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Fractional Casson Fluid Flow via Oscillating Motion of Plate and Microchannel

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ABSTRACT

The impact of the Caputo fractional derivative on the unsteady mixed convention boundary layer flow of Casson fluid is investigated. It is evaluated the flow via two different geometries which are plate and microchannel with oscillating motion. The problems are modelled using a set of partial differential equations with appropriate initial and boundary conditions. The dimensional equations are turned into dimensionless governing equations by using relevant dimensionless variables. The obtained solutions are transformed into fractional form using Caputo fractional derivative. The exact solutions are obtained using the Laplace transform approach. Inverse Laplace transform is applied to the oscillating plate problem while Zakian's explicit formula approach is used to obtain the results of temperature and velocity profiles. Both profiles are graphed and studied its behaviour in both geometries. The temperature profile is shown to have an opposite pattern of graph for both geometries. While when compared between both geometries on its velocity profile, oscillating plate has a higher velocity compared to oscillating plate. For both profiles, increasing the fractional parameter resulted in a greater pattern. This study aids in the comprehension of Casson fluid flows in fractional systems.

1. Introduction

Fractional calculus is used in a wide range of domains, including biology, engineering and economics field. As agreed by other researchers Jan *et al*., [1] and Qureshi [2] as compared to classical order models, fractional order derivatives are more practical. Since most of real-world problems are subjected to three major mathematical laws. According to Saqib *et al*., [3] and Abro [4], the laws include the power function, generalized Mittag-Leffler function and exponential decay law. Aside from that, fractional modelling is the sole way to express some of the most important rheological properties of industrialized fluids. Fractional partial differential equations have a variety of distinctive characteristics that make them as a useful mathematical tool for simulating the intricate behaviors of boundary layer flow. The most popular and commonly used fractional derivative operators are Riemann-Liouville and Caputo fractional derivative. However, since Riemann-Liouville fractional

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derivative has some limitations, the Caputo fractional derivative has been made to fill in the gap. The Caputo fractional derivative has received a lot of attention in the past [5-8].

Nowadays, it is critical to do research on the flow behavior of non-Newtonian fluids due to its extensive use in industrial and engineering area. Additionally, according to Abidin *et al*., [9] viscoelastic fluids exhibit peculiar patterns of instability that are neither predicted nor seen in Newtonian fluid. This fluid cannot be represented using Navier-Stokes equations. They behave as a fluid which its shear rate is unproportionally to the shear stress. One of the most important types of non-Newtonian fluid is Casson fluid. Casson first debuted this model in 1959 [10]. Yield stress is present in Casson fluid. Casson fluid is a shear thinning liquid with an infinite viscosity at zero shear rate while a zero viscosity at an infinite shear rate. In agreement with Pramanik [11] it behaves like a solid whenever the applied shear stress is less than the yield stress. Whereas when the situation is vice versa, it begun to move. Recent Casson fluid research includes the following in Reyaz *et al*., Sheikh *et al*., Naqvi *et al*., Halsted and Brown and Gudekote *et al*., [12-16].

Many researchers are actively interested in researching the flow of fluid over a variety of geometries such as a plate and microchannel, along with its applicability in nowadays concerns. An example usage of plates is in automotive industry while microfluidic can be seen in Micro Electromechanical Systems (MEMS) technology with microchannel as its geometry. The plate's and microchannel's motion may varies depending on its needed situation, such as static, oscillating, and others. Hamrelaine *et al*., [17] studied on a flow of Jeffrey fluid between two static plates with the presence of injection and suction.

As a result of the preceding research, the main objective of this study is to apply the Caputo fractional derivative operator on Casson fluid convective flow via oscillating plate as well as oscillating microchannel. Because it has yet to be addressed, this study will do so to fill in the gap. The Laplace transform method is used to acquire the exact solutions for fractional convective flow of Casson fluid. The impact of a variety of variables on fluid flow is investigated and addressed.

