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Application of Caputo Fractional Derivatives to the Convective Flow of Casson Fluids in a Microchannel with Thermal Radiation

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ABSTRACT

In this paper, the application of Caputo fractional derivative on unsteady boundary layer Casson fluid flow in a microchannel is studied. The partial differential equations which governed the problem are considered with the presence of thermal radiation. The fractional partial differential equations are transformed into dimensionless governing equations using appropriate dimensionless variables. It is then solved analytically using the Laplace transform technique which transforms the equations into linear ordinary differential equations. These transformed equations are then solved using the appropriate method, and the inverse Laplace transform technique is applied to obtain the solution in form of velocity and temperature profiles. Graphical illustrations are acquired using Mathcad software and the influence of important physical parameters on velocity and temperature profiles are analyzed. Results show that thermal radiation and fractional parameter have enhanced the velocity and temperature profiles.

1. Introduction

Generally, most of the motion in nature such as fluid flow, heat transfer, the wave of sound, and others can be mathematically described by using the partial differential equation (PDE). Natural phenomena including velocity and acceleration are normally described in derivatives [1]. Besides, PDEs are often used by scientists and engineers to explore a wide range of physical phenomena including fluid dynamics, electricity, and thermal transfer.

In 2007, Jumarie [2] has investigated PDE using fractional derivative and solve using modified Riemann-Liouville derivative. Based on Khalil *et al.,* [3], the fractional derivative has been existed a long time ago, even as old as calculus. The application of fractional derivatives has provided more general and accurate models of a system compared to the traditional calculus since it is concerned with the real or complex order generalization of integrals and derivatives. Due to order differentiation features, systems described with fractional calculus are non-linear and may exhibit a significantly richer dynamical behavior. Moreover, the most real-world problem is modeled by a

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fractional derivative equation. Ray *et al.,* [4] has mentioned the vast range of fields on the usage of fractional derivatives in science as well as engineering fields.

The most popular and frequently used definition of the fractional derivative is the Caputo definition. The kernel of the Caputo fractional derivative is singular. The development of Caputo is a great help in solving problems with the unusual initial condition without physical meaning and tough to operate. By modifying the fractional parameter of Caputo, the result obtained is more realistic since it can be compared to experimental data for an excellent agreement. Caputo fractional derivative operator has been investigated by many previous researchers [5-11].

In fluid flow problems, non-Newtonian fluid is a fluid that does not obey Newton's vicious law. The most common type of non-Newtonian fluid used by previous researchers is Casson fluid. Mukhopadhyay *et al.,* [12] explains that Casson fluid behaved as a shear-thinning liquid. The characteristics of Casson fluid are, firstly, at zero rates of shear, it is simulated to have an infinite viscosity. Secondly, yield stress below which there is no flow. Lastly, it has a zero viscosity at an infinite rate of shear. Hussanan *et al.,* [13] mentioned that non-Newtonian fluids were used in cosmetics, pharmaceutical, and food industries. Also supported by Khalid *et al.,* [14], applications of non-Newtonian fluids are broad including in geophysics and petroleum industries. Ali *et al.,* [6], Ahmad *et al.,* (2021) [8], Imran *et al.,* [15], Awan *et al.,* [16], Ullah *et al.,* [17], and Jamil *et al.,* [18] are some researchers who researched Casson fluid. Examples of Casson fluid that can be seen in real life such as ketchup, honey, and human blood.

The application of fluid problems in microdevices is currently in attention by many researchers due to its importance in real-life problems. Khan *et al.,* [19] agreed since its practical uses in space technology, engineering, and material processing operation as well as in high power density processors in supercomputers and many other devices. In an engineering context, microfluidics is the flows of fluid and gases in single or multiple phases through microdevices fabricated by Micro Electro Mechanical Systems (MEMS) technology [20]. As stated by Gad-el-Hak [21], MEMS devices that involve fluid flows are such as micro ducts, micropumps, microturbines, and microvalves. Some studies in microchannels also discussed in the studies by Cao *et al.,* [22], Ahmad *et al.,* [23], and Phu *et al.,* [24].

Bearing in mind the importance of fractional derivatives, very few studies have been reported on non-Newtonian fluid flowing in a microchannel with Caputo fractional derivative approach. Thus, this study is focused on solving the fractional-order time equation of motion in a microchannel. Thermal radiation is also considered to improve the understanding of thermal radiation on Casson fluid convective flow. The equations are then solved analytically using Laplace transform and inverse Laplace transform is used to solve velocity and temperature profiles. Mathcad is used to generate graphical findings for differential values of the fractional parameter as well as several crucial physical parameters such as the Casson parameter.

2. Modeling of the Problem

In this study, the fluid is considered flowing in a microchannel that is made up of two vertical parallel plates at a distance d apart. The movement of the fluid is in the x-axis direction and the y-axis is perpendicular to the movement of the fluid flow. The schematic diagram of the microchannel is shown in Figure 1.

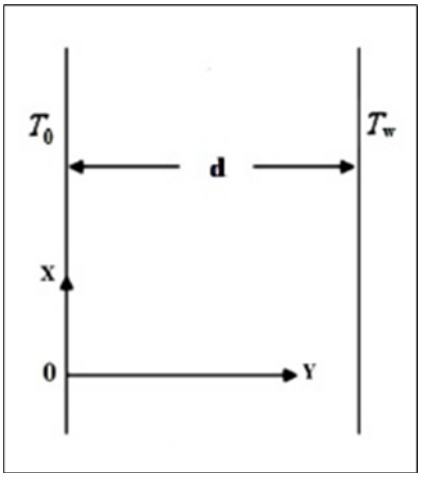

Fig. 1. Schematic diagram of the microchannel

The fluid is stationary at the time, $t=0$ and the temperature of the plate $y=0$ is at T_o. As t increasing, the temperature of the plate at $y=d$, the plate starts to oscillate at $u(y,t)=U_oH(t)\cos\omega t.$ Here U_o is the amplitude, ω is the frequency of oscillation and $H(t)$ is the Heaviside unit step function. For an unsteady Casson fluid flow, the shear stress is defined as follows:

$$\tau_{i,j}=\begin{cases}2\left(\mu_B+\dfrac{p_y}{\sqrt{2\pi}}\right)e_{ij},\pi>\pi_c\\[2ex]2\left(\mu_B+\dfrac{p_y}{\sqrt{2\pi_c}}\right)e_{ij},\pi<\pi_c\end{cases}$$

where τ represent the shear stress, μ is the dynamic viscosity, e_{ij} is the $(i,j)th$ component for deformation rate that is represented by $\pi=e_{ij}e_{ij}$. The yield stress of Casson fluid is shown by p_y. π_c depicts the critical value of the product whereas μ_B is the plastic dynamics viscosity.

