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An Integral Equation Method for Conformal Mapping of Doubly Connected Regions Involving the Neumann Kernel

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Abstract We present an integral equation method for conformal mapping of doubly connected regions onto a unit disc with a circular slit of radius $\mu < 1$. Our theoretical development is based on the boundary integral equation for conformal mapping of doubly connected region derived by Murid and Razali [15]. In this paper, using the boundary relationship satisfied by the mapping function, a related system of integral equations via Neumann kernel is constructed. For numerical experiment, the integral equation is discretized which leads to a system of linear equations, where μ is assumed known. Numerical implementation on a circular annulus is also presented.

Keywords Conformal mapping; integral equations; doubly connected regions; Neumann kernel.

1 Introduction

Numerical conformal mapping of multiply connected regions are presently still a subject of interest. Every region of connectivity p can be mapped conformally on each of the five canonical regions [2, 3, 16]. They are the disc with concentric circular slits; an annulus with concentric circular slits; the circular slit region; the radial slit region and the parallel slit region. In particular, if Ω is a multiply connected regions of connectivity $(p+1)$ inside the unit disc $|z| < 1$ where $\Gamma = |z| = 1$ is the boundary component of Ω, then there exists a univalent analytic function $w = f(z)$ in Ω such that (i) it maps Ω conformally onto a region G inside the unit disc $|w| < 1$ which has p circular slits centered at $w = 0$ and (ii) it maps the unit circle $|z| = 1$ conformally onto a unit circle $|w| = 1$. The images of the circular slits are traversed twice [6, 10].

Several methods for conformal mapping of multiply connected regions have been proposed in the literature [4, 7, 9, 11, 15, 17, 18, 19, 21, 22]. One of the methods is the integral equation method. Some notable ones are the integral equations of Warschawski, Gerschgorin, and Symm. All these integral equations are extensions of those maps for simply connected regions. Recently there are two integral equations for conformal mapping of simply connected regions derived by Kerzman and Trummer [8] and Razali et al. [20]. An effort for their extensions to doubly connected case has been given by Murid and Razali [15] but without any numerical experiment. Numerical conformal mapping of doubly connected regions onto an annulus based on [15] is discussed in [12].

In this paper, we adapted the work of Murid and Razali [15] to construct an integral equation involving the Neumann kernel for conformal mapping of doubly connected regions onto a unit disc with a circular slit of radius $\mu < 1$. For numerical experiment, the integral equation is discretized which leads to a system of linear equations provided μ is known. A numerical example is reported for a circular annulus as a test region.

2 A Boundary Integral Equation for Conformal Mapping of Doubly Connected Regions

Let Γ_0 and Γ_1 be two smooth Jordan curves in the complex z-plane such that Γ_1 lies in the interior of Γ_0. Denote the finite doubly connected region by Ω with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Let $w = f(z)$ be the analytic function which maps Ω conformally onto a unit disc with a circular slit of radius $\mu < 1$. The mapping function f is determined up to a factor of modulus 1. The function f could be made unique by prescribing that

$$f(a) = 0,\ f'(a) > 0,$$

where $a \in \Omega$ is a fixed point.

The boundary values of f can be represented in the form

$$f(z_0(t)) = e^{\mathrm{i}\theta_0(\mathrm{t})}, \quad \Gamma_0 : z = z_0(t), \quad 0 \le t \le \beta_0, \tag{1}$$

$$f(z_1(t)) = \mu e^{\mathrm{i}\theta_1(t)}, \quad \Gamma_1 : z = z_1(t), \quad 0 \le t \le \beta_1, \tag{2}$$

where $\theta_0(t)$ and $\theta_1(t)$ are the boundaries corresponding to the functions Γ_0 and Γ_1, respectively.

The unit tangent to Γ at $z(t)$ is denoted by $T(z(t)) = z'(t)/|z'(t)|$. Thus it can be shown that

$$f(z_0(t)) = \frac{1}{\mathrm{i}} T(z_0(t)) \frac{\theta_0'(t)}{|\theta_0'(t)|} \frac{f'(z_0(t))}{|f'(z_0(t))|} = \frac{1}{\mathrm{i}} T(z_0(t)) \frac{f'(z_0(t))}{|f'(z_0(t))|}, \tag{3}$$

$$f(z_1(t)) = \frac{\mu}{\mathrm{i}} T(z_1(t)) \frac{\theta_1'(t)}{|\theta_1'(t)|} \frac{f'(z_1(t))}{|f'(z_1(t))|} = \pm \frac{\mu}{\mathrm{i}} T(z_1(t)) \frac{f'(z_1(t))}{|f'(z_1(t))|}. \tag{4}$$

Note that $\theta_0'(t) > 0$ while $\theta_1'(t)$ may be positive or negative. Thus $\theta_1'(t)/|\theta_1'(t)| = \pm 1$.

