EXTENDED RUNGE-KUTTA FOURTH ORDER METHOD WITH POLYNOMIAL INTERPOLATION TECHNIQUE FOR FUZZY POPULATION MODELS

NOR ATIRAH IZZAH ZULKEFLI

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DEDICATION

To
my beloved husband, Mohd Faizal Yong and
my loving daughters, Nor Airis Farisa and Nor Ameena Dahlia

and

especially my loving parents,
Zulkefli Idris and Nor Zihan Jusoh.

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ABSTRACT

Uncertainty quantification plays an increasingly important role in the mathematical modeling of physical phenomena. One alternative of the mathematical modelings is provided by fuzzy sets. The main research of this thesis is the study of numerical method in solving fuzzy differential equations (FDEs). In this thesis, the problem of FDEs in one-dimensional problem and two-dimensional problem were considered, namely fuzzy logistic differential equation and fuzzy predator-prey systems. The problems were solved using extended Runge-Kutta fourth order (ERK4) method. Nevertheless, due to the lacking of numerical methods available for solving polynomial type of FDEs, the ERK4 method is incorporated with polynomial interpolation technique in order to reduce the high degree of polynomials during multiplication operation. Parameter estimation provides tools for the efficient use of data in the estimation of the parameters that appears in the mathematical models. Thus, this study presents the parameter estimation using two techniques of minimization which are center difference differentiation and robust gradient minimization. Stability analysis and convergence proof of the approximation methods had been carried out. Hence, this research is carried out in order to solve these problems. The obtained numerical results prove that the ERK4 method with incorporated polynomial interpolation technique produce higher accuracy results and may become an alternative method for other uncertainty problems.

ABSTRAK

Kuantifikasi ketidakpastian memainkan peranan yang semakin penting dalam pemodelan matematik bagi fenomena fizikal. Salah satu alternatif dalam pemodelan matematik adalah disediakan oleh set kabur. Kajian utama tesis ini ialah mengkaji tentang kaedah berangka dalam menyelesaikan persamaan pembezaan kabur (FDEs). Dalam tesis ini, FDEs dalam satu dimensi dan dua dimensi dipertimbangkan, iaitu persamaan pembezaan logistik kabur dan sistem pemangsa-mangsa kabur. Masalah-masalah ini diselesaikan dengan menggunakan kaedah lanjutan Runge-Kutta peringkat keempat (ERK4). Walau bagaimanapun, disebabkan kekurangan kaedah berangka yang tersedia dalam menyelesaikan FDEs jenis polinomial, kaedah ERK4 telah digabungkan dengan teknik interpolasi polinomial untuk mengurangkan kuasa yang tinggi dalam polynomial semasa operasi pendaraban. Anggaran parameter menyediakan alat untuk penggunaan data yang cekap dalam anggaran parameter yang ada dalam model matematik. Oleh itu, kajian ini mengemukakan pendekatan untuk anggaran parameter menggunakan dua teknik peminimuman iaitu pembezaan beza tengah dan peminimuman mantap berdasarkan kecerunan. Analisis kestabilan dan bukti penumpuan untuk kaedah penghampiran telah dijalankan. Maka, penyelidikan ini adalah dijalankan untuk menyelesaikan masalah-masalah ini. Hasil berangka yang diperoleh membuktikan bahawa kaedah ERK4 dengan gabungan teknik interpolasi polinomial menghasilkan penyelesaian berketepatan tinggi dan mungkin menjadi kaedah alternatif untuk masalah ketidakpastian yang lain.

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LIST OF ABBREVIATIONS

FDE	-	Fuzzy Differential Equation
FLDE	-	Fuzzy Logistic Differential Equation
FPPS	-	Fuzzy Predator-Prey System
RK	-	Runge-Kutta
RK4	-	Runge-Kutta Fourth Order
ERK	-	Extended Runge-Kutta
ERK4	-	Extended Runge-Kutta Fourth Order
CPU	-	Central Processing Unit
1D	-	One-dimensional
2D	-	Two-dimensional