The presence of thermal radiation in this study is optically thick fluid in one space coordinate with y. It is expressed as follows [19],

$$q_r=-\frac{4\sigma}{3k_1}\frac{\partial T^4}{\partial y} \tag{1}$$

where σ is the Stefan-Boltzmann constant $\left(5.670\times10^{-8}\dfrac{W}{m^2K^4}\right)$ and k_1 is the mean absorption coefficient. By using the Taylor series, T^4 is linearized about T_o and the other higher-order terms are abandoned since it has only a small difference between T to T_o. By neglecting the higher-order terms, T^4 can be simplified to a simpler form which is

$$T^4=4TT_o^3-3T_o^4 \tag{2}$$

Based on all assumptions above, the governing equation consists of momentum and energy equations can be written as

$$\frac{\partial u}{\partial t}=\upsilon\left(1+\frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2}+g\beta_T\left(T-T_o\right) \tag{3}$$

$$\rho C_p\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{\partial y^2}-\frac{\partial q_r}{\partial y} \tag{4}$$

subject to initial and boundary conditions

$$\begin{aligned} &u(y,0)=0 && T(y,0)=T_o \\ &u(0,t)=0 && T(0,t)=T_o \\ &u(d,t)=H(t)U\cos\omega t && T(d,t)=T_w \end{aligned} \tag{5}$$

Here, $u(y,t)$ is the fluid velocity, $T(y,t)$ is fluid temperature, υ is the kinematic viscosity, β is the Casson fluid parameter, g is the gravitational acceleration, β_T is the coefficient of thermal expansion, ρ is the density of the fluid, c_p is the specific heat at constant pressure, k is the thermal conductivity, q_r is the thermal radiation, T_∞ is the ambient temperature and T_w is the wall temperature. The velocity and temperature depend on space variables y and time t.

Combining the Eq. (1) and Eq. (2) into Eq. (4), the energy equation becomes

$$\rho C_p\frac{\partial T}{\partial t}=k\left(1+\frac{16\sigma T_o^3}{3kk_1}\right)\frac{\partial^2 T}{\partial y^2} \tag{6}$$

By substituting the following dimensionless variables [18],

$$v=\frac{u}{U},\quad \tau=\frac{t\upsilon}{d^2},\quad \xi=\frac{y}{d},\quad \theta=\frac{T-T_o}{T_w-T_o}, \tag{7}$$

into Eq. (3) and Eq. (6), the dimensionless equations are obtained as

$$\frac{\partial v}{\partial \tau}=\frac{1}{\beta_o}\frac{\partial^2 v}{\partial \xi^2}+Gr\theta \tag{8}$$

$$\Pr_{eff}\frac{\partial \theta}{\partial \tau}=\frac{\partial^2 \theta}{\partial \xi^2} \tag{9}$$

subject to

$$\frac{1}{\beta_o}=\left(\frac{\beta}{\beta+1}\right)\quad Gr=\frac{d^2 g\beta_T\left(T_w-T_o\right)}{\upsilon U}\quad \Pr=\frac{\mu_B C_p}{k}\quad R=\frac{16\sigma T_o^3}{3kk_1}\quad \Pr_{eff}=\frac{\Pr}{\left(1+R\right)}$$

Here, β_o is the dimensionless Casson fluid parameter, Gr is the thermal Grashof number, $\Pr$ is the Prandtl number, R is the radiation parameter, and $\Pr_{eff}$ is the effective Prandtl number, respectively. Also, the condition (5) in the form of dimensionless initial and boundary conditions are given as

$$\begin{aligned} v(\xi,0)&=0 \qquad \theta(\xi,0)=0\\ v(0,\tau)&=0 \qquad \theta(0,\tau)=0\\ v(1,\tau)&=H(\tau)\cos\Omega\tau\ \theta(d,\tau)=1 \end{aligned} \tag{10}$$

where $\Omega=\frac{\omega d^2}{\upsilon}$. Then, by using Caputo fractional, where the Caputo derivative operator is

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha}=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\int_0^t\left(-\frac{1}{(t-\tau)^\alpha}\right)\frac{\partial f(\tau)}{\partial\tau}\partial\tau;0<\alpha<1\\ \frac{\partial f(t)}{\partial t};\alpha=1\end{cases} \tag{11}$$

the transformed Eq. (8) and Eq. (9) in the form of fractional equations are obtained as

$$D_\tau^\alpha v=\frac{1}{\beta_o}\frac{\partial^2 v}{\partial\xi^2}+Gr\theta \tag{12}$$

$$\Pr_{eff} D_\tau^\alpha\theta=\frac{\partial^2\theta}{\partial\xi^2} \tag{13}$$

where the fractional parameter is denoted by α .

3. Solution of the Problem

The obtained fractional dimensionless equations are then solved analytically using the Laplace transform method. The technique will reduce the equations into a set of linear ordinary differential equations. Later, the solutions are acquired by applying the inverse Laplace technique.