The boundary relationships (3) and (4) can be unified as

$$f(z) = \pm \frac{|f(z)|}{\mathrm{i}} T(z) \frac{f'(z)}{|f'(z)|}, \quad z \in \Gamma, \tag{5}$$

where $\Gamma = \Gamma_0 \cup \Gamma_1$.

To eliminate the $\pm$ sign, we square both sides of the boundary relationship (5) and get

$$f(z)^2 = -|f(z)|^2 T(z)^2 \frac{f'(z)^2}{|f'(z)|^2}, \quad z \in \Gamma. \tag{6}$$

An integral equation can be constructed related to the boundary relationship (6) based on the following result of Murid and Razali [15]:

Suppose $D(z)$ is analytic and single-valued with respect to $z \in \Omega$ and is continuous on $\Omega \cup \Gamma$. Suppose further that D satisfies the boundary relationship

$$P(z) = \overline{c(z)}\,\frac{T(z)Q(z)D(z)^2}{|D(z)|^2}, \quad z \in \Gamma, \tag{7}$$

where $T(z(t)) = z'(t)/|z'(t)|$ is the complex unit tangent function at $z \in \Gamma$, while c, P and Q are complex-valued functions defined on Γ with the following properties:

(P1) $c(z) = \begin{cases} c_0, & \text{if } z \in \Gamma_0 \\ c_1, & \text{if } z \in \Gamma_1 \end{cases}$, where c_0 and c_1 are complex constants,

(P2) $P(z)$ is analytic and single-valued with respect to $z \in \Omega$,

(P3) $P(z)$ is continuous on $\Omega \cup \Gamma$,

(P4) $P(z)$ has a finite number of zeroes at $a_1, a_2, \ldots, a_n$ in Ω,

(P5) $P(z) \neq 0$, $Q(z) \neq 0$, $D(z) \neq 0$, $z \in \Gamma$.

Theorem 1 *Let u and v be any complex-valued function that are defined on Γ. Then*

$$\begin{aligned}
&\frac{1}{2}\left[v(z) + \frac{u(z)}{\overline{T(z)Q(z)}}\right] D(z) + \mathrm{PV}\frac{1}{2\pi \mathrm{i}}\int_\Gamma \left[\frac{u(z)}{(\overline{w}-\overline{z})\overline{Q(w)}} - \frac{v(z)T(w)}{w-z}\right] D(w)|dw| \\
&= -c(z)u(z)\left[\sum_{a_j\, inside\, \Gamma} \operatorname*{Res}_{w=a_j} \frac{D(w)}{(w-z)P(w)}\right]^- \\
&\quad - u(z)(c_0 - c_1)\left[\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_2} \frac{D(w)}{(w-z)P(w)} dw\right]^-, \quad z \in \Gamma,
\end{aligned} \tag{8}$$

where the minus sign in the superscript denotes complex conjugate and where

$$\Gamma_2 = \begin{cases} -\Gamma_1, & \text{if } z \in \Gamma_0, \\ \Gamma_0, & \text{if } z \in \Gamma_1. \end{cases}$$

Comparison of (6) and (7) leads to a choice of $c(z) = -|f(z)|^2$, $P(z) = f(z)^2$, $D(z) = f'(z)$, $Q(z) = T(z)$. Substituting these assignments into (8) along with the choice of $u(z) = \overline{T(z)Q(z)}$ and $v(z) = 1$, gives the integral equation

$$\begin{aligned}
f'(z) \quad &+ \quad \int_\Gamma M(z,w) f'(w)|dw| = |f(z)|^2\overline{\overline{T(z)^2}}\left[\operatorname*{Res}_{w=a} \frac{f'(w)}{(w-z)f(w)^2}\right]^- \\
&+ \quad \overline{T(z)^2}(1-\mu^2)\left[\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_2} \frac{f'(w)}{(w-z)f(w)^2} dw\right]^-, \quad z \in \Gamma,
\end{aligned} \tag{9}$$

where

$$M(z,w) = \begin{cases} \dfrac{T(w)}{2\pi \mathrm{i}}\left[\dfrac{\overline{T(z)^2}}{\overline{w}-\overline{z}} - \dfrac{1}{w-z}\right], & \text{if } w, z \in \Gamma, w \neq z, \\[2ex] \dfrac{1}{2\pi}\dfrac{\operatorname{Im}[z''(t)\overline{z'(t)}]}{|z'(t)|^3}, & \text{if } w = z \in \Gamma. \end{cases} \tag{10}$$

