THE COMMUTATIVITY DEGREE OF 3-ENGEL GROUPS UP TO ORDER 40

MARDHIAH BINTI ZAKARIA

UNIVERSITI TEKNOLOGI MALAYSIA
THE COMMUTATIVITY DEGREE OF 3-ENGEL GROUPS UP TO ORDER 40

MARDHIAH BINTI ZAKARIA

A dissertation submitted in fulfillment of the requirements for the award of the degree of
Master of Science (Mathematics)

Faculty of Science
Universiti Teknologi Malaysia

JUNE 2011
To my beloved mother and father
ACKNOWLEDGEMENTS

First of all, thanks to Allah, who has given me the strength to complete this dissertation. In preparing this thesis, I was in contact with many people. They have contributed towards my understanding and thoughts. In particular, I wish to express my sincere appreciation to my dissertation supervisor, Dr. Nor Muhainiah Mohd Ali for her encouragement, guidance and invaluable suggestions. She has helped me in many ways besides making sure that I had all necessary information as a reference to complete this dissertation. Besides, I would like to thank also Assoc. Prof. Dr. Nor Haniza Sarmin for her comments and suggestions to improve my research.

Moreover, I would like to express my thanks to my parents, Senah Binti Man and Zakaria Bin Mamat for their long lasting love. Lastly, I would like to thank my siblings, my friends; Fadila, Maryam and my very best friend, Mohd Afiq Aizat Bin Mat Ali. They really supported me and played an important role in the completion of my thesis. I would like to thank them for their encouragement, love and emotional support.
Friedrich Engel has discovered Engel groups in the 50s in Germany. The origin of this group lies in the theory of Lie algebras. As an example, one of the basic classical results for Engel Lie algebras is Engel’s theorem. In this research, the 3-Engel groups of order up to 40 will be the main focus. An application of 3-Engel groups on probability theory will be presented too. The commutativity degree can also be viewed as the probability that two elements in the group commute denoted by $P(G)$. This probability applied to the eighteen 3-Engel groups with order at most 40 is proven to be at most 5/8. A software named Groups, Algorithms and Programming (GAP) software is used to facilitate most of the calculations in this research.
TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>TITLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>TITLE PAGE</td>
<td></td>
<td>i</td>
</tr>
<tr>
<td>DECLARATION OF ORIGINALITY</td>
<td></td>
<td>ii</td>
</tr>
<tr>
<td>DEDICATION</td>
<td></td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENT</td>
<td></td>
<td>iv</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td></td>
<td>v</td>
</tr>
<tr>
<td>ABSTRAK</td>
<td></td>
<td>vi</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td></td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td></td>
<td>ix</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td></td>
<td>xi</td>
</tr>
<tr>
<td>LIST OF SYMBOLS/ ABBREVIATION/ NOTATION</td>
<td></td>
<td>xii</td>
</tr>
</tbody>
</table>

