.6$, $\nu = 1$, $c = 15.391$, $m = 0.1$, $G(\tau) = 0$

and $g(\tau) = \text{sech}(0.01\tau)$. Here α refers to material constant, λ_θ is the initial circumferential stretch ratio, λ_z is the initial axial stretch ratio, ν is kinematic viscosity, m is mass of artery and c is the scale parameter.

The application of the MOL to the FKdVB equation (1) will be presented in this paper. The MOL approach of the FKdVB equation led to a system of ODEs. Solution of the system was obtained by applying the RK4 method. The solution of the FKdVB equation that is obtained using the MOL with progressive wave solution conducted by (Tay, 2007) will be compared in terms of its maximum absolute error at certain time τ .

METHODOLOGY

The MOL

The MOL is a powerful method used to solve partial differential equations (PDEs). It involves making an approximation to the spatial derivatives and reducing the problem into a system of ODEs (Hall & Watt, 1976; Loeb & Schiesser, 1974; Schiesser, 1994). In addition, this system of ODEs can be solved by using time integrator. The most important advantage of the MOL approach is that it has not only the simplicity of the explicit methods (Dehgan, 2006) but also the superiority (stability advantage) of the implicit ones unless a poor numerical method for solution of ODEs is employed. It is possible to achieve higher-order approximations in the discretization of spatial derivatives without significant increases in the computational complexity. This method has wide applicability to physical and chemical systems modeled by PDEs such as delay differential equations (Koto, 2004), two-dimensional sine-Gordon equation (Bratsos, 2007), the Nwogu one-dimensional extended Boussinesq equation (Hamdi et al, 2005), the fourth-order Boussinesq equation, the fifth-order Kaup–Kupershmidt equation and an extended Fifth-Order Korteweg–de Vries (KdV5) equation (Saucez et al, 2004).

In this paper, the spatial derivatives are firstly discretized using central finite difference formulae as follows:

$$u_\xi \approx \frac{u_{j+1} - u_{j-1}}{2\Delta\xi},$$

$$u_{\xi\xi} \approx \frac{u_{j+1} - 2u_j + u_{j-1}}{(\Delta\xi)^2},$$

$$u_{\xi\xi\xi} \approx \frac{u_{j+2} - 2u_{j+1} + 2u_{j-1} - u_{j-2}}{2(\Delta\xi)^3}, \quad (4)$$

where ξ is the spatial variable, τ is the temporal variable, j is the index denoting the spatial position along ξ -axis

and $\Delta\xi$ is the step size along the spatial axis. The ξ -interval is divided into M points with $j = 1, 2, \dots, M-2, M$. Therefore, the MOL approximation of the equation (1) is given by

$$\begin{aligned} \frac{\partial u_j}{\partial \tau} = & -\frac{\mu_1}{2\Delta\xi} u_j (u_{j+1} - u_{j-1}) + \frac{\mu_2}{(\Delta\xi)^2} (u_{j+1} - 2u_j + u_{j-1}) \\ & - \frac{\mu_3}{2(\Delta\xi)^3} (u_{j+2} - 2u_{j+1} + 2u_{j-1} - u_{j-2}) - \frac{\mu_4(\tau)}{2\Delta\xi} (u_{j+1} - u_{j-1}) \\ & + \mu(\tau) \equiv f(u_j). \end{aligned} \quad (5)$$

Equation (5) is written as an ODE since there is only one independent variable, which is τ . Also, equation (5) represents a system of M equations of ODEs. The initial condition for equation (5) after discretization is given by

$$u(\xi_j, \tau = 0) = u_0(\xi_j), \quad j = 1, 2, \dots, M-1, M. \quad (6)$$

For the time integration, the RK4 method is applied. Thus, the numerical solution at time τ_{i+1} is

$$u_{i+1,j} = u_{i,j} + \frac{1}{6} (a_{i,j} + 2b_{i,j} + 2c_{i,j} + d_{i,j}), \quad (7)$$

where

$$\begin{aligned} a_{i,j} &= \Delta\tau f(u_{i,j}), \\ b_{i,j} &= \Delta\tau f\left(u_{i,j} + \frac{1}{2}a_{i,j}\right), \\ c_{i,j} &= \Delta\tau f\left(u_{i,j} + \frac{1}{2}b_{i,j}\right), \\ d_{i,j} &= \Delta\tau f(u_{i,j} + c_{i,j}) \end{aligned} \quad (8)$$

Here $\Delta\tau$ is the step size of the temporal coordinates.