3.1 Solution of the Energy Equation

The Laplace transform for energy Eq. (13) when applying the boundary conditions (10), we obtain

$$\frac{\partial^2\theta}{\partial\xi^2} - \Pr_{eff}\, q^{\alpha}\bar{\theta} = 0 \tag{14}$$

Since, it is shown to be a homogenous equation, the method of solving a homogenous equation, the characteristic technique is used. The solution of Eq. (14) is determined as

$$\bar{\theta} = \frac{1}{q\left[e^{\sqrt{\Pr_{eff}\, q^{\alpha}}} - e^{-\sqrt{\Pr_{eff}\, q^{\alpha}}}\right]}\left[e^{\xi\sqrt{\Pr_{eff}\, q^{\alpha}}} - e^{-\xi\sqrt{\Pr_{eff}\, q^{\alpha}}}\right] \tag{15}$$

3.2 Solution of the Momentum Equation

Similarly, the Laplace transform for momentum Eq. (12) when applying the subjected boundary conditions (10), we have

$$\frac{\partial^2\bar{v}}{\partial\xi^2} - \beta_o q^{\alpha}\bar{v} = -\beta_o Gr\frac{1}{q\left[e^{\sqrt{\Pr_{eff}\, q^{\alpha}}} - e^{-\sqrt{\Pr_{eff}\, q^{\alpha}}}\right]}\left[e^{\xi\sqrt{\Pr_{eff}\, q^{\alpha}}} - e^{-\xi\sqrt{\Pr_{eff}\, q^{\alpha}}}\right] \tag{16}$$

Eq. (16) is a non-homogenous equation. By choosing the method of undetermined coefficient, the velocity profile is obtained as

$$\begin{aligned}\bar{v}(\xi,q) = &\frac{q}{q^2+\Omega^2}\left(\frac{e^{-\xi\sqrt{\beta_o q^{\alpha}}} - e^{\xi\sqrt{\beta_o q^{\alpha}}}}{e^{-\sqrt{\beta_o q^{\alpha}}} - e^{\sqrt{\beta_o q^{\alpha}}}}\right)\\ &+\frac{Gr\left(e^{\sqrt{\Pr_{eff}\, q^{\alpha}}} - e^{-\sqrt{\Pr_{eff}\, q^{\alpha}}}\right)}{q\left(e^{\sqrt{\Pr_{eff}\, q^{\alpha}}} - e^{-\sqrt{\Pr_{eff}\, q^{\alpha}}}\right)\left(\Pr_{eff}\, q^{\alpha} - \beta_o q^{\alpha}\right)}\left(\frac{e^{-\xi\sqrt{\beta_o q^{\alpha}}} - e^{\xi\sqrt{\beta_o q^{\alpha}}}}{e^{-\sqrt{\beta_o q^{\alpha}}} - e^{\sqrt{\beta_o q^{\alpha}}}}\right)\\ &-\frac{Gr_o}{q\left(e^{\sqrt{\Pr_{eff}\, q^{\alpha}}} - e^{-\sqrt{\Pr_{eff}\, q^{\alpha}}}\right)\left(\Pr_{eff}\, q^{\alpha} - \beta_o q^{\alpha}\right)}\left(e^{\xi\sqrt{\Pr_{eff}\, q^{\alpha}}} - e^{-\xi\sqrt{\Pr_{eff}\, q^{\alpha}}}\right)\end{aligned} \tag{17}$$

3.3 Inverse Laplace Transform

The temperature and velocity functions are given in Eq. (15) and Eq. (17). In practice, explicit functions cannot be used to calculate the inverse Laplace transform. As a result, using Zakian's explicit formula approach, the inverse Laplace transform is computed numerically. The weighted function is used to approximate the time domain function. Zakian's algorithm-based technique is described by

$$f(t) = \frac{t}{2}\sum_{j=1}^{N}\mathrm{Re}\left\{K_j F\left(\frac{\alpha_j}{t}\right)\right\},$$

where K_j and α_j are constants that can be either real or complex conjugate pairs. Whereas N indicates the number of terms utilized in the summation where $f(t) \rightarrow f(t)$ when N is approaching infinity. The fact that the truncated error is ignorable for $N = 5$ is well-known as suggested by Halsted and Brown [25]. The values of K_j and α_j are seen as listed in the studies by Halsted and Brown [25], Wang and Zhan [26], and Zakian and Littlewood [27].

4. Graphical Results

This section shows the result obtained for the velocity and temperature profiles which have been displayed in a graphical form for the various values of parameters α, β_o, Gr, $\Pr$, and R.

The numerical findings have been compared with a study by Khan *et al.,* [19] those obtained using the same fluid with thermal radiation (Figure 2 and Figure 3). In the current work, we present the application of Caputo to the convective flow of Casson fluid flowing through an oscillating microchannel whereas Khan *et al.,* [19] considered the fluid flowing with the application of the Caputo-Fabrizio fractional derivative approach. Hence, for comparison purposes, the limiting case of the present study is considered where $Pr = 10, \tau = 1, \beta_o = 1.5, Gr = 5, \Omega\tau = \frac{\pi}{2}$ and $R = 1.5$ are used. As shown in Figure 2 and Figure 3, the results of the present study are in very good agreement with the outputs provided in a study by Khan *et al.,* [19].