If we multiply both sides of (9) by $T(z)$ and use the fact $T(z)\overline{T(z)} = |T(z)|^2 = 1$, we obtain

$$T(z)f'(z) + \int_\Gamma N(z,w)T(w)f'(w)|dw| = |f(z)|^2\overline{T(z)}\left[\operatorname*{Res}_{w=a}\frac{f'(w)}{(w-z)f(w)^2}\right]^- + \overline{T(z)}(1-\mu^2)\left[\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_2}\frac{f'(w)}{(w-z)f(w)^2}dw\right]^-, \quad z\in\Gamma, \tag{11}$$

where N is the Neumann kernel [6] defined by

$$N(z,w) = \begin{cases} \dfrac{1}{\pi}\mathrm{Im}\left[\dfrac{T(z)}{w-z}\right], & \text{if } w,z\in\Gamma, w\neq z,\\ \dfrac{1}{2\pi}\dfrac{\mathrm{Im}[z''(t)\overline{z'(t)}]}{|z'(t)|^3}, & \text{if } w=z\in\Gamma. \end{cases} \tag{12}$$

To evaluate the residue in equation (11), we use the fact that if $f(w) = g(w)/h(w)$, where g and h are analytic at a, and $g(a)\neq 0$, $h(a) = h'(a) = 0$, $h''(a)\neq 0$, which means a is a double pole of $f(w)$, then [5]

$$\operatorname*{Res}_{w=a} f(w) = 2\frac{g'(a)}{h''(a)} - \frac{2}{3}\frac{h'''(a)g(a)}{h''(a)^2}. \tag{13}$$

Applying (13) to (11) and after several algebraic manipulations, we obtain

$$\operatorname*{Res}_{w=a}\frac{f'(w)}{(w-z)f(w)^2} = -\frac{1}{(a-z)^2 f'(a)}. \tag{14}$$

Thus integral equation (11) becomes

$$T(z)f'(z) + \int_\Gamma N(z,w)T(w)f'(w)|dw| = |f(z)|^2\overline{T(z)}\left[-\frac{1}{(a-z)^2 f'(a)}\right]^- + \overline{T(z)}(1-\mu^2)\left[\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_2}\frac{f'(w)}{(w-z)f(w)^2}dw\right]^-, \quad z\in\Gamma. \tag{15}$$

Multiply (15) by $f'(a)$, we get

$$T(z)f'(z)f'(a) + \int_\Gamma N(z,w)T(w)f'(w)f'(a)|dw| = -|f(z)|^2\overline{T(z)}\frac{1}{(\overline{a}-\overline{z})^2} + \overline{T(z)}(1-\mu^2)\left[\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_2}\frac{f'(w)}{(w-z)f(w)^2}f'(a)dw\right]^-, \quad z\in\Gamma. \tag{16}$$

The single integral equation in (16) can be separated into a system of two integral equations given by

$$T(z_0)f'(z_0)f'(a) + \int_\Gamma N(z_0,w)T(w)f'(w)f'(a)|dw| = -\overline{T(z_0)}\frac{1}{(\overline{a}-\overline{z_0})^2} + \overline{T(z_0)}(1-\mu^2)\left[\frac{1}{2\pi\mathrm{i}}\int_{-\Gamma_1}\frac{f'(w)}{(w-z_0)f(w)^2}f'(a)dw\right]^-, \quad z_0\in\Gamma_0, \tag{17}$$

$$T(z_1)f'(z_1)f'(a) + \int_\Gamma N(z_1,w)T(w)f'(w)f'(a)|dw| = -\mu^2\overline{T(z_1)}\frac{1}{(\overline{a}-\overline{z_1})^2}$$
$$+\overline{T(z_1)}(1-\mu^2)\left[\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_0}\frac{f'(w)}{(w-z_1)f(w)^2}f'(a)dw\right]^-, \quad z_1\in\Gamma_1. \tag{18}$$

Taking boundary relationship (6) into account, (17) and (18) become

$$T(z_0)f'(z_0)f'(a) + \int_\Gamma N(z_0,w)T(w)f'(w)f'(a)|dw| = -\frac{\overline{T(z_0)}}{(\overline{a}-\overline{z_0})^2}$$
$$+\overline{T(z_0)}(1-\mu^2)\left[\frac{1}{2\pi\mathrm{i}}\int_{-\Gamma_1}\frac{f'(w)f'(a)}{(w-z_0)\left[-\mu^2T(w)^2\frac{f'(w)^2}{|f'(w)|^2}\right]}dw\right]^-, \quad z_0\in\Gamma_0, \tag{19}$$

$$T(z_1)f'(z_1)f'(a) + \int_\Gamma N(z_1,w)T(w)f'(w)f'(a)|dw| = -\frac{\mu^2\overline{T(z_1)}}{(\overline{a}-\overline{z_1})^2}$$
$$+\overline{T(z_1)}(1-\mu^2)\left[\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_0}\frac{f'(w)f'(a)}{(w-z_1)\left[-T(w)^2\frac{f'(w)^2}{|f'(w)|^2}\right]}dw\right]^-, \quad z_1\in\Gamma_1. \tag{20}$$