2. Methodology

2.1 Problem Formulation

The microchannel is represented by two plates separated by distance of d apart whereas the plate is considered as an infinite vertical flat plate. The flow is restricted to $y > 0$ in both geometries, where y is the coordinate measured in the plate's normal direction. Both the fluid and the geometries, the plate and the microchannel, at $t = 0$, are at rest. The temperature for the plate while at rest is at uniform temperature, T_∞, Meanwhile for the microchannel, it is at rest at constant ambient temperature T_o. For both geometries, they begins to oscillate in their respective planes at time $t = 0^+$, corresponding to its velocity,

$$V = fH(t)\cos\omega t, \tag{1}$$

where the constant f is the plate oscillations amplitude, $H(t)$ is the unit step function, and ω is the frequency with which the plate oscillates. For both geometries, the plate temperature is raised to T_w at the same moment as oscillating motion occurred. And then it is kept constant at that temperature. The temperature and velocity are regulated by the time variable t and space variable y. The momentum and energy equations following forms when utilizing unidirectional and one-dimensional Casson fluid flow and the Boussinesq's approximation.

$$\frac{\partial u(y,t)}{\partial t} = \nu\left(1+\frac{1}{\beta}\right)\frac{\partial^2 u(y,t)}{\partial y^2} + g\beta_T\left(T-T_\infty\right), \tag{2}$$

$$\rho C_p \frac{\partial T(y,t)}{\partial t} = k\frac{\partial^2 T(y,t)}{\partial y^2}, \tag{3}$$

Here, the fluid velocity is denoted by $u(y,t)$, the temperature of the fluid is symbolized by $T(y,t)$, β is the parameter of Casson fluid, the kinematic viscosity is expressed by ν, g is the acceleration of gravitational, β_T is the thermal expansion coefficient, C_p is the specific heat at constant pressure, and k is the conductivity of thermal. For oscillating plate, equations (2) and (3) are subjected to equation (4) whereas equation (5) depicts the initial and boundary conditions for oscillating microchannel.

$$\begin{aligned} u(y,0) &= 0, & T(y,0) &= T_\infty, \\ u(0,t) &= fH(t)\cos\omega t, & T(0,t) &= T_w, \\ u(\infty,t) &= 0, & T(\infty,t) &= T_\infty, \end{aligned} \tag{4}$$

$$\begin{aligned} u(y,0) &= 0, & T(y,0) &= T_o, \\ u(0,t) &= 0, & T(0,t) &= T_o, \\ u(d,t) &= fH(t)\cos\omega t, & T(d,t) &= T_w, \end{aligned} \tag{5}$$

The following non-dimensional variables are introduced to transform the governing equations into non-dimensional equations. Equation (6) are used for oscillating plates, while oscillating microchannel used non-dimensional variables as in equation (7)

$$v=\frac{u}{f}, \quad \xi=\frac{y}{\gamma}, \quad \gamma=\frac{\nu}{f}, \quad \tau=\frac{t}{\lambda}, \quad \theta=\frac{T-T_\infty}{T_w-T_\infty}, \tag{6}$$

$$v=\frac{u}{U}, \quad \tau=\frac{t\nu}{d^2}, \quad \xi=\frac{y}{d}, \quad \theta=\frac{T-T_o}{T_w-T_o}, \tag{7}$$

The momentum and energy equations, as well as the initial and boundary conditions, are derived in non-dimensional form using the non-dimensional variables as in equation (7). Therefore, the non-dimensional governing equations are as follows

$$\frac{\partial v}{\partial \tau} = \frac{1}{\beta_o}\frac{\partial^2 v}{\partial \xi^2} + Gr\theta, \tag{8}$$

$$\Pr\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \xi^2}, \tag{9}$$

subjected to

$$\begin{aligned} v(\xi,0)&=0, & \theta(\xi,0)&=0, \\ v(0,\tau)&=H(\tau)\cos\omega\tau, & \theta(0,\tau)&=1, \\ v(\infty,\tau)&=0, & \theta(\infty,\tau)&=0, \end{aligned} \tag{10}$$

for oscillating plate. While the initial and boundary conditions for oscillating microchannel are

$$\begin{aligned} v(\xi,0)&=0, & \theta(\xi,0)&=0, \\ v(0,\tau)&=0, & \theta(0,\tau)&=0, \\ v(1,\tau)&=H(\tau)\cos\Omega\tau, & \theta(1,\tau)&=1, \end{aligned} \tag{11}$$

where $\Omega=\dfrac{\omega d^2}{\nu}$. $\Omega\tau$ shows the phase angle. Then, the non-dimensional governing equations are then transformed into fractional non-dimensional governing equations using the Caputo fractional derivative. The definition of the Caputo fractional derivative are as follows

