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Perfect Codes in the Spanning and Induced Subgraphs of the Unity Product Graph

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Abstract The unity product graph of a ring R is a graph which is obtained by setting the set of unit elements of R as the vertex set. The two distinct vertices r_i and r_j are joined by an edge if and only if $r_i \cdot r_j = e$. The subgraphs of a unity product graph which are obtained by the vertex and edge deletions are said to be its induced and spanning subgraphs, respectively. A subset C of the vertex set of induced (spanning) subgraph of a unity product graph is called perfect code if the closed neighbourhood of c, $S_1(c)$ forms a partition of the vertex set as c runs through C. In this paper, we determine the perfect codes in the induced and spanning subgraphs of the unity product graphs associated with some commutative rings R with identity. As a result, we characterize the rings R in such a way that the spanning subgraphs admit a perfect code of order cardinality of the vertex set. In addition, we establish some sharp lower and upper bounds for the order of C to be a perfect code admitted by the induced and spanning subgraphs of the unity product graphs.

Keywords Commutative Ring, Unity Product Graph, Induced Subgraph, Spanning Subgraph, Perfect Code

1 Introduction

All graphs studied in this research are simple and undirected. Let $\Gamma = (V, E)$ be a graph. A graph S is called a subgraph of Γ, written $S \subseteq \Gamma$, if $V(S) \subseteq V(\Gamma)$ and $E(S) \subseteq E(\Gamma)$. A subgraph of Γ is called spanning if it is obtained only by edge deletions and is called induced if it is obtained only by vertex deletions.

Associating simple and undirected graphs with rings has been thoroughly studied, for instance, the well-known zero divisor graph has a very long background in the past research on graphs of rings. The research on combinatorial structure (graph) associated with algebraic structures (ring) has first been proposed in [1]. It has further been investigated by Anderson and Livingston in [2]. In 2010, another combinatorial structure, named unit graph for a ring was introduced by [3] and its properties such as domination number [4], planarity [5], girth [6], diameter [7] and Hamiltonian [8] were further investigated.

Given $\Gamma = (V, E)$. For every $x \in V(\Gamma)$, the closed neighbourhood of the vertex x is denoted by $S_1(x)$ and is defined as $S_1(x) = \{y \in V(\Gamma) : d(x, y) \leq 1\}$. For the given Γ, every $C \subseteq V(\Gamma)$ is called a code. A code C is said to be perfect if $S_1(x) \cap S_1(y) = \emptyset$ for all distinct $x, y \in C$and $\bigcup S_1(x) = V(\Gamma)$ for all $x \in C$. The study of perfect codes in various graphs [9, 10, 11, 12, 13] is evolved from the research in [14], which in turn has roots in the theory of coding [15]. This concept forms an interesting branch of combinatorics which can be linked with group theory, ring theory and module theory among others. In recent years, the research of perfect codes in algebraic graphs got much attention and this notion have been extensively investigated in Cayley graphs of groups [16, 17, 18, 19], power and power reduced power graphs of groups [20, 21]. However, there are limited number of researches concentrating on investigating the perfect codes in graphs of rings (see, [22, 23]).

In this paper, the perfect codes in the induced and spanning subgraphs of unity product graphs of commutative rings with identity are studied. We characterize all commutative rings whose induced and spanning subgraphs of unity product graphs admit a perfect code of order cardinality of the vertex set. Moreover, we establish some sharp lower and the upper bounds for the order of C to be a perfect code in induced and spanning subgraphs of the unity product graphs.

2 Perfect Codes in Spanning Subgraphs of Unity Product Graphs

In this section, some results are established on determining the perfect codes in spanning subgraphs of the unity product graphs associated with some commutative rings with identity.

Proposition 2.1. *Let $\Gamma^s(R)$ be a spanning subgraph of $\Gamma^{'}(R)$ and C a perfect code in $\Gamma^s(R)$, then $C = U(R)$ if and only if any of the following hold:*
(1) $R \cong \mathbb{Z}_2$,
(2) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$,
(3) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$.
(4) $R \cong \mathbb{Z}_k$, where $k > 2$ is a divisor of 24.
(5) R is a reduced ring of $Char(R) = 0$ with $|U(R)| = 2$.