Using the facts that $|f'(w)|^2 = f'(w)\overline{f'(w)}$, $T(w)|dw| = dw$, and $T(w)\overline{T(w)} = |T(w)|^2 = 1$, the two integral equations (19) and (20) become

$$T(z_0)f'(z_0)f'(a) + \int_\Gamma N(z_0,w)T(w)f'(w)f'(a)|dw| = -\frac{\overline{T(z_0)}}{(\overline{a}-\overline{z_0})^2}$$
$$+\frac{1}{2\pi\mathrm{i}\mu}\overline{T(z_0)}(1-\mu^2)\int_{-\Gamma_1}\frac{T(w)}{\overline{w}-\overline{z_0}}f'(w)f'(a)|dw|, \quad z_0\in\Gamma_0, \tag{21}$$

$$T(z_1)f'(z_1)f'(a) + \int_\Gamma N(z_1,w)T(w)f'(w)f'(a)|dw| = -\frac{\mu^2\overline{T(z_1)}}{(\overline{a}-\overline{z_1})^2}$$
$$+\frac{1}{2\pi\mathrm{i}}\overline{T(z_1)}(1-\mu^2)\int_{\Gamma_0}\frac{T(w)}{\overline{w}-\overline{z_1}}f'(w)f'(a)|dw|, \quad z_1\in\Gamma_1. \tag{22}$$

Since $\Gamma = \Gamma_0\cup\Gamma_1$, equations (21) and (22) can be written as

$$T(z_0)f'(z_0)f'(a) + \int_{\Gamma_0} N(z_0,w)T(w)f'(w)f'(a)|dw|$$
$$-\int_{-\Gamma_1} N(z_0,w)T(w)f'(w)f'(a)|dw|$$
$$= -\frac{\overline{T(z_0)}}{(\overline{a}-\overline{z_0})^2} + \frac{\overline{T(z_0)}}{2\pi\mathrm{i}\mu^2}\int_{-\Gamma_1}\frac{1}{\overline{w}-\overline{z_0}}T(w)f'(w)f'(a)|dw|$$
$$-\frac{\overline{T(z_0)}}{2\pi\mathrm{i}}\int_{-\Gamma_1}\frac{1}{\overline{w}-\overline{z_0}}T(w)f'(w)f'(a)|dw|, \quad z_0\in\Gamma_0, \tag{23}$$

$$\begin{aligned}
T(z_1)f'(z_1)f'(a) &+ \int_{\Gamma_0} N(z_1,w)T(w)f'(w)f'(a)|dw| \\
&- \int_{-\Gamma_1} N(z_1,w)T(w)f'(w)f'(a)|dw| \\
= -\frac{\mu^2\overline{T(z_1)}}{(\overline{a}-\overline{z_1})^2} &+ \frac{\overline{T(z_1)}}{2\pi\mathrm{i}}\int_{\Gamma_0}\frac{1}{\overline{w}-\overline{z_1}}T(w)f'(w)f'(a)|dw| \\
&- \frac{\mu^2\overline{T(z_1)}}{2\pi\mathrm{i}}\int_{\Gamma_0}\frac{1}{\overline{w}-\overline{z_1}}T(w)f'(w)f'(a)|dw|, \quad z_1\in\Gamma_1. \qquad (24)
\end{aligned}$$

Using the fact that any $z\in\mathbb{C}$, $\mathrm{Im}(z)=(z-\overline{z})/(2\mathrm{i})$, the Neumann kernel which is defined in (12) can be written as

$$N(z,w)=\begin{cases}\dfrac{1}{2\pi\mathrm{i}}\left[\dfrac{T(z)}{z-w}-\dfrac{\overline{T(z)}}{\overline{z}-\overline{w}}\right], & \text{if } w,z\in\Gamma, w\neq z,\\[2ex] \dfrac{1}{2\pi}\dfrac{\mathrm{Im}[z''(t)\overline{z'(t)}]}{|z'(t)|^3}, & \text{if } w=z\in\Gamma.\end{cases} \qquad (25)$$

Applying definition (25) to $N(z_0,w)$ in $\int_{-\Gamma_1}$ of equation (23) and to $N(z_1,w)$ in $\int_{\Gamma_0}$ of equation (24), we obtain

$$\begin{aligned}
T(z_0)f'(z_0)f'(a) &+ \int_{\Gamma_0} N(z_0,w)T(w)f'(w)f'(a)|dw| \\
&- \int_{-\Gamma_1}\frac{1}{2\pi\mathrm{i}}\left[\frac{T(z_0)}{z_0-w}-\frac{\overline{T(z_0)}}{\overline{z_0}-\overline{w}}\right]T(w)f'(w)f'(a)|dw| \\
= -\frac{\overline{T(z_0)}}{(\overline{a}-\overline{z_0})^2} &- \frac{1}{2\pi\mathrm{i}\mu^2}\int_{-\Gamma_1}\frac{\overline{T(z_0)}}{\overline{z_0}-\overline{w}}T(w)f'(w)f'(a)|dw| \\
&+ \frac{1}{2\pi\mathrm{i}}\int_{-\Gamma_1}\frac{\overline{T(z_0)}}{\overline{z_0}-\overline{w}}T(w)f'(w)f'(a)|dw|, \quad z_0\in\Gamma_0, \qquad (26)
\end{aligned}$$