1. RESEARCH FRAMEWORK 1

1.1 Introduction 1

1.2 Research Background 2

1.3 Problem Statement 2

1.4 Research Objectives 2

1.5 Scope of Research 3

1.6 Significance of Study 3

1.7 Research Methodology 4

1.8 Dissertation Report Organization 4
<table>
<thead>
<tr>
<th>Table No.</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>All 3-Engel groups of order at most 40</td>
<td>18</td>
</tr>
<tr>
<td>3.2</td>
<td>Some groups of order at most 40 which are not 3-Engel</td>
<td>88</td>
</tr>
<tr>
<td>4.1</td>
<td>Cayley Table of $D_8$</td>
<td>105</td>
</tr>
<tr>
<td>4.2</td>
<td>0,1- Table of $D_8$</td>
<td>106</td>
</tr>
<tr>
<td>4.3</td>
<td>Cayley Table of $QD_{16}$</td>
<td>106</td>
</tr>
<tr>
<td>4.4</td>
<td>0,1- Table of $QD_{16}$</td>
<td>107</td>
</tr>
<tr>
<td>4.5</td>
<td>Cayley Table of $Q_{16}$</td>
<td>107</td>
</tr>
<tr>
<td>4.6</td>
<td>0,1- Table of $Q_{16}$</td>
<td>108</td>
</tr>
<tr>
<td>4.7</td>
<td>Cayley Table of $(Z_8</td>
<td></td>
</tr><tr>
<td>times Z_2)</td>
<td></td>
<td></td>
</tr><tr>
<td>times Z_2$</td>
<td>109</td>
<td></td>
</tr>
<tr>
<td>4.8</td>
<td>0,1- Table of $(Z_8</td>
<td></td>
</tr><tr>
<td>times Z_2)</td>
<td></td>
<td></td>
</tr><tr>
<td>times Z_2$</td>
<td>110</td>
<td></td>
</tr>
<tr>
<td>4.9</td>
<td>Cayley Table of $Z_2, ((Z_4 	imes Z_2)</td>
<td></td>
</tr><tr>
<td>times Z_2) = (Z_2 	imes Z_2). (Z_4 	imes Z_2)$</td>
<td>112</td>
<td></td>
</tr>
<tr>
<td>4.10</td>
<td>0,1- Table of $Z_2, ((Z_4 	imes Z_2)</td>
<td></td>
</tr><tr>
<td>times Z_2) = (Z_2 	imes Z_2). (Z_4 	imes Z_2)$</td>
<td>113</td>
<td></td>
</tr>
<tr>
<td>4.11</td>
<td>Cayley Table of $(Z_8</td>
<td></td>
</tr><tr>
<td>times Z_2)</td>
<td></td>
<td></td>
</tr><tr>
<td>times Z_2$</td>
<td>114</td>
<td></td>
</tr>
<tr>
<td>4.12</td>
<td>0,1- Table of $(Z_8</td>
<td></td>
</tr><tr>
<td>times Z_2)</td>
<td></td>
<td></td>
</tr><tr>
<td>times Z_2$</td>
<td>115</td>
<td></td>
</tr>
<tr>
<td>4.13</td>
<td>Cayley Table of $Q_8</td>
<td></td>
</tr><tr>
<td>times Z_4$</td>
<td>117</td>
<td></td>
</tr>
<tr>
<td>4.14</td>
<td>0,1- Table of $Q_8</td>
<td></td>
</tr><tr>
<td>times Z_4$</td>
<td>118</td>
<td></td>
</tr>
<tr>
<td>4.15</td>
<td>Cayley Table of $(Z_4 	imes Z_4)</td>
<td></td>
</tr><tr>
<td>times Z_2$</td>
<td>119</td>
<td></td>
</tr>
<tr>
<td>4.16</td>
<td>0,1- Table of $(Z_4 	imes Z_4)</td>
<td></td>
</tr><tr>
<td>times Z_2$</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>4.17</td>
<td>Cayley Table of $Z_8</td>
<td></td>
</tr><tr>
<td>times Z_4$</td>
<td>122</td>
<td></td>
</tr>
<tr>
<td>4.18</td>
<td>0,1- Table of $Z_8</td>
<td></td>
</tr><tr>
<td>times Z_4$</td>
<td>123</td>
<td></td>
</tr>
<tr>
<td>4.19</td>
<td>Cayley Table of $Z_8</td>
<td></td>
</tr><tr>
<td>times Z_4$</td>
<td>124</td>
<td></td>
</tr>
</tbody>
</table>
The number and lists of conjugacy classes, $k(G)$ of all 3-Engel groups of order at most 40.
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure No.</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Friedrich Engel</td>
<td>5</td>
</tr>
<tr>
<td>2.2</td>
<td>Flowchart on detecting 3-Engel groups</td>
<td>13</td>
</tr>
<tr>
<td>2.3</td>
<td>A simulation model on detecting 3-Engel groups</td>
<td>14</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS/ ABBREVIATIONS/ NOTATION