Proof. (1)-(3) If R is isomorphic to either $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$, then $U(R) = \{e\}$ and thus $\Gamma^{'}(R) = K_1$. Hence, $\Gamma^s(R)$ is also a K_1, because $V(\Gamma^{'}(R)) = V(\Gamma^s(R))$. Since, $V(\Gamma^s(R)) = \{e\}$, it follows that $S_1(e) = V(\Gamma^s(R))$. Therefore, $C = U(R) = \{e\}$, which is trivial.
(4) If $R \cong \mathbb{Z}_k$, $k > 2$ and $k|24$, then every elements of $U(R)$ is self-inverse. Since $V(\Gamma^{'}(R)) = U(R)$, it follows that $\Gamma^{'}(R)$ is a $\bar{K}_r$ graph, where $r = |U(R)|$. Similarly, since $V(\Gamma^{'}(R)) = V(\Gamma^s(R))$, this implies that $\Gamma^s(R)$ is also a $\bar{K}_r$ graph. Let $V(\Gamma^s(R)) = \{a_1, a_2, a_3, \cdots, a_r\}$, then the closed neighbourhoods of $a_i \in V(\Gamma^s(R))$ yield that $S_1(a_i) \cap S_1(a_j) = \emptyset$ for all $a_i \neq a_j,\ a_i, a_j \in V(\Gamma^s(R))$ and $\bigcup_{k=1}^{r} S_1(a_i) = V(\Gamma^s(R))$. Hence, $C = U(R)$. (5) Assume R is a reduced ring of $Char(R) = 0$ with $|U(R)| = 2$, then $V(\Gamma^s(R) = V(\Gamma^{'}(R) = \{e, -e\}$. Since $e \cdot (-e) \neq e$, it follows that $\Gamma^s(R) = 2K_1$. To show $C = U(R)$ is the perfect code in $\Gamma^s(R)$, it follows from the proof of part(4).
Conversely if $C = U(R)$, thus $S_1(a_i) = \{a_i\}$ for all $a_i \in C$. Hence, based on $S_1(a_i)$ the spanning subgraph $\Gamma^s(R)$ is either a K_1 or a $\bar{K}_r$ graph, which implies R is isomorphic to either $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ or $R \cong \mathbb{Z}_k$, where $k > 2$ is a divisor of 24 or R is a reduced ring of $Char(R) = 0$ with $|U(R)| = 2$.

□

Theorem 2.1. *Let $\Gamma^s(R)$ be a spanning subgraph of $\Gamma^{'}(R)$ and C a perfect code in $\Gamma^s(R)$. Then, $2^{m-2}+2 \leq |C| \leq 2^{m-1}$ if $R \cong \mathbb{Z}_{2^m}$ for $m \geq 3$.*

Proof. If $R \cong \mathbb{Z}_{2^m}$, $m \geq 3$, then $|U(R)| = 2^{m-1}$. Since, R contains four elements which are self-inverses, it follows that $\Gamma^{'}(R) = \bigcup_{i=1}^{2^{m-2}-2} K_2 \cup \bar{K}_4$. Suppose that $\Gamma^s(R)$ is the spanning subgraph of $\Gamma^{'}(R)$, thus $V(\Gamma^s(R)) = V(\Gamma^{'}(R))$. By deleting $2^{m-2} - 2$ edges of $\Gamma^{'}(R)$, then $2^{m-2} - 2 + 1 = 2^{m-2} - 1$ distinct spanning subgraphs including $\Gamma^{'}(R)$ are obtained. By deleting one edge of $\Gamma^{'}(R)$ at each stage, the edges of $\Gamma^{'}(R)$ will be deleted in $2^{m-2} - 2$ stages. This yields the following spanning subgraphs:

$$\Gamma_1^s(R) = \bigcup_{i=1}^{2^{m-2}-2} K_2 \cup \bar{K}_4 = \Gamma^{'}(R) \qquad (1)$$