LIST OF SYMBOLS

$\Re$	-	Real number
C	-	Constant number
f	-	Function of the problem
f'	-	First derivative
f''	-	Second derivative
x_0, y_0	-	Initial value problem
G	-	Growth rate
K	-	Carrying capacity
k	-	Stage number
a, b, c, d	-	Positive real parameters
$P(x)$	-	Polynomial of x
P_m	-	Polynomial degree m
P_n	-	Polynomial degree n
$\mu_{\tilde{a}}$	-	Membership function of fuzzy set $\tilde{a}$
L	-	Lower bound
U	-	Upper bound
γ	-	Fuzzy width
h	-	Step size
t	-	Time
V	-	Vector space
$\|x\|$	-	Modulos of x
$\|\|x\|\|$	-	Norm of x
$E(G, K)$	-	Objective function
y_{SE}	-	Semi exact solution

$\subseteq$	-	Subset or is included in
$\subset$	-	Subset
$\in$	-	Element of
$\notin$	-	Not element of
$\cap$	-	Intersection
$\cup$	-	Union
$\wedge$	-	Wedge product
$\neq$	-	Not equal to
$<$	-	Less than
$\leq$	-	Less than or equal to
$>$	-	Greater than
$\geq$	-	Greater than or equal to

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CHAPTER 1

INTRODUCTION

1.1 Research Background

Although ecological system modeling has been applied for decades, still inherent uncertainties, which need to be taken into account in order to improve the accuracy and predictability of estimates. The reasoning generally used is that the mathematical description of the uncertainty should be able to represent the lack of information and the type of information given. If, for example, a real parameter is only known to be within certain bounds, then an interval is a perfect representation of this uncertainty.

In applied problems of population model, for instance, it is not always possible to know exactly the initial number of individuals in a given environment. In general, one gets information by means of linguistic statements as the initial condition is approximately y_0. To the extent that the label approximately is imprecise, it can be modeled as a fuzzy set. Thus, linguistic statements like these can be regarded as fuzzy restrictions on the values taken by the variable of interest [1].

Since Zadeh [2] introduced the concept of fuzzy set and corresponding fuzzy operations, enormous efforts have been dedicated to the development of various aspects of the theory and applications of fuzzy systems, particularly on the theory of differential equations with uncertainty. Therefore, fuzzy differential equations (FDEs) have been adapted as natural approach to model dynamical systems under the possibility of uncertainty. Fuzzy dynamical systems based on FDEs are also widely applied in many fields such as fuzzy control systems [3, 4], bifurcations of fuzzy nonlinear dynamical systems [5], and artificial system [6].

In the last few years, many works have been performed by several authors on numerical solutions of FDEs. One of the most common methods used to numerically solve these equations is the RK method. Most efforts to increase the order of RK method are accomplished by increasing the number of Taylors series terms used, and thus the number of function evaluations [7]. Here, the extended Runge-Kutta (ERK) method for departure point calculation is introduced. The ERK method in order to enhance the order of accuracy of the solutions is by evaluations of both f and f'. Specifically, the proposed formulae with f' is more efficient for cases where f' is easier to evaluate than f [8].

Fuzzy equation can be regarded as a generalized form of the fuzzy polynomial, which can be immediately used to validate the convexness in real time. Fuzzy polynomials are used to form a suitable setting for mathematical modeling of real world problems which often come with uncertainties and vagueness. One approach is by using interpolation technique. Interpolation of the function $f(x)$ includes $O(n)$ time complexity at n data points [9]. The most importance advantage of this approach is that, it is capable of greatly reduce the degree of polynomials while still maintaining high accuracy of the numerical solution.

The extraction of information from data is one of the fundamental tasks in engineering and science. Parameter estimation is a discipline that provides tools for efficient use of data in the estimation of constants in mathematical models, besides as an aid in the modeling of phenomena.

Parameter estimation is necessary to assist problem solving in diverse areas related to the modeling of ecology. Example and application of parameter estimation in this research, however, are directed to the common problems in the engineering and science fields, where FDEs and ordinary differential equations are commonly used in constructing population model.

The increasing need of parameter estimation has been made easy by emergence of computers, resulting to more practicable solutions of parameter estimation for a great array of applications. Estimation was first extensively discussed by Legendre 1806 [10], who first considered comprehensive treatment of the method of least squares, although priority for its discovery was shared with Gauss. Gauss is recognized as the first to use parameter estimation, which is the method of least squares, in connection with the orbit determination of minor planets [11].

