

# Epidermiological Approach to Predator-Prey Interaction

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**Abstract** An epidemiological approach is used to model predator-prey interaction. The concept is borrowed from SIS model of disease transmission. In the predator-prey interaction, predation is viewed as a “disease” of the predator transmitted by the prey. In this approach, the predator population is partitioned into hungry and satisfied subpopulation. The stability of the steady states from the equations is analyzed.

**Keywords** Predator-prey interaction, Model SIS

**Abstrak** Pendekatan penyakit berjangkit digunakan untuk memodelkan interaksi mangsa-pemangsa. Konsep ini dipinjam daripada Model SIS jangkitan penyakit. Dalam interaksi mangsa-pemangsa, pemangsaan dianggap sebagai “penyakit” pemangsa yang dijangkiti daripada mangsa. Pendekatan ini membahagikan populasi pemangsa kepada subpopulasi lapar dan kenyang. Kestabilan titik-titik keseimbangan daripada persamaan dianalisa.

**Katakunci** Interaksi mangsa-pemangsa, Model SIS

## 1 Introduction

In mathematical models of predation, the hardest phenomenon to capture is the effect of predation on predator reproduction. The more or less standard assumption in the literature ever since the pioneering work of Volterra and Lotka, is classical convention that predator fertility is dependent on its food intake. Mathematically this assumption allows us to use ODE techniques in the analysis of predator prey equations. Volterra himself [3] was aware

of the limitations of this approach (which basically presupposes zero gestation time) and suggested using delay-differential equations instead.

In this paper, we suggest a simple way of overcoming this difficulty, a way that is also suggestive of many generalizations. We suggest a different approach to predation effect on predator growth rate. We call this approach an *epidemiological* one as it borrows concepts from the SIS model of disease transmission [2]. In other words, we view predation as a “disease” of the predator transmitted by the prey. We partition the predator population into hungry  $H(t)$  and satisfied  $S(t)$  subpopulation and make the following assumption:

- Only satisfied predators will reproduce
- A newborn predator will automatically be in the hungry class.

## 2 The Model

The resulting equations are

$$\begin{aligned}\frac{dN}{dt} &= \alpha N \left(1 - \frac{N}{K}\right) - \beta NH \\ \frac{dH}{dt} &= \lambda S - \mu H - \gamma NH \\ \frac{dS}{dt} &= \gamma NH - \nu S.\end{aligned}\tag{1}$$

Here,  $N, H$  and  $S$  are the densities of the prey, the hungry predator and the satisfied predator respectively.  $\alpha$  is the birth rate of the prey and  $\nu$  and  $\mu$  are the death rate of the satisfied and the hungry predators respectively.  $\lambda$  is a combination of the birth rate of the satisfied predator and the rate at which satisfied predators become hungry again while  $\gamma$  is the efficiency of predation, i.e. the rate at which an encounter with prey leads to satisfaction, and  $\beta$  is the prey death rate due to predation. To analyze the system (1), we non-dimensionalize the equations by setting  $a = \frac{\lambda}{\alpha}$ ,  $b = \frac{\mu}{\alpha}$ ,  $c = \frac{\gamma K}{\alpha}$ ,  $\tau = \alpha t$ ,  $n = \frac{N}{K}$ ,  $h = \frac{\beta}{\alpha} H$ , and  $s = \frac{\beta}{\alpha} S$ , to obtain

$$\begin{aligned}\frac{dn}{d\tau} &= n(1-n) - nh \\ \frac{dh}{d\tau} &= as - bh - cnh \\ \frac{ds}{d\tau} &= cnh - ds\end{aligned}\tag{2}$$

with  $a, b, c, d > 0$

**Lemma 1** *If  $a, b, c, d > 0$  and  $n(0), h(0), s(0) > 0$ , the solution of 2 remains in the positive octant and is bounded.*