$$\Gamma_2^s(R) = \bigcup_{i=1}^{2^{m-2}-3} K_2 \cup \bar{K}_6 \qquad (2)$$

$$\Gamma_3^s(R) = \bigcup_{i=1}^{2^{m-2}-4} K_2 \cup \bar{K}_8 \qquad (3)$$

$$\vdots$$

$$\Gamma_{(r)}^s(R) = \bar{K}_{2^{m-1}} \qquad (r),$$

where $r = 2^{m-2} - 1$. According to (1), (2), $\cdots$, (r), $\Gamma_1^s(R)$ has the minimum independence number and $\Gamma_{(2^{m-2}-1)}^s(R)$ has the maximum independence number. Suppose that $\Gamma_1^s(R) = \Gamma^{'}(R)$, thus $\Gamma_1^s(R)$ contains $2^{m-2} - 2 + 4 = 2^{m-2} + 2$ number of independence vertices. Let $C \subseteq \Gamma^{'}(R)$ be a code, then $C = \{r_j : r_j \text{ is an independence number of } \Gamma_1^s(R)\}$ is an order $2^{m-2} + 2$ perfect code in $\Gamma_1^s(R)$ since it satisfies the conditions $S_1(r_j) \cap S_1(r_k) = \emptyset$ for all distinct $r_j, r_k \in C$ and $\bigcup_{j=1}^{2^{m-2}+2} S_1(r_j) = V(\Gamma^s(R))$ for all $r_j \in C$. Hence, $|C| \geq 2^{m-2} + 2$, and therefore, we got a lower bound for $|C|$.
Similarly, let $\Gamma_{2^{m-2}-1}^s(R) = \bar{K}_{2^{m-1}}$, then $\Gamma_{2^{m-2}-1}^s(R)$ contains 2^{m-1} independence vertices, that is in this case all the vertices of $\Gamma_{2^{m-2}-1}^s(R)$ are isolated. Let $C \subseteq \Gamma^{'}(R)$ be a code, thus $C = \{r_j : r_j \text{ is an independence number of } \Gamma_{(2^{m-2}-1)}^s(R)\}$ is an order 2^{m-1} perfect code in $\Gamma_{2^{m-2}-1}^s(R)$ since it satisfies the conditions $S_1(r_j) \cap S_1(r_k) = \emptyset$ for all distinct $r_j, r_k \in C$ and $\bigcup_{j=1}^{2^{m-1}} S_1(r_j) = V(\Gamma^s(R))$ for all $r_j \in C$. Hence, $|C| \leq 2^{m-1}$, and therefore, we got an upper bound for $|C|$. Therefore, $2^{m-2} + 2 \leq |C| \leq 2^{m-1}$. □

The results found in Theorem 2.1 can be generalized for any commutative ring whose associated unity product graph contains only four isolated vertices. This result is stated in Corollary 2.1.

Corollary 2.1. *Let $\Gamma^s(R)$ be the spanning subgraph of $\Gamma^{'}(R)$ and C a perfect code in $\Gamma^s(R)$. Then, $k + 2 \leq |C| \leq 2k$ if $\Gamma^{'}(R)$ contains four isolated vertices and $|V(\Gamma^{'}(R))| = 2k$, for all $k \geq 2$.*

Theorem 2.2. *If $\Gamma^s(R)$ is a spanning subgraph of $\Gamma^{'}(R)$ and C a perfect code in $\Gamma^s(R)$. Then, $\frac{p+1}{2} \leq |C| \leq p - 1$ if R is a division ring with $O(R) = p$, p is an odd prime.*

Proof. Assume R is a division ring with $O(R) = p$, where $p \geq 3$ is prime. Then, all non-zero elements of R are units, i.e. $U(R) = \{r_i : r_i \neq 0\}$. Since R is a division ring, it follows that it contains the identity element e such that $e \cdot r_i \neq e$ for all $r_i \neq e$. Similarly, R contains the element $-e$ such that $-e \cdot r_i \neq e$ for all $r_i \neq -e$. Also, there exist distinct unique elements $r_i, r_j \in V(\Gamma^{'}(R)) \setminus \{e, -e\}$ such that $r_i \cdot$

$r_j = e$, meaning that $\{r_i, r_j\}$ form an edge of $\Gamma^{'}(R)$. Hence, $\Gamma^{'}(R) = \bigcup_{i=1}^{\frac{p-3}{2}} K_2 \cup \bar{K}_2$. Suppose that $\Gamma^s(R)$ is the spanning subgraph of $\Gamma^{'}(R)$, then $V(\Gamma^s(R)) = V(\Gamma^{'}(R))$. It means that by deleting $\frac{p-3}{2}$ edges of $\Gamma^{'}(R)$, we get that $\frac{p-3}{2}+1 = \frac{p-1}{2}$ distinct spanning subgraphs including $\Gamma^{'}(R)$ are obtained. If we delete one edge of $\Gamma^{'}(R)$ at each stage, then the edges of $\Gamma^{'}(R)$ can be deleted in $\frac{p-3}{2}$ stages. This yields the following spanning subgraphs:

