SOLVING FRACTIONAL DIFFUSION EQUATION USING VARIATIONAL ITERATION METHOD AND ADOMIAN DECOMPOSITION METHOD

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A dissertation submitted in partial fulfilment of
the requirements for the award of the degree of
Master of Science (Mathematics)

Faculty of Science
Universiti Teknologi Malaysia

JUNE 2013

Dedicated to my beloved father, Dzulkarnain Abdullah and my mother, Nurhaedah binti Sining, all my siblings, and all my friends. Thank you to my supervisor, Professor Dr. Mohd Nor Mohamad who guided and give me support to do this research.

ACKNOWLEDGEMENTS

This thesis has been with the support of moral by many people. First, I wish to express my sincere appreciation to my supervisor Prof. Dr. Mohd Nor Mohamad for his continuous guidance, motivation and support.

I am also thankful to all my family members especially my parents for their motivation, patience and encouragement.

Not to forget to all my fellow friends that giving me some thought in this study, thank you so much. Their moral support has helped me greatly. Their views, tips and comment have helped me improved myself. I am grateful to all my friends.

The most importance all, I am grateful to God that enables me to realize these goals and for the strength God has given me to complete my thesis writing.

ABSTRACT

Fractional calculus has been used in many areas of sciences and technologies. This is the consequences of the elementary calculus. The order of the derivative in elementary calculus is integer, *n*. The *n*th derivative was changed to α for fractional calculus, where α is a fraction number or complex number. Fractional diffusion equation is one of the examples of fractional derivative equation. This study will focus on the solving fractional diffusion equation using variational iteration method and Adomian decomposition method to obtain an approximate solution to the fractional differential equation. Graphical output may explain further the results obtained. In certain problems the use of fractional differential equation gives more accurate representation rather than using elementary differential equation. Adomian decomposition method is easier in solving fractional diffusion equation since there is no nonlinear term in the equation. However, variational iteration method is more suitable to be applied in solving fractional derivative equation that consists of nonlinear term.

ABSTRAK

Kalkulus pecahan telah digunakan dengan meluas dalam bidang sains dan teknologi. Ini merupakan kesinambungan daripada kalkulus asas. Peringkat bagi terbitan dalam kalkulus asas adalah integer, n. Peringkat bagi terbitan ke-n di tukar kepada α untuk kalkulus pecahan, dimana α adalah pecahan ataupun nombor kompleks. Persamaan resapan peringkat pecahan merupakan persamaan pembezaan dengan peringkat pecahan. Kajian ini tertumpu kepada penyelesaian persamaan resapan dengan peringkat pecahan menggunakan kaedah lelaran variasi dan kaedah penguraian Adomian untuk mendapatkan penyelesaian yang hampir bagi persamaan pembezaan peringkat pecahan. Hasil grafik yang diperoleh menjelaskan lagi keputusan yang diperoleh. Dalam masalah tertentu, penggunaan persamaan pembezaan peringkat pecahan memberikan keputusan yang lebih baik berbanding persamaan pembezaan asas. Kaedah penguraian Adomian merupakan kaedah yang lebih mudah untuk menyelesaikan persamaan resapan peringkat pecahan kerana persamaan ini tidak mempunyai sebutan tidak linear. Walaubagaimanapun, kaedah lelaran variasi adalah lebih sesuai diaplikasikan semasa menyelesaikan persamaan pembezaan peringkat pecahan yang mempunyai sebutan tidak linear.

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INTRODUCTION

1.0 Background of the Study

Fractional calculus is the consequences of the elementary calculus. It has the fractional order of derivatives or integrations. This can be defined as

$$D^{\alpha} f(x) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}$$

where,

α is the fractional order of derivatives or an arbitrary order,

$f(x)$ is any suitable functions [1].

There are various field of science and engineering used the applications in fractional calculus. Electric transmission lines, ultrasonic wave propagation in human cancellous bone, modelling of speech signals, modelling the cardiac tissue electrode interface, lateral and longitudinal control of autonomous vehicles, viscoelasticity, edge detection and some application in fluid mechanics are the applications using fractional calculus [2].

The difference between ordinary derivative and fractional derivative can be seen when we differentiate a constant. Let k be any constant. From the equation

(1.1) and (1.2) we can say that the fractional derivation of a constant produced a function, $f(x)$.

$$\frac{d}{dx}(k) = 0 \tag{1.1}$$

$$\frac{d^{1/2}}{dx^{1/2}}(k) = \sqrt{\pi} f(x) \tag{1.2}$$

In this research, the method of variational iteration and Adomian decomposition method are used in solving the fractional diffusion equation. This research will discuss the solution by analytical and numerical.

1.1 Statement of the Problem

The study in fractional derivative problem is still new in science and engineering area. There are many applications need to have fractional order of derivative such as in biopotential recording and functional electrical stimulation. This fills the gaps in calculus area of study.

Instead of solving the differential with *n*th derivative, we go further to solve differential equation with fractional order of derivative. As a parable, when we write x^3 we know that $x^3 = x \times x \times x$ which the variable x is multiple by three times. But, when we have $x^{3.4}$ we cannot describe the physical meaning simply as x^3. This concept applied to fractional calculus. We can write $\frac{d^3x}{dt^3} = \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{dx}{dt}\right)\right)$ and called as differentiating x three times with respect to *t*. But, when it comes to $\frac{d^{3/4}x}{dt^{3/4}}$ we need the other way to solve the derivative.

In that case, some methods were found to be suitable in solving some fractional derivative problems. This study will discuss about the method of variational iteration and Adomian decomposition in solving the fractional diffusion equation.

1.2 Objectives of the Study

By doing this research, the objectives of the study are:

1. To solve fractional diffusion equation using variational iteration method and Adomian decomposition method.

2. To obtained the graphical output different cases of $f(x)$.

1.3 Scope of the Study

This research will focus on the algorithm for the methods of variational iteration and Adomian decomposition method. The methods are used to solve fractional diffusion equation. The general equation of fractional diffusion equation is

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = A\frac{\partial^{2} u(x,t)}{\partial x^{2}} - \frac{\partial}{\partial x}(F(x)u(x,t)), \quad 0<\alpha\leq 1,\ D>0 \qquad (1.3)$$

where,

α is a fraction,

$\frac{\partial^{\alpha}}{\partial t^{\alpha}}(.)$ is the Caputo derivative of order α,

$u(x,t)$ represents the probability density function of finding a particle at the x in the time t,

A is the positive constant depends on the temperature, the friction coefficient, the universal gas constant and on the Avagadro number,

$F(x)$ is the external force.

This research only consider for the case $A=1$, $\alpha = \frac{1}{2}$ and $F(x) = -x$ with the initial condition $f(x)$.

1.4 Significance of the Study

Fractional calculus has been used in many areas of science and engineering. For the elementary calculus, we just consider for the *n*th derivatives which *n*=1,2,3,… But now some of applications need to consider a fraction order of derivative, α.

This idea starts in 1695 when L'Hopital inquired Leibniz that invents the notation $D^n f(x) = \frac{d^n f(x)}{dx^n}$, what if the n be a fraction. By then, the research has been continued by Langrange, Lacroix, Fourier, Abel, Liouville, Riemann and many other famous mathematicians. Recently there are so many researches in fractional calculus have been done in physics, continuum mechanics, signal processing and electromagnetics applications.

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