The Application of GAP Software in Constructing the Non-Normal Subgroup Graphs of Alternating Groups

N F Rahin[1, 2] , N H Sarmin[2] and S Ilangovan[3]

1 Centre for Defence Foundation Studies, Universiti Pertahanan Nasional Malaysia, Kem Sg Besi, 57000 Kuala Lumpur, Malaysia.
2 Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Skudai, Johor Bahru, Johor, Malaysia.
3 Faculty of Science and Engineering, University of Nottingham Malaysia, Jalan Broga, 43500 Semenyih, Selangor, Malaysia.
E-mail: Sheila.Ilangovan@nottingham.edu.my

Abstract. A graph in group theory is constructed by using any elements of a group as a set of vertices. Some of the properties of a group are used to form the edges of the graph. A finite group can be represented in a graph by its subgroup structure. A subgroup H of a group G is a subset of G, where H itself is a group under the same operation as in G, whereas a subgroup H is said to be a normal subgroup if its left and right cosets coincide. The non-normal subgroup graph of a group G is defined as a directed graph with a vertex set G and two distinct elements x and y are adjacent if $xy \in H$. In this paper, the non-normal subgroups graph of alternating groups for order twelve is determined by using GAP software. The graphs are found to be a union of complete digraphs and directed graphs with the same pattern depending on the order of the non-normal subgroups.

1. Introduction

A graph associated with a finite group shows a relationship among the group elements. It is constructed by how we define the adjacent vertices using the properties of the finite group. In this paper, the finite group discussed is the alternating group which is represented geometrically by its subgroup structure. Some of the graphs which are related to its subgroup were cyclic subgroup graphs defined by Devi and John Arul Singh [1]. These graphs were using the cyclic subgroups as vertices and two distinct subgroups are adjacent if one of them is a subset of the other. Then, more graphs were constructed using subgroups which are stable subgroup graph defined by Tolue [2] and normal subgroup-based power graph defined by Bhuniya and Sudip Bera [3]. Most of the research on graphs associated with the finite group were simple undirected graphs. But there were some researches on the directed graph related to group theory. For instance, the order graph was defined by Bilal and Ahmad [4] as a directed graph whose vertices are the elements of the group order classes. Next, the subgroup graph was formally defined in 2015 by Kakeri and Erfanian [5]. The subgroup graph is a simple directed graph associated with subgroups of a finite group and this graph was extended to a non-normal subgroup graph defined by Nabilah et al. [6]. Therefore, non-normal subgroup graphs are constructed for the alternating group of order twelve in this paper.
Computer software is widely used in mathematical research as a tool to compute complex computation. Besides, computation related to group theory is one of the first area in pure mathematics to be computerized too. Groups, Algorithms, and Programming software, GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory. This software is commonly

used in research and teaching for studying groups and their representations, rings, vector spaces, algebras, combinatorial structures, codes, Lie algebras, and more. The other famous software which is mostly used by pure mathematicians are MAGMA and Maple to observe the properties of groups, constructing groups and graphs for visualization.
The following section is the preliminary section where some basic definitions in graph theory and group theory related are stated. The third section discusses the main result of this research followed by the conclusion section.

2. Preliminaries
In this section, some definitions related to this research are stated. The definitions for the symmetric and the alternating group are as follows:
Definition 1 [7] (Symmetric group)
Let A be the finite set $\{1, 2, \dots, n\}$. The group of all permutations of A is the symmetric group on n letters and is denoted by S_n. Note that S_n has $n!$ elements, where $n! = n(n-1)(n-2) \dots (3)(2)(1)$.
Definition 2 [8] (Alternating group)
The set of all even permutations in S_n forms a subgroup of S_n for each $n \geq 2$. This subgroup is called the alternating group of degree n, denoted by A_n.
Definition 3 [6] (The non-normal subgroup graph)
Let H be a non-normal subgroup of a group G. The non-normal subgroup graph $\Gamma_H^{NN}(G)$ is a graph in which the directed edges are formed by the vertex set G; such that the edge starts from x to y if and only if $x \neq y$ and $xy \in H$.