1, e  Identity
GAP  Groups, Algorithms and Programming
\( P(G) \)  Commutativity degree
\( k(G) \)  Number of conjugacy classes
\( G \)  A finite group \( G \)
\(|G|\)  Order of a finite group \( G \)
\([x, y]\)  Commutator of \( x \) and \( y \), \( x^{-1}y^{-1}xy \)
\([x, _y]\)  \( [x, y, y, y] = [[[x, y], y], y] \)
\( H \triangleleft G \)  \( H \) is normal in \( G \)
\( \langle a \rangle \)  Cyclic group
\( cl(a) \)  The conjugacy class of \( a \) in \( G \)
\( x^y \)  The conjugate of \( x \) in \( y \), \( y^{-1}xy \)
\( x^G \)  The set of all conjugates, \( g^{-1}xg, \forall g \in G \)
\( \in \)  Element of
\( \neq \)  Not equal to
\( \forall \)  For all
\( \times \)  Semi direct product
\( \blacksquare \)  End of proof
CHAPTER 1

RESEARCH FRAMEWORK

1.1 Introduction

Friedrich Engel has discovered Engel groups in the 50's in Germany. The origin of this group lies in the theory of Lie algebras. As an example, one of the basic classical results for Engel Lie algebras is Engel’s theorem.

This research begins on viewing the history of Friedrich Engel and the basic properties of \(n\)-Engel groups. In this research, the 3-Engel groups will be our main focus. More characteristics of this group will be discussed and several examples will be presented and used. An application of 3-Engel groups on probability theory will be presented too as again as our main research. A software named Groups, Algorithms and Programming (GAP) software is use to facilitate some of the calculations in this research.
1.2 Research Background

This research begins with an overview of the history of Friedrich Engel. Some basic properties of $n$-Engel groups are given especially on 3-Engel groups. Some characteristics of this group will be presented with the proofs. These include some definitions, theorems and examples. A software called Groups, Algorithms and Programming (GAP) is used to identify the 3-Engel groups of small order (groups of order up to 40). The commutativity degree of a group can also be viewed as the probability that two random elements in the group commute.

1.3 Problem Statement

Which groups are 3-Engel groups? What type of characteristics do they have? How commutative are they?

1.4 Research Objectives

The main objectives of this research are:
1. to find the characteristics of 3-Engel groups and present the proofs.
2. to use GAP in order to determine small groups (groups up to order 40) that are 3-Engel.
3. to calculate the degree of commutativity degree, $P(G)$ of given 3-Engel groups by using two different approaches;
   (i) Cayley table,
   (ii) Conjugacy classes.
1.5 **Scope of Research**

This research will focus only on all 3-Engel groups of order up to 40.

1.6 **Significance of Study**

The result of this research can be used for further research in related areas. Research paper will also be sent to national and international indexed journal such as journal of “Sains Malaysiana”, Bulletin of the Malaysian Mathematical Sciences Society, and World Scientific. Furthermore, this research will enhance contribution from mathematicians in Malaysia especially in Pure Mathematics fields and Statistics.

1.7 **Research Methodology**

This dissertation will be carried out according to the following steps:

1. Study and understand the concepts of 3-Engel groups.
2. Review and prove some given basic properties of 3-Engel groups.
3. Find more characterization of 3-Engel groups and present the proofs.
4. Identify some specific small groups (groups up to order 40) that fulfill the characteristics of 3-Engel.
5. Find the commutativity degree of these 3-Engel groups.
6. Write up of dissertation.
7. Presentation of dissertation.
1.8 Dissertation Report Organization

This dissertation organized into five chapters. Chapter 1 includes the research framework.

In Chapter 2, the history of Friedrich Engel is overviewed and some definitions and basic concepts that will be used throughout the dissertation are introduced. Besides, a simulation model on detecting 3-Engel groups by GAP is included too.

Chapter 3 reviews on some characteristics of 3-Engel groups. Some definitions and theorems involving 3-Engel groups are presented with complete proof. All examples of 3-Engel groups of order at most 40 are included together with some of non examples of 3-Engel group with order less than 40.

Chapter 4 discusses the application of 3-Engel groups in probability theory. This chapter reviews on some theorems involving conjugacy classes, $k(G)$ and the commutativity degree of a group, $P(G)$. The result on the commutativity degree of all 3-Engel groups of order at most 40 with different approaches is presented and analyses.

Lastly, in Chapter 5, the obtaining results are summarized. Suggestions for further research are also included.
REFERENCES