$$\begin{aligned}
T(z_1)f'(z_1)f'(a) &+ \int_{\Gamma_0}\frac{1}{2\pi\mathrm{i}}\left[\frac{T(z_1)}{z_1-w}-\frac{\overline{T(z_1)}}{\overline{z_1}-\overline{w}}\right]T(w)f'(w)f'(a)|dw| \\
&- \int_{-\Gamma_1} N(z_1,w)T(w)f'(w)f'(a)|dw| \\
= -\frac{\mu^2\overline{T(z_1)}}{(\overline{a}-\overline{z_1})^2} &- \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_0}\frac{\overline{T(z_1)}}{\overline{z_1}-\overline{w}}T(w)f'(w)f'(a)|dw| \\
&+ \frac{\mu^2}{2\pi\mathrm{i}}\int_{\Gamma_0}\frac{\overline{T(z_1)}}{\overline{z_1}-\overline{w}}T(w)f'(w)f'(a)|dw|, \quad z_1\in\Gamma_1. \qquad (27)
\end{aligned}$$

After simplications, we get

$$T(z_0)f'(z_0)f'(a) + \int_{\Gamma_0} N(z_0,w)T(w)f'(w)f'(a)|dw|$$
$$- \int_{-\Gamma_1} \frac{1}{2\pi\mathrm{i}} \left[\frac{T(z_0)}{z_0 - w}\right] T(w)f'(w)f'(a)|dw|$$
$$= -\frac{\overline{T(z_0)}}{(\overline{a}-\overline{z_0})^2} - \frac{1}{2\pi\mathrm{i}\mu^2}\int_{-\Gamma_1} \frac{\overline{T(z_0)}}{\overline{z_0}-\overline{w}} T(w)f'(w)f'(a)|dw|, \quad z_0 \in \Gamma_0, \quad (28)$$

$$T(z_1)f'(z_1)f'(a) + \int_{\Gamma_0} \frac{1}{2\pi\mathrm{i}} \left[\frac{T(z_1)}{z_1 - w}\right] T(w)f'(w)f'(a)|dw|$$
$$- \int_{-\Gamma_1} N(z_1,w)T(w)f'(w)f'(a)|dw|$$
$$= -\frac{\mu^2\overline{T(z_1)}}{(\overline{a}-\overline{z_1})^2} + \frac{\mu^2}{2\pi\mathrm{i}}\int_{\Gamma_0} \frac{\overline{T(z_1)}}{\overline{z_1}-\overline{w}} T(w)f'(w)f'(a)|dw|, \quad z_1 \in \Gamma_1. \quad (29)$$

Rearranging (28) and (29) yields

$$T(z_0)f'(z_0)f'(a) + \int_{\Gamma_0} N(z_0,w)T(w)f'(w)f'(a)|dw|$$
$$- \int_{-\Gamma_1} \frac{1}{2\pi\mathrm{i}} \left[\frac{T(z_0)}{z_0 - w} - \frac{\overline{T(z_0)}}{\mu^2(\overline{z_0}-\overline{w})}\right] T(w)f'(w)f'(a)|dw|$$
$$= -\frac{\overline{T(z_0)}}{(\overline{a}-\overline{z_0})^2}, \quad z_0 \in \Gamma_0, \quad (30)$$

$$T(z_1)f'(z_1)f'(a) + \int_{\Gamma_0} \frac{1}{2\pi\mathrm{i}} \left[\frac{T(z_1)}{z_1 - w} - \frac{\mu^2\overline{T(z_1)}}{\overline{z_1}-\overline{w}}\right] T(w)f'(w)f'(a)|dw|$$
$$- \int_{-\Gamma_1} N(z_1,w)T(w)f'(w)f'(a)|dw| = -\frac{\mu^2\overline{T(z_1)}}{(\overline{a}-\overline{z_1})^2}, \quad z_1 \in \Gamma_1. \quad (31)$$

Defining

$$\begin{aligned}
g(z,a) &= T(z)f'(z)f'(a), \\
h(a,z) &= -\frac{\overline{T(z)}}{(\overline{a}-\overline{z})^2}, \\
P(z,w) &= \frac{1}{2\pi\mathrm{i}}\left[\frac{T(z)}{z-w} - \frac{\overline{T(z)}}{\mu^2(\overline{z}-\overline{w})}\right], \\
Q(z,w) &= \frac{1}{2\pi\mathrm{i}}\left[\frac{T(z)}{z-w} - \frac{\mu^2\overline{T(z)}}{\overline{z}-\overline{w}}\right],
\end{aligned}$$