*Proof:*

If  $h = 0, n > 0, s > 0, \frac{dh}{d\tau} > 0$ . Similarly, if  $s = 0, h > 0, n > 0, \frac{ds}{d\tau} > 0$ , and if  $n = 0, \frac{dn}{d\tau} = 0$ , so the positive octant is positively invariant. To see that the solution through  $n(0), h(0), s(0)$  is bounded, note that the equation for  $s(\tau)$  can be solved in terms of  $n, h$  to give

$$s(\tau) = s(0)e^{-d\tau} + ce^{-d\tau} \int_0^\tau e^{dt} n(t)h(t) dt.$$

Hence for  $h$  we have

$$\frac{dh}{d\tau} \leq as(0)e^{-d\tau} + ace^{-d\tau} \int_0^\tau e^{dt} n(t)h(t)dt - bh(\tau).$$

However, from the first equation we have that

$$e^{d\tau} \frac{dn}{d\tau} = e^{d\tau} n(1-n) - e^{d\tau} nh$$

so

$$\int_0^\tau e^{dt} \frac{dn}{dt} dt = \int_0^\tau e^{dt} n(t)(1-n(t)) dt - \int_0^\tau e^{dt} n(t)h(t) dt.$$

Integrating by part,

$$[e^{dt} n]_0^\tau - d \int_0^\tau e^{dt} n(t) dt = \int_0^\tau e^{dt} n(t) dt - \int_0^\tau e^{dt} n^2(t) dt - \int_0^\tau e^{dt} n(t)h(t) dt.$$

This means that

$$\int_0^\tau e^{dt} n(t)h(t) dt \leq K_1 \int_0^\tau e^{dt} dt + K_2$$

where  $K_1, K_2$  are constants depending on  $n(0)$ , since  $n(\tau)$  is bounded above by the solution of  $z' = z(1-z), z(0) = n(0)$ . Therefore

$$\frac{dh}{d\tau} \leq f(\tau) - bh(\tau)$$

where  $f(\tau)$  is a bounded function of  $\tau$ . But then  $h(\tau)$  is bounded, and therefore so is  $s(\tau)$  from the third equation of (2).  $\diamond$

### 3 The Analysis

The rest points for the systems of (2) are

- The trivial solution  $(0, 0, 0)$ ,
- The semi-trivial solution,  $n_2 = 1, h_2 = s_2 = 0$ , where only the prey survives,
- The coexistence rest point,  $n_3 = \frac{db}{c(a-d)}, h_3 = 1 - n_3, s_3 = \frac{b(1-n_3)}{a-d}$ .

To have a positive coexistence rest point, we must have

$$a > d \quad \text{and} \quad c > \frac{db}{a-d}$$

The Jacobian for the system (2) is

$$\text{Jacobian} = \begin{pmatrix} 1 - 2n - h & -n & 0 \\ -ch & -b - cn & a \\ ch & cn & -d \end{pmatrix}$$

The Jacobian at the trivial rest point is

$$J(0, 0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -b & a \\ 0 & 0 & -d \end{pmatrix}$$

and the eigenvalues are  $\lambda_{1,2,3} = 1, -b, -d$ , thus the trivial rest point is unstable (one of the eigenvalues is positive).

The Jacobian for  $(n_2, h_2, s_2)$  is

$$J(1, 0, 0) = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -b - c & a \\ 0 & c & -d \end{pmatrix}$$

and its characteristic polynomial is

$$\lambda^3 + (d + b + 1 + c)\lambda^2 + (dc + db + b - ca + c + d)\lambda + (dc + db - ca) = 0.$$

From the characteristic polynomial, we can see that the rest point  $(n_2, h_2, s_2)$  is locally asymptotically stable if  $c < \frac{db}{a-d}$ . This condition violates the condition for the existence of a positive coexistence rest point. Hence, if  $(n_3, h_3, s_3)$  is negative, then  $(n_2, h_2, s_2)$  is locally asymptotically stable, and vice versa. In other words, as  $c$  passes through the value  $\frac{db}{a-d}$ , we have a transcritical bifurcation (The open square in Figure 1).