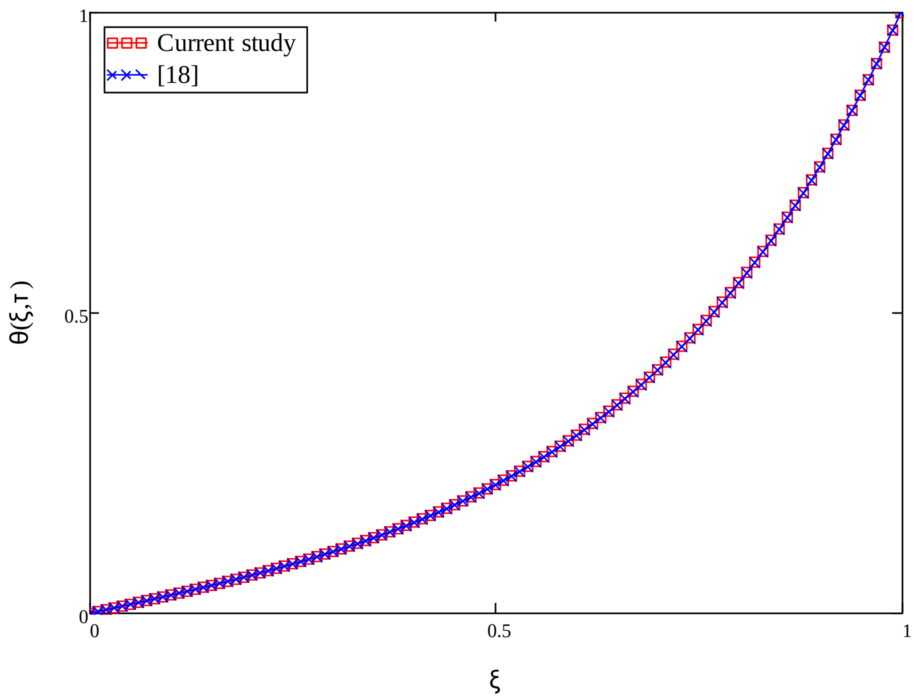

Fig. 2. Comparison of the temperature profile

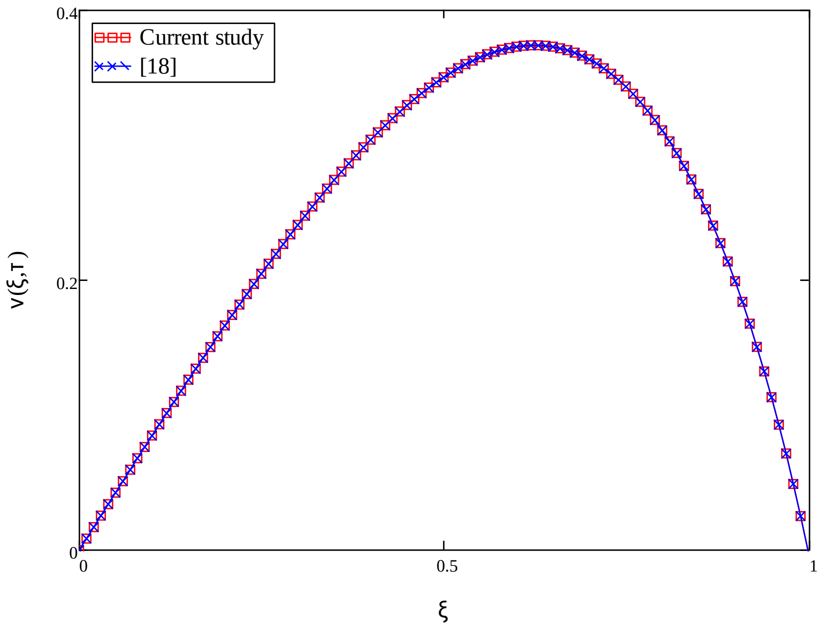

Fig. 3. Comparison of the velocity profile

4.1 Temperature Results

The influence of three parameters α, $\Pr$, and R on the temperature profiles is explored in this section.

Figure 4 depicts the effects of α on the fluid temperature. It shows the influence of α at $\theta(\xi,\tau)$ when $\Pr=12$, $R=1.5$, and $\tau = 1$. As α increases, the fluid temperature increases, and maximum curves obtained at $\alpha=1$. This result shows that the properties of classical Casson fluid have a higher temperature compared to the fractional Casson fluid model. Physically, it is true since a larger value of α will helps in strengthening the buoyancy forces, which then will increase the thermal boundary layer, and then increases the temperature profile.

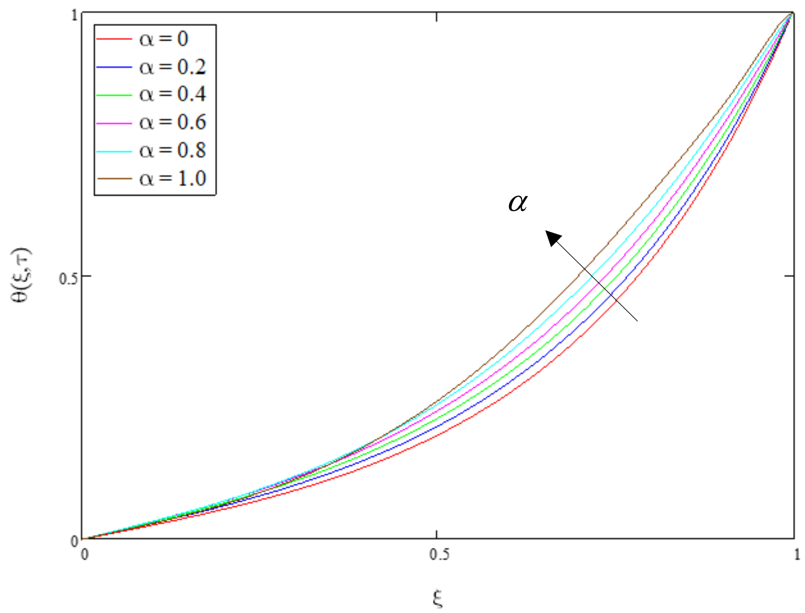

Fig. 4. Temperature profiles with different values of α

Figure 5 and Figure 6 describe the temperature profiles with various values of $\Pr$ and R respectively. Figure 5 displays the profiles of $\theta(\xi,\tau)$ when $\alpha=0.5$, $R=1.5$, and $\tau=1$ while Figure 6 illustrates the distributions of $\theta(\xi,\tau)$ when $\alpha=0.5$, $\Pr=10$, and $\tau=1$. By analyzing the graph below, increasing in $\Pr$ has caused the temperature profile to decrease. Whereas increasing in R shows a contradicting effect on temperature profiles where it shows an increasing behavior.