(30) and (31) can be written as

$$g(z_0,a)+\int_{\Gamma_0} N(z_0,w)g(w,a)|dw| - \int_{-\Gamma_1} P(z_0,w)g(w,a)|dw| = h(a,z_0), \quad z_0\in\Gamma_0, \quad (32)$$

$$g(z_1,a)+\int_{\Gamma_0} Q(z_1,w)g(w,a)|dw| - \int_{-\Gamma_1} N(z_1,w)g(w,a)|dw| = \mu^2 h(a,z_1), \quad z_1\in\Gamma_1. \quad (33)$$

3 Numerical Implementation

Using parametric representations $z_0(t)$ of Γ_0 for $t : 0 \le t \le \beta_0$ and $z_1(t)$ of $-\Gamma_1$ for $t : 0 \le t \le \beta_1$, equations (32) and (33) become

$$\begin{aligned} g(z_0(t),a) &+ \int_0^{\beta_0} N(z_0(t),z_0(s))g(z_0(s),a)|z_0'(s)|ds \\ &- \int_0^{\beta_1} P(z_0(t),z_1(s))g(z_1(s),a)|z_1'(s)|ds = h(a,z_0(t)), \quad z_0(t)\in\Gamma_0, \quad (34) \\ g(z_1(t),a) &+ \int_0^{\beta_0} Q(z_1(t),z_0(s))g(z_0(s),a)|z_0'(s)|ds \\ &- \int_0^{\beta_1} N(z_1(t),z_1(s))g(z_1(s),a)|z_1'(s)|ds = \mu^2 h(a,z_1(t)), \quad z_1(t)\in\Gamma_1. \quad (35) \end{aligned}$$

Multiplying (34) and (35) by $|z_0'(t)|$ and $|z_1'(t)|$, respectively, gives

$$\begin{aligned} |z_0'(t)|g(z_0(t),a) &+ \int_0^{\beta_0} |z_0'(t)|N(z_0(t),z_0(s))g(z_0(s),a)|z_0'(s)|ds \\ &- \int_0^{\beta_1} |z_0'(t)|P(z_0(t),z_1(s))g(z_1(s),a)|z_1'(s)|ds = |z_0'(t)|h(a,z_0(t)), \quad z_0(t)\in\Gamma_0, \quad (36) \\ |z_1'(t)|g(z_1(t),a) &+ \int_0^{\beta_0} |z_1'(t)|Q(z_1(t),z_0(s))g(z_0(s),a)|z_0'(s)|ds \\ &- \int_0^{\beta_1} |z_1'(t)|N(z_1(t),z_1(s))g(z_1(s),a)|z_1'(s)|ds = \mu^2|z_1'(t)|h(a,z_1(t)), \quad z_1(t)\in\Gamma_1. \quad (37) \end{aligned}$$

Defining

$$\begin{aligned} \phi_0(t) &= |z_0'(t)|g(z_0(t),a), \\ \phi_1(t) &= |z_1'(t)|g(z_1(t),a), \\ \gamma_0(t) &= |z_0'(t)|h(a,z_0(t)), \\ \gamma_1(t) &= \mu^2|z_1'(t)|h(a,z_1(t)), \\ K_{00}(t_0,s_0) &= |z_0'(t)|N(z_0(t),z_0(s)), \\ K_{01}(t_0,s_1) &= |z_0'(t)|P(z_0(t),z_1(s)), \\ K_{10}(t_1,s_0) &= |z_1'(t)|Q(z_1(t),z_0(s)), \\ K_{11}(t_1,s_1) &= |z_1'(t)|N(z_1(t),z_1(s)), \end{aligned}$$

equations (36) and (37) can be written briefly as

$$\phi_0(t) + \int_0^{\beta_0} K_{00}(t_0, s_0)\phi_0(s)ds - \int_0^{\beta_1} K_{01}(t_0, s_1)\phi_1(s)ds = \gamma_0(t), \quad z_0 \in \Gamma_0, \tag{38}$$

$$\phi_1(t) + \int_0^{\beta_0} K_{10}(t_1, s_0)\phi_0(s)ds - \int_0^{\beta_1} K_{11}(t_1, s_1)\phi_1(s)ds = \gamma_1(t), \quad z_1 \in \Gamma_1. \tag{39}$$