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha}=\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t \frac{1}{(t-\tau)^\alpha}\frac{\partial f(\tau)}{\partial\tau}\partial\tau; 0<\alpha<1 \\ \dfrac{\partial f(t)}{\partial t}; \alpha=1 \end{cases} \tag{12}$$

where Γ represents the Euler Gamma function and the fractional parameter is described by α.

2.2 Solutions
2.2.1 Solution for oscillating plate

The fractional non-dimensional governing equations are applied with Laplace transform method. The changed system is then produced as follows:

$$\frac{\partial^2\bar{v}}{\partial\xi^2}-\beta_o q^\alpha \bar{v}(\xi,q)=-\beta_o Gr\frac{1}{q}e^{-\xi\sqrt{\Pr q^\alpha}}, \tag{13}$$

$$\Pr q^\alpha\bar{\theta}=\frac{\partial^2\theta}{\partial\xi^2}, \tag{14}$$

$$\bar{v}(0,q)=\frac{q}{q^2+\omega^2}, \quad \bar{\theta}(0,q)=\frac{1}{q}, \tag{15}$$

Based on condition (15), the following is the solution to equations (13) and (14) respectively.

$$\bar{v}(\xi,q)=\frac{q}{q^2+\omega^2}e^{-\xi\sqrt{\beta_o q^\alpha}}+\frac{Gr_o}{(\Pr q^\alpha-\beta_o q^\alpha)}\frac{1}{q}e^{-\xi\sqrt{\beta_o q^\alpha}}-\frac{Gr_o}{(\Pr q^\alpha-\beta_o q^\alpha)}\frac{1}{q}e^{-\xi\sqrt{\Pr q^\alpha}}, \tag{16}$$

$$\bar{\theta}(\xi,q)=\frac{1}{q}e^{-\xi\sqrt{\Pr q^\alpha}}, \tag{17}$$

The inverse Laplace transform of equation (16) and (17) gives

$$\mathrm{v}=\cos\,\omega\tau\times\tau^{-1}\phi\left(0,-\frac{\alpha}{2},-\xi\sqrt{\frac{\beta_o}{\tau^{\alpha}}}\right)+\frac{Gr_o}{(\Pr q^{\alpha}-\beta_o q^{\alpha})}\times\begin{bmatrix}\phi\left(1,-\frac{\alpha}{2},-\xi\sqrt{\frac{\beta_o}{\tau^{\alpha}}}\right)\\ \phi\left(1,-\frac{\alpha}{2},-\xi\sqrt{\frac{\Pr}{\tau^{\alpha}}}\right)\end{bmatrix}, \tag{18}$$

$$\theta=\phi\left(1,-\frac{\alpha}{2},-\xi\sqrt{\frac{\Pr}{\tau^{\alpha}}}\right). \tag{19}$$

2.2.2 Solution for oscillating microchannel

The same method as in oscillating plate, the Laplace transform method is also used to solve the fractional non-dimensional governing equations for the oscillating microchannel. The modified system is created as follows:

$$\beta_o q^{\alpha}\bar{v}(\xi,q)=\frac{\partial^2\bar{v}}{\partial y^2}+\beta_o Gr\frac{1}{q\left(e^{\sqrt{\Pr_{eff}\,q^{\alpha}}}-e^{-\sqrt{\Pr_{eff}\,q^{\alpha}}}\right)}\left(e^{\xi\sqrt{\Pr_{eff}\,q^{\alpha}}}-e^{-\xi\sqrt{\Pr_{eff}\,q^{\alpha}}}\right), \tag{20}$$

$$\Pr_{eff}\,q^{\alpha}\bar{\theta}=\frac{\partial^2\theta}{\partial\xi^2}, \tag{21}$$

$$\bar{v}(1,q)=\frac{q}{q^2+\Omega^2},\quad \bar{\theta}(0,q)=0,\quad \bar{\theta}(1,q)=\frac{1}{q}, \tag{22}$$