1.2 Statement of Problem

When a physical problem is transformed into a deterministic initial value problem

$$y = (t, y(t)), \quad y(0) = y_0, \tag{1.1}$$

the modeling is not entirely perfect, since the initial value may not be known exactly and the function may contain unknown parameters. If these values are known through some measurements, they are necessarily subjected to errors. Analysis on the effect of these errors will lead to the study on the qualitative behavior of the solution, as in Equation (1.1). If the nature of errors is random, then instead of employing deterministic problem as given in the Equation (1.1), a random differential equation with random initial value or random coefficients will be used. However, if the underlying structure is not probabilistic, due to subjective choices, it would be appropriate to employ FDEs.

In the physical problem given by Equation (1.1), it is assumed that the initial value is the fuzzy number, to obtain the FDE.

In solving the problem of FDEs, some limitations need to be addressed. From the basic operation of fuzzy with polynomial operation, degree of polynomial will develop very fast. In this research, the extended Runge-Kutta fourth order (ERK4) method has been incorporated with polynomial interpolation technique to reduce the degree of polynomial, which is the objective of this thesis.

To achieve the objective, the following questions need to be clarified:

(a) What is the ERK4 scheme for FDEs?
(b) How to incorporated the ERK4 method with the polynomial interpolation technique to FDEs?
(c) What is the accuracy, stability analysis, and convergence of ERK4 method with polynomial FDEs?
(d) How to estimate the unknown parameter of FDEs?

1.3 Research Objectives

The objectives of this research are as follows:

(a) To apply a numerical algorithm for solving FDEs using ERK4 method.
(b) To couple ERK4 method with polynomial interpolation technique in numerical computation.
(c) To analyze the accuracy, stability analysis, and convergence of ERK4 method with polynomial FDEs.
(d) To estimate parameters using parameter estimation technique via for prediction of population model.

1.4 Scope of the Study

For this research, FDEs have been taken into consideration. The problems considered in this research involve the one-dimensional (1D) FDEs and two-dimensional (2D) FDEs, which are fuzzy logistic differential equation (FLDE) and fuzzy predator-prey systems (FPPS). The development of logistic equation and predator-prey equation with fuzzy initial value problem is based on the concept of fuzzy algorithm. ERK4 method has been employed to find a numerical solutions for the problems. The formula uses both function, f and derivative of function, f' in order to improve the numerical solution. The basic operation of fuzzy will lead to emergence of degree of polynomial. Therefore, polynomial interpolation technique governed by Vandermonde matrix has been coupled with ERK4 method to reduce the degree of polynomial. Numerical algorithm has been derived for simulation using Microsoft Visual C++ compiler. Next, accuracy, stability analysis, and convergence of the proposed method have been measured. Lastly, minimization techniques via center difference differentiation and robust gradient minimization have been employed to estimate the parameters of the FLDE.

1.5 Significance of Findings

The influence of uncertainty in many fields of applications such as engineering, physics, and biology contributes to an accelerating interest in the development of population model with fuzzy problems. Therefore, numerical results obtained for all problems presented in this thesis are significant as reference for future investigations, prediction, and even for validation purposes. The modification of algorithms in this research are also significant contribution; for calculation of polynomials in solving FDEs which require numerical methods.

1.6 Thesis Organization

This thesis is divided into six chapters. The content of each chapter is briefly delineated below.

Chapter 1 introduces the purpose of this research, covering the statement of problem, research objectives, scope, as well as significance of research. This chapter also addresses the identification of research gap and methods to achieve the research objective.

Chapter 2 clarifies the definitions and properties related to fuzzy set. This chapter reviews population model and numerical methods of RK method in detail. Polynomial interpolation, with related important technique, is reviewed. In order to solve the FDEs, these reviews have been significantly used as a background study, as described in Chapter 3.

Chapter 3 presents the derivation for fuzzy initial value problem for 1D FDEs and 2D system FDEs, including derivation of ERK4 method for FDEs. Also highlighted is the development of numerical algorithm to perform numerical example, so that the efficiency of the ERK4 method incorporated with polynomial interpolation can be assured. Next, the accuracy, stability analysis, and convergence proof for the proposed method are presented.

Chapter 4 addresses numerical example for FLDE and FPPS for the validation of the numerical solution.

Chapter 5 highlights the modeling a real phenomena due to the fact that the natural systems in ecology have the after effect property and are subjected to the uncertainty phenomena. Also discussed in this chapter are parameter estimations obtained through minimization technique via center difference differentiation and robust gradient minimization, as well as validation of the minimization techniques efficiency.

Chapter 6 presents the conclusion of the whole content of the thesis. Some recommendations for future study based on present solution are also highlighted in this chapter.

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