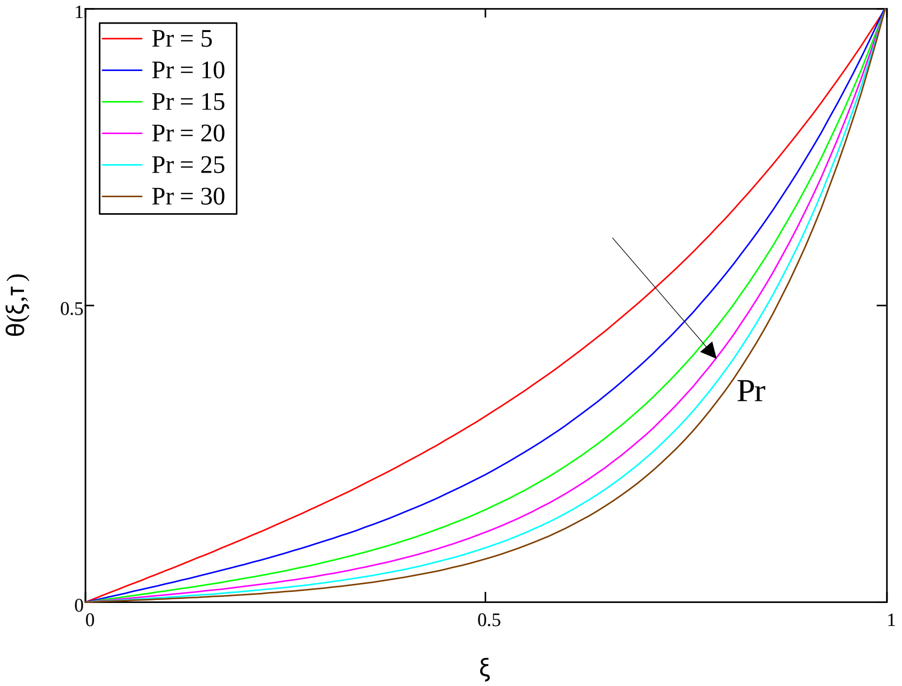

Fig. 5. Temperature profiles with different values of $\Pr$

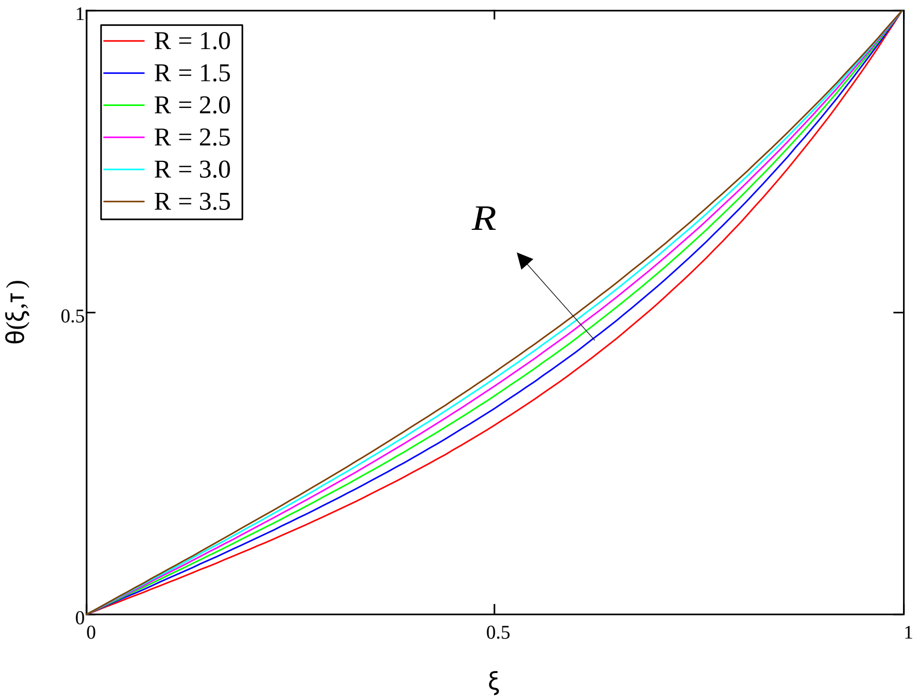

Fig. 6. Temperature profiles with different values of R

4.2 Velocity Results

In this section, the effect of α, β_o, $\Pr$, Gr, and R on the velocity profile is analyzed. The influence of α on the fluid velocity is plotted and shown in Figure 7. As α increases, the velocity distribution increases. Since the thickness of the momentum boundary layer is smaller than the thermal boundary layer at $\alpha = 1$, it is said that the classical velocity is at its maximum velocity distribution. Figure 8 shows the influence of β_o while other parameters are kept constant. Figure 8 shows that larger values of β_o, tending to decrease the velocity of the fluid. This is due to the physical effect of β_o, where the higher value of β_o will enhance the viscous forces compared to the thermal forces. Hence, the fluid velocity will tend to decrease. The influence of $\Pr$ on the velocity profile is depicted in Figure 9. The increment values of $\Pr$ have decreased the fluid velocity. $\Pr$ gives a great impact on the velocity of fluid since it measures the ratio of vicious and thermal forces. Consequently, as $\Pr$ increases, the velocity of fluid decreases.