Since the functions ϕ, γ, and K are β-periodic, an appealing procedure for solving (38) and (39) numerically is using the Nyström's method with the trapezoidal rule [1]. The trapezoidal rule is the most accurate method for integrating periodic functions numerically. We choose $\beta_0 = \beta_1 = 2\pi$ and n equidistant collocation points $t_i = (i-1)\beta_0/n$, $1 \le i \le n$ on Γ_0 and m equidistant collocation points $t_\imath = (\imath - 1)\beta_1/m$, $1 \le \imath \le m$, on Γ_1. Applying the Nyström's method with trapezoidal rule to discretize (38) and (39), gives

$$\phi_0(t_i) + \frac{\beta_0}{n}\sum_{j=1}^{n} K_{00}(t_i, t_j)\phi_0(t_j) - \frac{\beta_1}{m}\sum_{k=1}^{m} K_{01}(t_i, t_k)\phi_1(t_k) = \gamma_0(t_i), \tag{40}$$

$$\phi_1(t_\imath) + \frac{\beta_0}{n}\sum_{j=1}^{n} K_{10}(t_\imath, t_j)\phi_0(t_j) - \frac{\beta_1}{m}\sum_{k=1}^{m} K_{11}(t_\imath, t_k)\phi_1(t_k) = \gamma_1(t_\imath). \tag{41}$$

Equations (40) and (41) lead to a system of $(n+m)$ complex equations in n unknowns $\phi_0(t_i)$ and m unknowns $\phi_1(t_\imath)$. By defining the matrices

$$\begin{aligned}
B_{ij} &= \frac{\beta_0}{n}K_{00}(t_i, t_j),\\
C_{ik} &= \frac{\beta_1}{m}K_{01}(t_i, t_k),\\
E_{\imath j} &= \frac{\beta_0}{n}K_{10}(t_\imath, t_j),\\
D_{\imath k} &= \frac{\beta_1}{m}K_{11}(t_\imath, t_k),\\
x_{0i} &= \phi_0(t_i),\\
x_{1\imath} &= \phi_1(t_\imath),\\
\gamma_{0i} &= \gamma_0(t_i),\\
\gamma_{1\imath} &= \gamma_1(t_\imath),
\end{aligned}$$

the system of equations (40) and (41) can be written as $n+m$ by $n+m$ system of equations

$$[I_{nn} + B_{nn}]\mathbf{x}_{0n} - C_{nm}\mathbf{x}_{1m} = \gamma_{0n}, \tag{42}$$

$$E_{mn}\mathbf{x}_{0n} + [I_{mm} - D_{mm}]\mathbf{x}_{1m} = \gamma_{1m}. \tag{43}$$

The result in matrix form for the system of equations (42) and (43) is

$$\begin{pmatrix} I_{nn} + B_{nn} & \cdots & -C_{nm} \\ \vdots & \cdots & \vdots \\ E_{mn} & \cdots & I_{mm} - D_{mm} \end{pmatrix} \begin{pmatrix} \mathbf{x}_{0n} \\ \vdots \\ \mathbf{x}_{1m} \end{pmatrix} = \begin{pmatrix} \gamma_{0n} \\ \vdots \\ \gamma_{1m} \end{pmatrix}. \tag{44}$$

Defining

$$A = \begin{pmatrix} I_{nn}+B_{nn} & \cdots & -C_{nm} \\ \vdots & \cdots & \vdots \\ E_{mn} & \cdots & I_{mm}-D_{mm} \end{pmatrix}, \ \mathbf{x}= \begin{pmatrix} \mathbf{x}_{0n} \\ \vdots \\ \mathbf{x}_{1m} \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} \gamma_{0n} \\ \vdots \\ \gamma_{1m} \end{pmatrix},$$

the $(n+m)\times(n+m)$ system can be written briefly as $A\mathbf{x} = \mathbf{y}$. Separating A, $\mathbf{x}$ and $\mathbf{y}$ in terms of the real and imaginary parts, the system can be written as

$$\operatorname{Re} A \operatorname{Re}\mathbf{x} - \operatorname{Im} A \operatorname{Im}\mathbf{x} + \mathrm{i}(\operatorname{Im} A \operatorname{Re}\mathbf{x} + \operatorname{Re} A \operatorname{Im}\mathbf{x}) = \operatorname{Re}\mathbf{y} + \mathrm{i}\operatorname{Im}\mathbf{y}. \tag{45}$$

The single $(n+m)\times(n+m)$ complex system (45) can also be written as $2(n+m)\times 2(n+m)$ system matrix involving the real (Re) and imaginary (Im) of the unknown functions, i.e.,

$$\begin{pmatrix} \operatorname{Re} A & \cdots & \operatorname{Im} A \\ \vdots & \cdots & \vdots \\ \operatorname{Im} A & \cdots & \operatorname{Re} A \end{pmatrix} \begin{pmatrix} \operatorname{Re}\mathbf{x} \\ \vdots \\ \operatorname{Im}\mathbf{x} \end{pmatrix} = \begin{pmatrix} \operatorname{Re}\mathbf{y} \\ \vdots \\ \operatorname{Im}\mathbf{y} \end{pmatrix}. \tag{46}$$