3. Results and discussion
In this section, the results of the non-normal subgroup graphs are determined for the alternating group of order 12, A_4. The elements of A_4 are

$$A_4 = \{(1), (13)(24), (12)(34), (14)(23), (123), (132), (234), (142), (243), (143), (134), (124)\}$$

Groups, Algorithms and Programming (GAP) is used to identify the non-normal subgroups and the directions of the edges for constructing the non-normal subgroup graph. The GAP coding is shown in the first section, while the second section is on the non-normal subgroup graph obtained for the alternating group of order 12.

3.1. GAP coding
In this subsection, the coding for the non-normal subgroups and the direction of the edges of the graph in this research are presented. The GAP software is used to obtain the direction of the edges instead of doing it manually since there are 12 elements in the alternating group. Based on definition 3, the vertices of the non-normal subgroup graph are all the elements of A_4 and the direction of the edges is determined by the product of the two distinct elements. The following GAP coding is for one of the subgroups of the alternating group of order 12.

```
gap> A4:=AlternatingGroup(4);
Alt( [ 1 .. 4 ] )
gap> e:=Elements(A4);
[ (), (2,3,4), (2,4,3), (1,2)(3,4), (1,2,3), (1,2,4), (1,3,2), (1,3,4),
(1,3)(2,4), (1,4,2), (1,4,3), (1,4)(2,3) ]
gap> AllSubgroups(A4);
[ Group(()), Group([ (1,2)(3,4) ]), Group([ (1,3)(2,4) ]),
Group([ (1,4)(2,3) ]), Group([ (2,4,3) ]),
  Group([ (1,2,3) ]), Group([ (1,4,2) ]), Group([ (1,3,4) ]),
Group([ (1,3)(2,4), (1,2)(3,4) ]), Group([ (1,3)
  (2,4), (1,2)(3,4), (2,4,3) ]) ]
gap> N:=NormalSubgroups(A4);
[ Alt( [ 1 .. 4 ] ), Group([ (1,3)(2,4), (1,2)(3,4) ]), Group(()) ]
gap> H1:=Subgroup(A4,[(1,2)(3,4)]);
```

```
Group([ (1,2)(3,4) ])
gap> Elements(H1);
[ (), (1,2)(3,4) ]
gap> for i in e do
> for j in e do;
> if i<>j then
> c:=i*j;
> d:=c in H1;
> if (d=true) then
> Print ("{",i,",",j,",",c,",",d,"},");
> fi;
> fi;
> od;
> od;
{(),(1,2)(3,4),(1,2)(3,4),true},{(2,3,4),(2,4,3),(),true},{(2,3,4),(1,2
,3),(1,2)(3,4),true},{(2,4,3),
(2,3,4),(),true},{(2,4,3),(1,2,4),(1,2)(3,4),true},{(1,2)(3,4),(),(1,2)
(3,4),true},{(1,2,3),(1,3,2),(),true},{(1,2,3),
(1,4,3),(1,2)(3,4),true},{(1,2,4),(1,3,4),(1,2)(3,4),true},{(1,2,4),(1,
4,2),(),true},{(1,3,2),(2,4,3),(1,2)
(3,4),true},{(1,3,2),(1,2,3),(),true},{(1,3,4),(1,3,2),(1,2)(3,4),true}
,{(1,3,4),(1,4,3),(),true},{(1,3)(2,4),(1,4)
(2,3),(1,2)(3,4),true},{(1,4,2),(2,3,4),(1,2)(3,4),true},{(1,4,2),(1,2,
4),(),true},{(1,4,3),(1,3,4),(),true},{(1,4,3),
(1,4,2),(1,2)(3,4),true},{(1,4)(2,3),(1,3)(2,4),(1,2)(3,4),true},
gap> LogTo();
```

From this GAP coding, there are ten subgroups of the alternating group of order twelve which consists of three normal and seven non-normal subgroups, respectively. The order of the non-normal subgroups are two and three for instance, $H_1 = \{(1),(12)(34)\}$, $H_2 = \{(1),(13)(24)\}$, $H_3 = \{(1),(14)(23)\}$, $H_4 = \{(1),(234),(243)\}$, $H_5 = \{(1),(123),(132)\}$, $H_6 = \{(1),(124),(142)\}$ and $H_7 = \{(1),(134),(143)\}$. The direction of the edges for the subgroup H_1 is given and the similar GAP coding can be applied to the other non-normal subgroups which are H_2 until H_7. Moreover, this coding also can be applied for a higher order of the alternating groups. In the next section, the non-normal subgroup graphs are constructed for each subgroup of the alternating group of order 12.