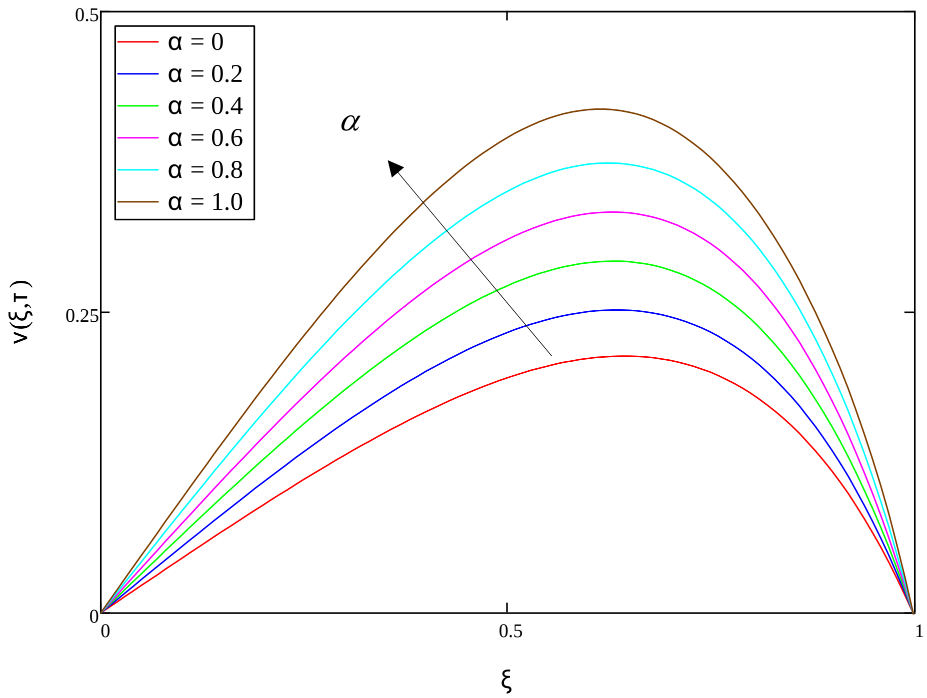

Fig. 7. Velocity profiles with different values of α

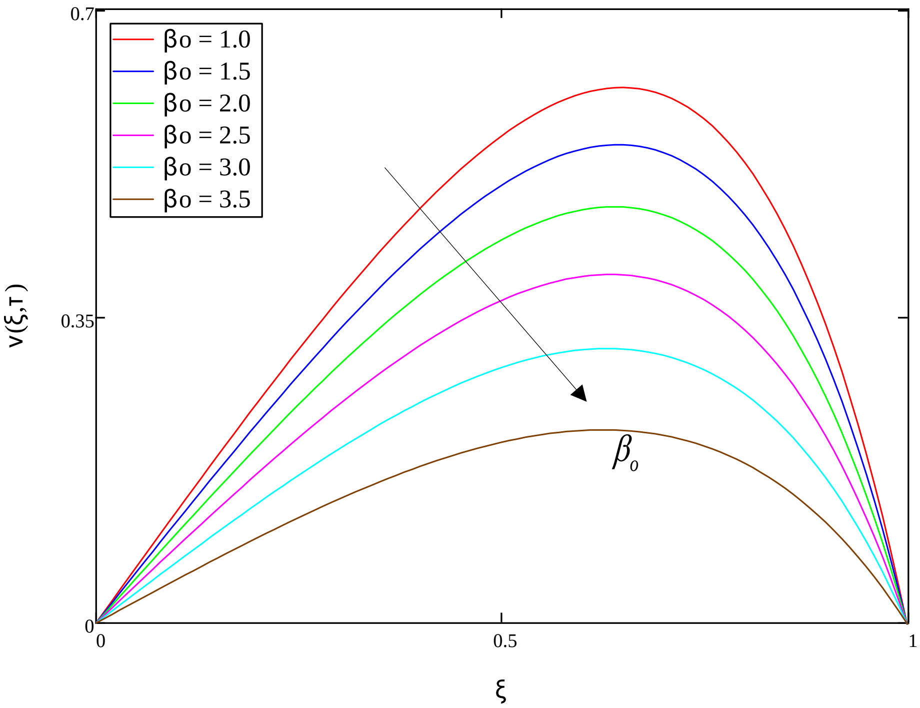

Fig. 8. Velocity profiles with different values of β_o

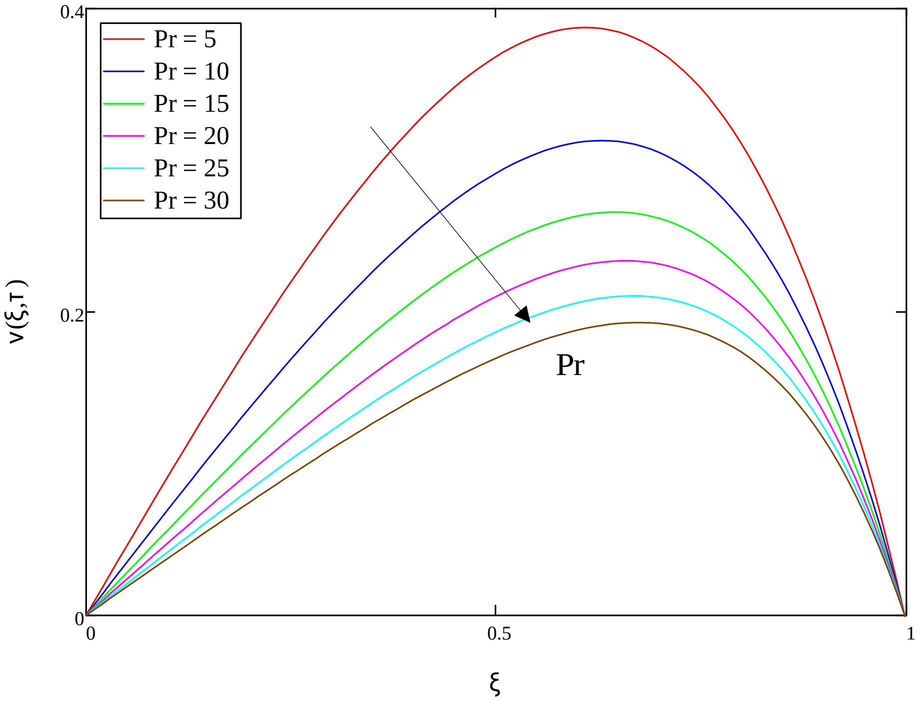

Fig. 9. Velocity profiles with different values of Pr

Figure 10 shows, as Gr increases, the velocity profiles increase. The value of Gr gives a positive effect to the buoyancy force. Hence, it gives a significant impact on the fluid velocity. Figure 11 is plotted to study the influence of R on fluid velocity. As R increases, the fluid velocity also increases. The greater the value of R, the stronger the convection effect which results in an increase in the velocity profile.