At the coexistence rest point  $(n_3, h_3, s_3)$ , the Jacobian is

$$J(n_3, h_3, s_3) = \begin{pmatrix} 1 - \frac{2db}{c(a-d)} - \frac{c(a-d) - db}{c(a-d)} & -\frac{db}{c(a-d)} & 0 \\ -\frac{c(a-d) - db}{c(a-d)} & -b - \frac{db}{a-d} & a \\ \frac{c(a-d) - db}{c(a-d)} & \frac{db}{a-d} & -d \end{pmatrix}$$

and its characteristic polynomial is

$$a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

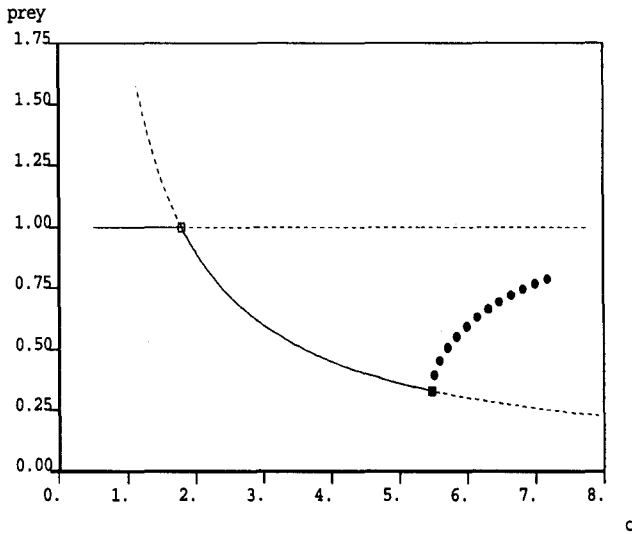


Figure 1: Prey vs  $c$  for  $b = 1.8$ ,  $a = 1.0$  and  $d = 0.5$ . Dashed line is unstable and solid line is stable

where

$$\begin{aligned}
 a_0 &= c(a-d)^2 \\
 a_1 &= (a-d)(cd(a-d) + acb + db) \\
 a_2 &= db(db + ba - d^2 + dc + ad - ca) \\
 a_3 &= bd(a-d)(c(a-d) - db)
 \end{aligned}$$

Using the Routh-Hurwitz criterion [1],  $(n_3, h_3, s_3)$  is locally asymptotically stable if and only if

$$\begin{aligned}
 a_i &> 0 \quad i = 1, 2, 3 \\
 \Delta_2 &= \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0 \\
 \Delta_3 &= \begin{vmatrix} a_0 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{vmatrix} > 0
 \end{aligned}$$

The above conditions are satisfied if  $a_1 a_2 - a_0 a_3 > 0$ , and that is when

$$b_0 c^2 + b_1 c + b_2 > 0 \quad \text{where}$$

$$\begin{aligned}
 b_0 &= -a(a-d)(a-d+b) \\
 b_1 &= (3db + b^2 + d^2)a^2 + (db^2 - 3d^2b - db - 2d^3)a + d^2b + d^4 \\
 b_2 &= db(d(a-d) + b(a+d))
 \end{aligned}$$

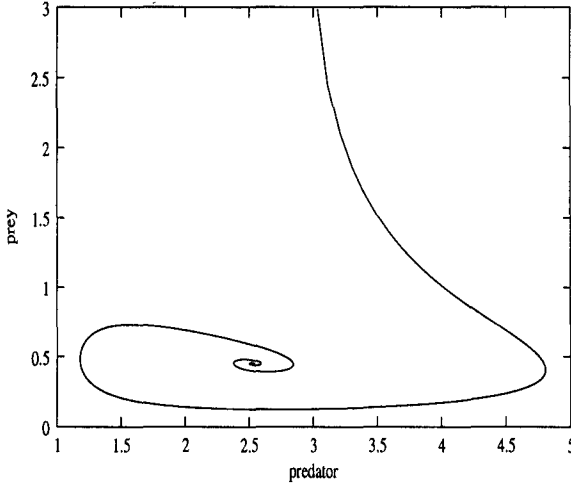


Figure 2: The system converges to a stable coexistence rest point with  $a = 1$ ,  $b = 1.8$ ,  $c = 4$  and  $d = 0.5$ .