Progressive Wave Solution

The progressive wave solution of the FKdVB equation as given by (Tay, 2007) is

$$u = \frac{a}{\mu_1} + \frac{3\mu_2^2}{25\mu_1\mu_3} (\text{sech}^2 \zeta - 2 \tanh \zeta) + \frac{1}{2} \left[g(\tau) - \frac{\lambda_0 \lambda_1}{\beta_1} G(\tau) \right], \quad (9)$$

where a is a constant. The phase function ζ can be expressed as

$$\zeta = \frac{\mu_2}{10\mu_3} \left\{ \xi - a\tau - \int_0^\tau \left[\left(\frac{3}{4\lambda_0} - \frac{\beta_2}{2\beta_1} \right) g(s) + \frac{\lambda_0}{\beta_1} \left(\gamma_2 - \frac{\mu_1 \gamma_1}{2} \right) G(s) \right] ds \right\}. \quad (10)$$

RESULTS AND DISCUSSION

To test MOL on the FKdVB equation, we need the initial condition as follows:

$$u(\xi, 0) = \frac{a}{\mu_1} + \frac{3\mu_2^2}{25\mu_1\mu_3} \text{sech}^2 \left(\frac{\mu_2}{10\mu_3} \xi \right) - \frac{6\mu_2^2}{25\mu_1\mu_3} \tanh \left(\frac{\mu_2}{10\mu_3} \xi \right) + 0.5. \quad (11)$$

Figure 1 (a) gives the MOL solution of the FKdVB equation (1) with spatial parameters at certain time τ , while Figure 1 (b) represents the progressive wave solution of the FKdVB equation (1) with spatial parameters at certain time τ . The solution of the FKdVB equation (1) with space ξ shows a decreasing shock profile propagating to the right with a decrease in wave amplitude as time τ increases.

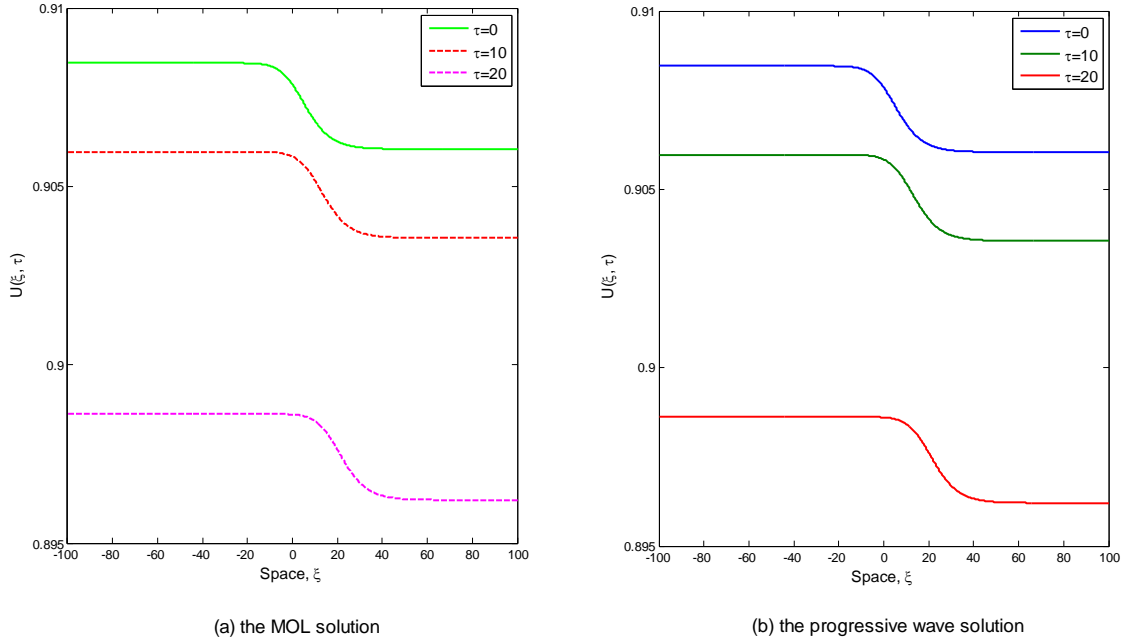


FIGURE 1.Solutions of the FKdVB equation versus space ξ for different time τ at $\Delta\xi = 0.1$

We then computed the absolute error between the progressive wave and MOL solutions for each discretized spatial point at certain time τ and later find the maximum absolute error. The maximum absolute errors between the progressive wave and MOL solutions are calculated based on the formula:

$$L_{\infty} = \max |U_{progressive} - U_{MOL}|. \quad (12)$$

Table a) gives the maximum absolute error between the progressive wave solution and MOL solution. It shows the maximum absolute errors are in order of 10^{-3} .

TABLE a). Maximum absolute error of the FKdVB equation for different time τ at $\Delta\tau = \partial\xi^{-3}$.

Time, τ	0	10	20
L_{∞}	0	0.76506×10^{-6}	1×10^{-6}

CONCLUSION

The MOL was employed to solve the FKdVB equation. It involved replacing the spatial derivatives in the PDE with finite-difference approximations and by doing that, the spatial derivatives are longer stated explicitly in terms of spatial independent variables. This leads to a system of ODEs. The system is then solved by using the RK4 method. This paper describes the effect of computational effort with respect to the accuracy of results. The MOL solution of the FKdVB equation (1) is plotted versus its progressive wave solution. From the observation, it was found that there were no differences for both MOL and progressive wave solutions. The maximum absolute errors between both MOL and progressive wave solutions at certain time τ are computed. Results revealed that the maximum absolute errors are in the order of 10^{-6} for $\Delta\xi = 0.1$ and $\Delta\tau = 1 \times 10^{-3}$. Hence, it can be concluded that the FKdVB equation can be solved successfully using the MOL.

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