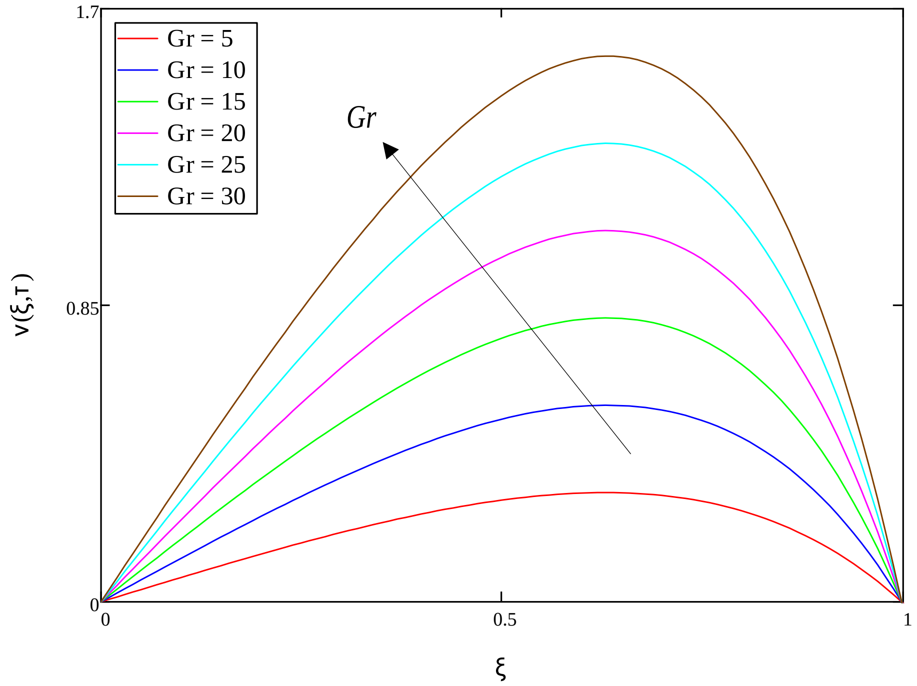

Fig. 10. Velocity profiles with different values of Gr

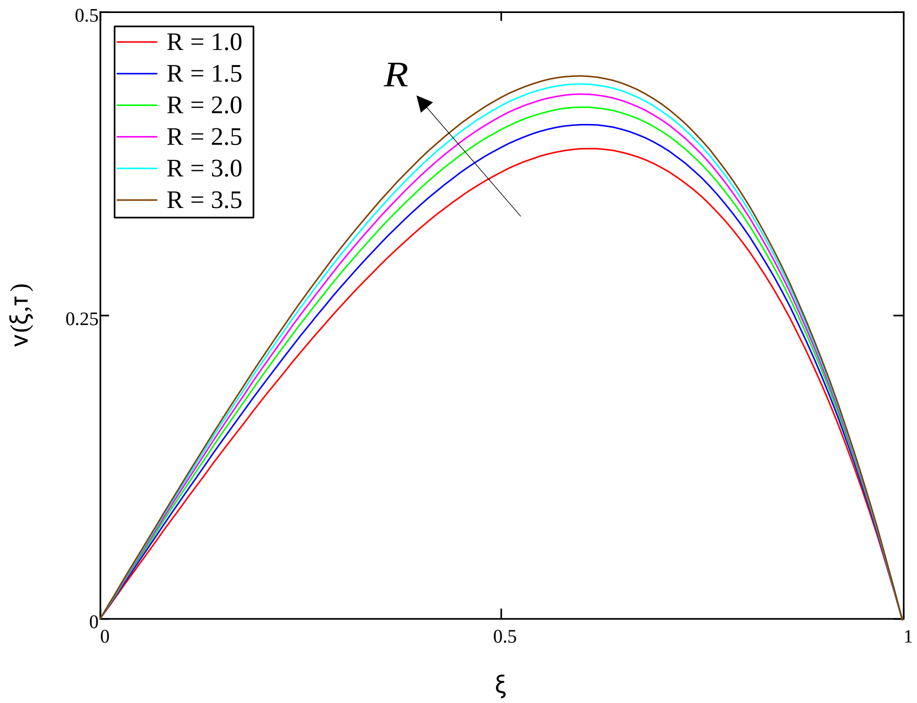

Fig. 11. Velocity profiles with different values of R

5. Conclusion

In this article, the application of Caputo fractional derivative to the convective flow of Casson fluid in a microchannel with the presence of thermal radiation is studied. Failla and Zingales [28] states that the fractional model fits the data better than the classical model $(\alpha = 1)$, as evidenced by the result. Analytical solutions are obtained by applying the Laplace transform technique. Importance physical parameters including the thermal radiation parameter, Casson parameter, and fractional parameter have been studied graphically. Hence, several major key points that can be extracted from this study are

i. The temperature and velocity profiles increase for a larger value of α.
ii. Greater values of $\Pr$ have enhanced the viscous force and consequently decreases the temperature and velocity profile.
iii. As R increase, the flow of the temperature and velocity profiles also increase.
iv. Increasing of β_o has caused the velocity profile to show a decreasing behavior.
v. The velocity profile increase for a larger value of Gr.

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References
[1] Farlow, Stanley J. *Partial differential equations for scientists and engineers*. Courier Corporation, 1993.
[2] Jumarie, Guy. "Fractional partial differential equations and modified Riemann-Liouville derivative new methods for solution." *Journal of Applied Mathematics and Computing* 24, no. 1 (2007): 31-48. https://doi.org/10.1007/BF02832299
[3] Khalil, Roshdi, Mohammed Al Horani, Abdelrahman Yousef, and Mohammad Sababheh. "A new definition of fractional derivative." *Journal of Computational and Applied Mathematics* 264 (2014): 65-70. https://doi.org/10.1016/j.cam.2014.01.002
[4] Ray, Santanu Saha, Abdon Atangana, S. C. Noutchie, Muhammet Kurulay, Necdet Bildik, and Adem Kilicman. "Fractional calculus and its applications in applied mathematics and other sciences." *Mathematical Problems in Engineering* 2014 (2014). https://doi.org/10.1155/2014/849395
[5] Ghalib, Muhammad Mansha, Azhar Ali Zafar, Muhammad Farman, Ali Akgül, M. O. Ahmad, and Aqeel Ahmad. "Unsteady MHD flow of Maxwell fluid with Caputo-Fabrizio non-integer derivative model having slip/non-slip fluid flow and Newtonian heating at the boundary." *Indian Journal of Physics* (2021): 1-10. https://doi.org/10.1007/s12648-020-01937-7
[6] Ali, Farhad, Nadeem Ahmad Sheikh, Ilyas Khan, and Muhammad Saqib. "Magnetic field effect on blood flow of Casson fluid in axisymmetric cylindrical tube: A fractional model." *Journal of Magnetism and Magnetic Materials* 423 (2017): 327-336. https://doi.org/10.1016/j.jmmm.2016.09.125
[7] Moosavi, Rouhollah, Reza Moltafet, Cheng-Xian Lin, and Po-Ya Abel Chuang. "Numerical modeling of fractional viscoelastic non-Newtonian fluids over a backward facing step-Buoyancy driven flow and heat transfer." *Thermal Science and Engineering Progress* 21 (2021): 100767. https://doi.org/10.1016/j.tsep.2020.100767
[8] Ahmad, Mushtaq, Muhammad Imran Asjad, Ali Akgül, and Dumitru Baleanu. "Analytical solutions for free convection flow of Casson nanofluid over an infinite vertical plate." *AIMS Mathematics* 6, no. 3 (2021): 2344-2358. https://doi.org/10.3934/math.2021142
[9] Saqib, Muhammad, Hanifa Hanif, T. Abdeljawad, Ilyas Khan, Sharidan Shafie, and K. Sooppy Nisar. "Heat transfer in mhd flow of maxwell fluid via fractional cattaneo-friedrich model: A finite difference approach." *Computers Materals & Continua* 65, no. 3 (2020): 1959-1973. https://doi.org/10.32604/cmc.2020.011339
[10] Ali, Rizwan, Muhammad Imran Asjad, and Ali Akgül. "An analysis of a mathematical fractional model of hybrid viscous nanofluids and its application in heat and mass transfer." *Journal of Computational and Applied Mathematics* 383 (2021): 113096. https://doi.org/10.1016/j.cam.2020.113096
[11] Reyaz, Ridhwan, Yeou Jiann Lim, Ahmad Qushairi Mohamad, Muhammad Saqib, and Sharidan Shafie. "Caputo fractional MHD Casson fluid flow over an oscillating plate with thermal radiation." *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences* 85, no. 2 (2021): 145-158. https://doi.org/10.37934/arfmts.85.2.145158
[12] Mukhopadhyay, Swati, Prativa Ranjan De, Krishnendu Bhattacharyya, and G. C. Layek. "Casson fluid flow over an unsteady stretching surface." *Ain Shams Engineering Journal* 4, no. 4 (2013): 933-938. https://doi.org/10.1016/j.asej.2013.04.004
[13] Hussanan, Abid, Ilyas Khan, and Sharidan Shafie. "An exact analysis of heat and mass transfer past a vertical plate with Newtonian heating." *Journal of Applied Mathematics* 2013 (2013). https://doi.org/10.1155/2013/434571