Thus,  $(n_3, h_3, s_3)$  is locally asymptotically stable if

$$\frac{db}{a-d} < c < \frac{-b_1 - \sqrt{b_1^2 - 4b_0b_2}}{2b_0}.$$

When  $c = \frac{-b_1 - \sqrt{b_1^2 - 4b_0b_2}}{2b_0}$ , it is easily checked that the rest point  $(n_3, h_3, s_3)$  loses its stability via a Hopf bifurcation. Using AUTO, we see in figure 1, that the Hopf bifurcation is supercritical, so that for all  $c > \frac{-b_1 - \sqrt{b_1^2 - 4b_0b_2}}{2b_0}$ , there is a family of locally asymptotically stable periodic solution (represented by  $\bullet$  in Figure 1).

In figure 2 and 3 we present the projection of the phase portrait onto the plane of  $(n, h + s)$ . The results are summarized in Figure 4.

## 4 Conclusion

The model presented here incorporates memory into the system in a very simple manner. It relates the predator's ability to reproduce to the history of predation. At the same time, it suffers from many drawbacks of most predator-prey models such as paradox of enrichment and the inability of the predator to go extinct, no matter how small the initial density of the predator,  $h(0) + s(0)$ , is.

The above model can obviously be extended in a number of ways. For example, it would be of interest to see how the system behaves under seasonal fluctuation, expressed, for example, in the parameter  $\lambda$  in (1), as the amount of energy needed for self maintenance may vary.

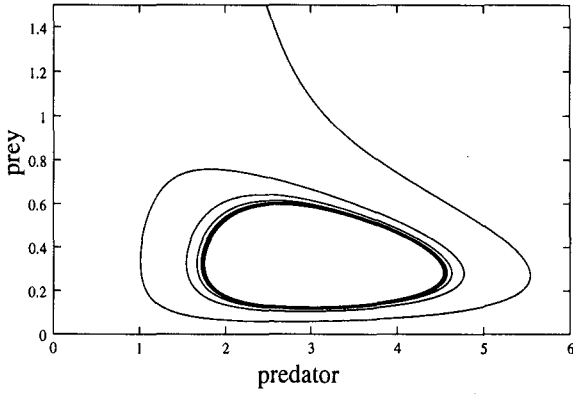


Figure 3: The system goes to a stable limit cycle with  $a = 1, b = 1.8, c = 6$  and  $d = 0.5$ .

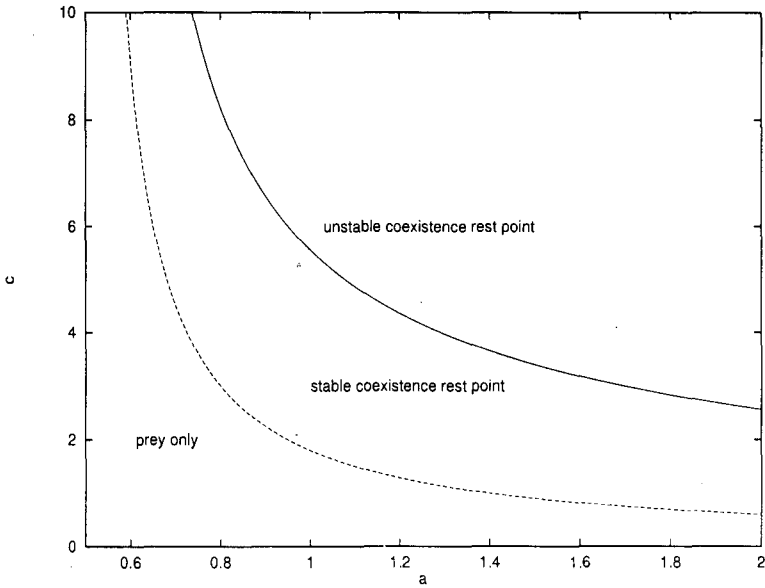


Figure 4: Region of stability for different  $a$  and  $c$  for  $b = 1.8$  and  $d = 0.5$ .

## References

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