Since the parameter μ is assumed known, the system (46) can be solved simultaneously for the unknown functions,

$$\phi_0(t) = |z_0'(t)|T(z_0(t))f'(z_0(t))f'(a), \tag{47}$$
$$\phi_1(t) = |z_1'(t)|T(z_1(t))f'(z_1(t))f'(a). \tag{48}$$

The boundary correspondence functions $\theta_0(t)$ and $\theta_1(t)$ are then computed approximately by the formulas

$$\theta_0(t) = \operatorname{Arg} f(z_0(t)) \approx \operatorname{Arg}(-\mathrm{i}\phi_0(t)), \tag{49}$$
$$\theta_1(t) = \operatorname{Arg} f(z_1(t)) \approx \operatorname{Arg}(\pm\mathrm{i}\phi_1(t)). \tag{50}$$

4 Numerical Results

For numerical experiment, we have used the frame of circular annulus $A = \{z : r < |z| < 1\}$, $r = q = e^{-\pi\tau}$, $\tau > 0$, as a test region. The exact mapping function is given by [23]

$$f(z) = -e^{2\sigma} \frac{\theta_4\left(\frac{1}{2\mathrm{i}}\log z + \frac{\mathrm{i}\pi\tau}{2} - \mathrm{i}\sigma\right)}{\theta_4\left(\frac{1}{2\mathrm{i}}\log z + \frac{\mathrm{i}\pi\tau}{2} + \mathrm{i}\sigma\right)}, \tag{51}$$

with $\mu = e^{-2\sigma}$ and θ_4 being the Jacobi Theta-functions. We have chosen $\tau = 0.50$ and $\sigma = 0.20$. Since $\theta_4(\pi\tau\mathrm{i}/2) = 0$ [24], this implies $a = e^{-2\sigma} = \mu$. Figure 1 shows the region and image based on our method. The results for the sub-norm error between the exact boundary correspondence functions $\theta_0(t)$, $\theta_1(t)$ and the computed boundary correspondence functions $\theta_{0n}(t)$, $\theta_{1m}(t)$ is shown in Table 4 All the computations were done using MATHEMATICA package [25] in single precision (16 digit machine precision).

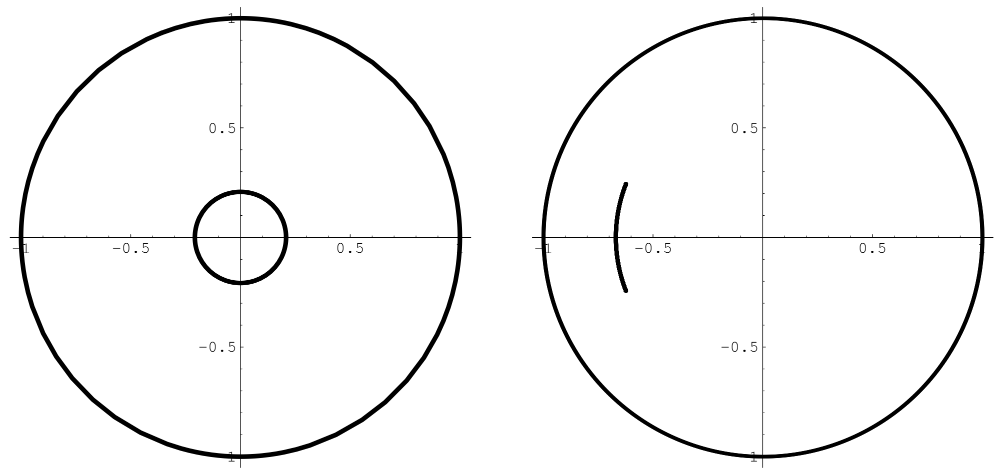

Figure 1: Conformal mapping of a circular annulus onto the unit disc with a circular slit : $\tau = 0.50, \sigma = 0.20, r = e^{-\pi\tau}, a = \mu = e^{-2\sigma}$.

Table 1: Error Norm

$n = m$	$\|\theta_0(t) - \theta_{0n}(t)\|_\infty$	$\|\theta_1(t) - \theta_{1m}(t)\|_\infty$
32	6.3(−05)	3.2(−04)
64	3.5(−10)	1.9(−09)
128	1.2(−14)	8.7(−13)

5 Conclusion

In this paper we have constructed a system of integral equations for numerical conformal mapping of doubly connected regions onto a unit disc with a concentric circular slit of radius μ. The system involved the Neumann kernel and is linear if μ is assumed to be known. The numerical example illustrates that the present method can be used to produce approximations of high accuracy provided μ is known. In practice, however, μ is unknown and has to be determined in the course of numerical computation. For unknown μ, the discretized system presented in this paper becomes a system of nonlinear equations. Therefore, any solution method must be iterative. For such treatment of our method, see the forthcoming papers [13, 14].

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