[14] Khalid, Asma, Ilyas Khan, Arshad Khan, and Sharidan Shafie. "Unsteady MHD free convection flow of Casson fluid past over an oscillating vertical plate embedded in a porous medium." *Engineering Science and Technology, an International Journal* 18, no. 3 (2015): 309-317. https://doi.org/10.1016/j.jestch.2014.12.006

[15] Imran, M. A., Fizza Miraj, I. Khan, and I. Tlili. "MHD fractional Jeffrey's fluid flow in the presence of thermo diffusion, thermal radiation effects with first order chemical reaction and uniform heat flux." *Results in Physics* 10 (2018): 10-17. https://doi.org/10.1016/j.rinp.2018.04.008

[16] Awan, Aziz Ullah, Qasim Ali, Samia Riaz, Nehad Ali Shah, and Jae Dong Chung. "A thermal optimization throughan innovative mechanism of free convection flow of Jeffrey fluid using non-local kernel." *Case Studies in Thermal Engineering* 24 (2021): 100851. https://doi.org/10.1016/j.csite.2021.100851

[17] Ullah, Imran, Sharidan Shafie, and Ilyas Khan. "Effects of slip condition and Newtonian heating on MHD flow of Casson fluid over a nonlinearly stretching sheet saturated in a porous medium." *Journal of King Saud University-Science* 29, no. 2 (2017): 250-259. https://doi.org/10.1016/j.jksus.2016.05.003

[18] Jamil, Dzuliana Fatin, Salah Uddin, Muhamad Ghazali Kamardan, and Rozaini Roslan. "The Effects of Magnetic Casson Blood Flow in an Inclined Multi-stenosed Artery by using Caputo-Fabrizio Fractional Derivatives." *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences* 82, no. 2 (2021): 28-38. https://doi.org/10.37934/arfmts.82.2.2838

[19] Khan, Ilyas, Muhammad Saqib, and Farhad Ali. "Application of time-fractional derivatives with non-singular kernel to the generalized convective flow of Casson fluid in a microchannel with constant walls temperature." *The European Physical Journal Special Topics* 226, no. 16 (2017): 3791-3802. https://doi.org/10.1140/epjst/e2018-00097-5

[20] Tabeling, P. "Some basic problems of microfluidics." In *14th Australasian Fluid Mechanics Conference*, p. 57-62. Adelaide University, Adelaide, Australia, 2001.

[21] Gad-el-Hak, Mohamed. "The fluid mechanics of microdevices-the Freeman scholar lecture." *ASME Journal of Fluids Engineering* 121, no. 1 (1999): 5-33. https://doi.org/10.1115/1.2822013

[22] Cao, Limei, Peipei Zhang, Botong Li, Jing Zhu, and Xinhui Si. "Numerical study of rotating electro-osmotic flow of double layers with a layer of fractional second-order fluid in a microchannel." *Applied Mathematics Letters* 111 (2021): 106633. https://doi.org/10.1016/j.aml.2020.106633

[23] Ahmad, M., M. A. Imran, and M. Nazar. "Mathematical modeling of ($Cu-Al_2O_3$) water based Maxwell hybrid nanofluids with Caputo-Fabrizio fractional derivative." *Advances in Mechanical Engineering* 12, no. 9 (2020): 1-11. https://doi.org/10.1177/1687814020958841

[24] Phu, Nguyen Minh, Pham Ba Thao, and Duong Cong Truyen. "Heat and Fluid Flow Characteristics of Nanofluid in A Channel Baffled Opposite to The Heated Wall." *CFD Letters* 13, no. 1 (2021): 33-44. https://doi.org/10.37934/cfdl.13.1.3344

[25] Halsted, D. J., and D. E. Brown. "Zakian's technique for inverting Laplace transforms." *The Chemical Engineering Journal* 3 (1972): 312-313. https://doi.org/10.1016/0300-9467(72)85037-8

[26] Wang, Quanrong, and Hongbin Zhan. "On different numerical inverse Laplace methods for solute transport problems." *Advances in Water Resources* 75 (2015): 80-92. https://doi.org/10.1016/j.advwatres.2014.11.001

[27] Zakian, V., and R. K. Littlewood. "Numerical inversion of Laplace transforms by weighted least-squares approximation." *The Computer Journal* 16, no. 1 (1973): 66-68. https://doi.org/10.1093/comjnl/16.1.66

[28] Failla, Giuseppe, and Massimiliano Zingales. "Advanced materials modelling via fractional calculus: challenges and perspectives." *Philosophical Transactions of the Royal Society A* 378, no. 2172 (2020): 20200050. https://doi.org/10.1098/rsta.2020.0050