$$\Gamma_1^s(R) = \bigcup_{i=1}^{\frac{p-3}{2}} K_2 \cup \bar{K}_2 = \Gamma^{'}(R) \quad (1)$$

$$\Gamma_2^s(R) = \bigcup_{i=1}^{\frac{p-5}{2}} K_2 \cup \bar{K}_4 \quad (2)$$

$$\Gamma_3^s(R) = \bigcup_{i=1}^{\frac{p-7}{2}} K_2 \cup \bar{K}_6 \quad (3)$$

$$\vdots$$

$$\Gamma^s_{\frac{p-1}{2}}(R) = \bar{K}_{p-1} \quad (\frac{p-1}{2})$$

Based on Equations (1), (2), $\cdots$, $\frac{p-1}{2}$, $\Gamma_1^s(R)$ has the minimum independence number and $\Gamma^s_{(\frac{p-1}{2})}(R)$ has the maximum independence number. Let $\Gamma_1^s(R) = \Gamma^{'}(R)$, thus $\Gamma_1^s(R)$ contains $\frac{p+1}{2}$ number of independence vertices. Let $C \subseteq \Gamma^{'}(R)$ be a code, thus $C = \{r_k : r_k \text{ is an independence number of } \Gamma_1^s(R)\}$ is an order $\frac{p+1}{2}$ perfect code in $\Gamma_1^s(R)$ since it satisfies the conditions $S_1(r_k) \cap S_1(r_l) = \emptyset$ for all distinct $r_k, r_l \in C$ and $\bigcup_{k=1}^{\frac{p+1}{2}} S_1(r_k) = V(\Gamma_1^s(R))$ for all $r_k \in C$. Hence, $|C| \geq \frac{p+1}{2}$, and therefore, we got a lower bound for $|C|$.

Similarly, let $\Gamma^s_{(\frac{p-1}{2})}(R) = \bar{K}_{p-1}$, then $\Gamma^s_{(\frac{p-1}{2})}(R)$ contains $p-1$ independence vertices. That is in this case, all the vertices of $\Gamma^s_{(\frac{p-1}{2})}(R)$ are isolated. Let $C \subseteq \Gamma^{'}(R)$ be a code, thus $C = \{r_k : r_k \text{ is an independence number of } \Gamma^s_{\frac{p-1}{2}}(R)\}$ is an order $p-1$ perfect code in $\Gamma^s_{(\frac{p-1}{2})}(R)$ since it satisfies the conditions $S_1(r_k) \cap S_1(r_l) = \emptyset$ for all distinct $r_k, r_l \in C$ and $\bigcup_{k=1}^{p-1} S_1(r_k) = V(\Gamma^s_{(\frac{p-1}{2})}(R))$ for all $r_k \in C$. Hence, $|C| \leq p-1$, and therefore, we got an upper bound for $|C|$. Therefore, $\frac{p+1}{2} \leq |C| \leq p-1$. □

The obtained results in Theorem 2.2 can be generalized for any commutative ring whose associated unity product graph contains only two isolated vertices. This result is given in the following corollary.

Corollary 2.2. *Let $\Gamma^s(R)$ be the spanning subgraph of $\Gamma^{'}(R)$ and C a perfect code in $\Gamma^s(R)$. Then, $k+1 \leq |C| \leq 2k$ if $\Gamma^{'}(R)$ contains two isolated vertices and $|V(\Gamma^{'}(R))| = 2k$, for all $k \geq 1$.*

3 Perfect Codes in Induced Subgraphs of Unity Product Graphs

In this section, some results are established on determining the perfect codes in induced subgraphs of the unity product graphs associated with some commutative rings with identity.