3.2. Main results

In this subsection, the results obtained from GAP software are used to construct the non-normal subgroup graph for each subgroup of the alternating group of order 12.

Proposition 3.2.1

Let G be the alternating group of order 12, A_4 and H_i be a non-normal subgroup of G with order two. Then the non-normal subgroup graph for H_i, $\Gamma_{H_i}^{NN}(G)$ is a union of two complete digraphs with two vertices and a bipartite graph as shown in figure 1.

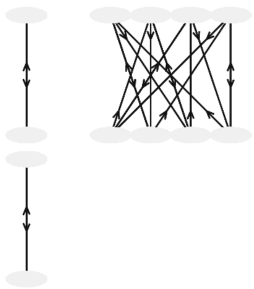

Figure 1. $\Gamma_{H_i}^{NN}(G)$ for H_i of order two.

Proposition 3.2.2
Let G be the alternating group of order 12, A_4 and H_i be a non-normal subgroup of G with order three. Then the non-normal subgroup graph for H_i, $\Gamma_{H_i}^{NN}(G)$ is shown in figure 2.

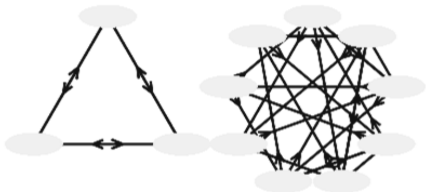

Figure 2. $\Gamma_{H_i}^{NN}(G)$ for H_i of order three.

4. Conclusion

The non-normal subgroup graph for subgroups of the alternating groups, A_4 are determined by using GAP software and the coding are presented in this paper. The non-normal subgroup graphs of A_4 have two different graphs depending on the order of the subgroups. The subgroups of A_4 with order two is a union of two complete digraphs with two vertices and a bipartite graph. Meanwhile, the subgroups of A_4 with order three is a union of a complete digraph with three vertices and a directed graph with nine vertices and 24 directed edges.

References

[1] Devi S and John Arul Singh J 2017 A Note on Cyclic Subgroup Graph of a Finite Groups *Communications in Applied Analysis* **21** No. 3 425-430
[2] Tolue B 2018 The Stable Subgroup Graph *Boletim da Sociedade Paranaense de Matematica* **36** 3 129–139
[3] Bhuniya A K and Bera S 2016 Normal subgroup Based Power Graphs of a Finite Group *Communication in Algebra* DOI: 10.1080/00927872.2016.1236122
[4] Al-Hasanat B N and Al-Hasanat A S 2019 Order Graph: A New Representation of Finite Groups *International Journal of Mathematics and Computer Science* **14** 809-819
[5] Kakeri F and Erfanian A 2015 The Complement of Subgroup Graph of a Group *Journal of Prime Research in Mathematics* **11** 55-60
[6] Rahin N F, Sarmin N H and Ilangovan S 2020 The Non-Normal Subgroup Graph of Some Dihedral Groups *ASM Science Journal* **13** 1-7
[7] Fraleigh J B 2003 *A First Course in Abstract Algebra 7th Edition* Addison-Wesley Boston 78
[8] Durbin J R 2009 *Modern Algebra: An Introduction 6th Edition* John Wiley & Sons 35

Acknowledgments

The first author would like to acknowledge Universiti Teknologi Malaysia (UTM) for the funding of this research through UTM Fundamental Research Grant (UTMFR) Vote Number 20H70 and Universiti Pertahanan Nasional Malaysia (UPNM) for the support. The authors would also like to thank the Ministry of Higher Education Malaysia (MoHE) for the financial support through Fundamental Research Grant Scheme (FRGS/1/2020/STG06/UTM/01/2) and University of Nottingham Malaysia.