Proposition 3.1. *Let $\Gamma^i(R)$ be a connected induced subgraph of $\Gamma^{'}(R)$ and C a perfect code in $\Gamma^i(R)$. Then, C is trivial.*

Proof. Assume $\Gamma^{'}(R)$ is the unity product graph of R, thus $V(\Gamma^{'}(R)) = U(R)$ and $E(\Gamma^{'}(R)) = \{\{r_i, r_j\} : r_i \cdot r_j = e \text{ for all } r_i \neq r_j\}$. Based on the vertex adjacency of $\Gamma^{'}(R)$, the only connected induced subgraph of $\Gamma^{'}(R)$ is K_2. Let r_1 and r_2 be the endpoints vertices of $\Gamma^i(R)$, it follows that an independence vertex set of $\Gamma^i(R)$ is $C = \{r_1\}$. Consequently, C is a trivial perfect code, since $S_1(r_1) = V(\Gamma^i(R))$. □

Theorem 3.1. *Let $\Gamma^i(R)$ be an induced subgraph of $\Gamma^{'}(R)$ which contains no isolated vertices and C a perfect code in $\Gamma^i(R)$. Then, $1 \leq |C| \leq \frac{p-3}{2}$ if R is a division ring with $O(R) = p$, $p \geq 5$ is prime.*

Proof. Assume $\Gamma^{'}(R)$ is the unity product graph associated with a division ring R with $O(R) = p$, where $p \geq 5$ is prime, thus $V(\Gamma^{'}(R)) = U(R)$ and $E(\Gamma^{'}(R)) = \{\{r_i, r_j\} : r_i \cdot r_j = e \text{ for all } r_i \neq r_j\}$. Based on the vertex adjacency of $\Gamma^{'}(R)$, then $\Gamma^{'}(R) = \bigcup_{i=1}^{\frac{p-3}{2}} K_2 \cup \bar{K}_2$. Since $\Gamma^i(R)$ contains no isolated vertices, it follows that $\Gamma^i(R) = \bigcup_{i=1}^{\frac{p-3}{2}} K_2$. For $p \geq 5$, we list all induced subgraphs as follows:

$$\Gamma_1^i(R) = K_2. \quad (1)$$

$$\Gamma_2^i(R) = \bigcup_{m=1}^{2} K_2. \quad (2)$$

$$\Gamma_4^i(R) = \bigcup_{m=1}^{4} K_2. \quad (4)$$

$$\vdots$$

$$\Gamma^i_{\frac{p-3}{2}}(R) = \bigcup_{m=1}^{\frac{p-3}{2}} K_2. \quad (\frac{p-3}{2})$$

Let $\Gamma_1^i(R) = K_2$, thus $\Gamma_1^i(R)$ contains one independence number. Suppose $C \subseteq V(\Gamma_1^i(R))$ is a code, then $C = \{r_k : r_k \text{ is an independence number of } \Gamma_1^i(R)\}$ is an order 1 perfect in $\Gamma_1^i(R)$, since $S_1(r_k) = V(\Gamma_1^i(R))$ for $r_k \in C$. This forms the lower bound for C, i.e. $|C| \geq 1$.

Similarly, let $\Gamma^i_{\frac{p-3}{2}}(R) = \bigcup_{m=1}^{\frac{p-3}{2}} K_2$, thus $\Gamma^i_{\frac{p-3}{2}}(R)$ contains $\frac{p-3}{2}$ independence number. Suppose that $C = \{r_k : r_k \text{ is an independence number of } \Gamma^i_{\frac{p-3}{2}}(R)\}$, thus the closed neighbourhood of the vertices $r_k \in C$ with radius 1, $S_1(r_k)$, partitions the vertex set $V(\Gamma^i_{\frac{p-3}{2}}(R))$ into $\frac{p-3}{2}$ disjoint sets, meaning that $S_1(r_k) \cap S_1(r_l) = \emptyset$ for all $r_k \neq r_l,\ r_k, r_l \in C$ and $\bigcup_{k=1}^{\frac{p-3}{2}} S_1(r_k) = V(\Gamma^i_{\frac{p-3}{2}}(R))$ for all $r_k \in C$. Hence, C is an $\frac{p-3}{2}$ order perfect code, which forms the upper bound for C, i.e. $|C| \leq \frac{p-3}{2}$. □

Theorem 3.2. *Let $\Gamma^i(R)$ be an induced subgraph of $\Gamma^{'}(R)$ and C a perfect code in $\Gamma^i(R)$. Then, $1 \leq |C| \leq k+1$, if $|V(\Gamma^{'}(R))| = 2k$, $k \geq 1$ and $\Gamma^{'}(R)$ contains two isolated vertices.*

Proof. Suppose $\Gamma^{'}(R)$ contains two isolated vertices, it follows that $\Gamma^{'}(R)$ contains $2k-2$ non-isolated vertices, since $|V(\Gamma^{'}(R))| = 2k$. Since $U(R)$ is an inverse closed set, it shows that $\Gamma^{'}(R)$ is consisting of $\frac{2k-2}{2} = k-1$ copies of K_2 with two copies of K_1, that is $\Gamma^{'}(R) = \bigcup_{m=1}^{k-1} K_2 + 2K_1$. Let $\Gamma^i(R)$ be an induced subgraph of $\Gamma^{'}(R)$, thus by deleting one endpoint vertex for each K_2, an induced subgraph with a maximum independence number is obtained, which is $\Gamma^i(R) = (k+1)K_1$. Similarly, by deleting $k-2$ copies of K_2 with two isolated vertices, an induced subgraph with a minimum independence number is obtained, which is $\Gamma^i(R) = K_2$. If $\Gamma^i(R) = K_2$, then by Theorem 3.1, $|C| = 1$, which forms the lower bound for C. Similarly, if $\Gamma^i(R) = (k+1)K_1$, then $S_1(r_m) = \{r_m\}$ for every $r_m \in \Gamma^i(R)$, which satisfies the conditions $S_1(r_m) \cap S_1(r_l) = \emptyset$ for all $r_m \neq r_l,\ r_m, r_l \in V(\Gamma^i(R))$ and $\bigcup_{m=1}^{k+1} S_1(r_m) = V(\Gamma^i(R))$. Hence, $C = V(\Gamma^i(R))$ is an order $k+1$ perfect code, which forms the upper bound for C, i.e. $|C| \leq k+1$. □

Theorem 3.3. *Let $\Gamma^i(R)$ be an induced subgraph of $\Gamma^{'}(R)$ and C a perfect code in $\Gamma^i(R)$. Then, $1 \leq |C| \leq k+2$, if $|V(\Gamma^{'}(R))| = 2k$, $k \geq 2$ and $\Gamma^{'}(R)$ contains four isolated vertices.*

Proof. Suppose $\Gamma^{'}(R)$ contains four isolated vertices, thus according to $V(\Gamma^{'}(R))$ the number of non-isolated vertices of $\Gamma^{'}(R)$ is $2k-4$. Since $U(R)$ is an inverse closed set, it follows that $\Gamma^{'}(R)$ is a graph consisting of $\frac{2k-4}{2} = k-2$ copies of K_2 with four copies of K_1, that is $\Gamma^{'}(R) = \bigcup_{m=1}^{k-2} K_2 + 4K_1$. Let $\Gamma^i(R)$ be the induced subgraph of $\Gamma^{'}(R)$, thus by deleting one endpoint vertex for each K_2, an induced subgraph with a maximum independence number is obtained, which is $\Gamma^i(R) = (k+2)K_1$. Similarly, by deleting $k-3$ copies of K_2 with four isolated vertices, an induced subgraph with a minimum independence number is obtained, which is $\Gamma^i(R) = K_2$. If $\Gamma^i(R) = K_2$, then by Theorem 3.1, $|C| = 1$, which forms the lower bound for C. Similarly, if $\Gamma^i(R) = (k+2)K_1$, then $S_1(r_m) = \{r_m\}$ for every $r_m \in \Gamma^i(R)$, which satisfies the conditions $S_1(r_m) \cap S_1(r_l) = \emptyset$ for all $r_m \neq r_l,\ r_m, r_l \in V(\Gamma^i(R))$ and $\bigcup_{m=1}^{k+2} S_1(r_m) = V(\Gamma^i(R))$. Hence, $C = V(\Gamma^i(R))$ is an order $k+2$ perfect code, which forms the upper bound for C, i.e. $|C| \leq k+2$. □

4 Conclusion

In this paper, the perfect codes in the induced and the spanning subgraphs of the unity product graphs associated with some commutative rings R with identity have been determined. We find that the the spanning subgraphs admit a perfect code of order cardinality of the vertex set if $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ or $R \cong \mathbb{Z}_k$, where $k > 2$ is a divisor of 24. However, for the rings R which are not as the same forms, we established some sharp lower and upper bounds for the order of C to be a perfect code admitted by the induced and spanning subgraphs of the unity product graphs.

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