The following is the solution to equation (20) and (21) based on condition (22):

$$\begin{aligned}\bar{v}(\xi,q)=&\frac{q}{q^2+\Omega^2}\left(\frac{e^{-\xi\sqrt{\beta_o q^{\alpha}}}-e^{\xi\sqrt{\beta_o q^{\alpha}}}}{e^{-\sqrt{\beta_o q^{\alpha}}}-e^{\sqrt{\beta_o q^{\alpha}}}}\right)\\&+\frac{Gr\left(e^{\sqrt{\Pr_{eff}\,q^{\alpha}}}-e^{-\sqrt{\Pr_{eff}\,q^{\alpha}}}\right)}{q\left(e^{\sqrt{\Pr_{eff}\,q^{\alpha}}}-e^{-\sqrt{\Pr_{eff}\,q^{\alpha}}}\right)\left(\Pr_{eff}\,q^{\alpha}-\beta_o q^{\alpha}\right)}\left(\frac{e^{-\xi\sqrt{\beta_o q^{\alpha}}}-e^{\xi\sqrt{\beta_o q^{\alpha}}}}{e^{-\sqrt{\beta_o q^{\alpha}}}-e^{\sqrt{\beta_o q^{\alpha}}}}\right)\\&-\frac{Gr_o}{q\left(e^{\sqrt{\Pr_{eff}\,q^{\alpha}}}-e^{-\sqrt{\Pr_{eff}\,q^{\alpha}}}\right)\left(\Pr_{eff}\,q^{\alpha}-\beta_o q^{\alpha}\right)}\left(e^{\xi\sqrt{\Pr_{eff}\,q^{\alpha}}}-e^{-\xi\sqrt{\Pr_{eff}\,q^{\alpha}}}\right),\end{aligned} \tag{23}$$

$$\bar{\theta}(\xi,q)=\frac{1}{q\left(e^{\sqrt{\Pr_{eff}\,q^{\alpha}}}-e^{-\sqrt{\Pr_{eff}\,q^{\alpha}}}\right)}\left(e^{\xi\sqrt{\Pr_{eff}\,q^{\alpha}}}-e^{-\xi\sqrt{\Pr_{eff}\,q^{\alpha}}}\right). \tag{24}$$

Since the equations obtained (23) and (24) cannot be inverted analytically using tabular forms, therefore the inverse Laplace transform is numerically computed using the Zakian's explicit formula approach, to display the temperature and velocity profiles, respectively.

$$f(t)=\frac{t}{2}\sum_{j=1}^{N}\mathrm{Re}\left\{K_j F\left(\frac{\alpha_j}{t}\right)\right\}, \tag{25}$$

where K_j and α_j are constants either in real or complex conjugate pairs form. The values for K_j and α_j are taken from Wang *et al.*, [18].

3. Results

In this section, the velocity and temperature profiles for the influence of the fractional parameter study are described. The effect of fractional parameter, α, on the fluid temperature is represented in Fig. 1. It can be seen that both of the geometries have an opposite pattern of graph. The oscillating plates have depicted a decreasing exponentially curve. Meanwhile for the oscillating microchannel, the curves increase exponentially. The findings indicate as those since the surface area for the plate is much wider compared to microchannel. Whereas, since the surface area for the microchannel is enclosed, the temperature favor to be kept in surrounding. The influence of α, for both geometries show an increasing pattern for the temperature profile. The temperature profile is higher for the classical Casson fluid model as compared for the fractional Casson fluid model. At $\alpha = 1$, the temperature profile's maximum curve is attained.

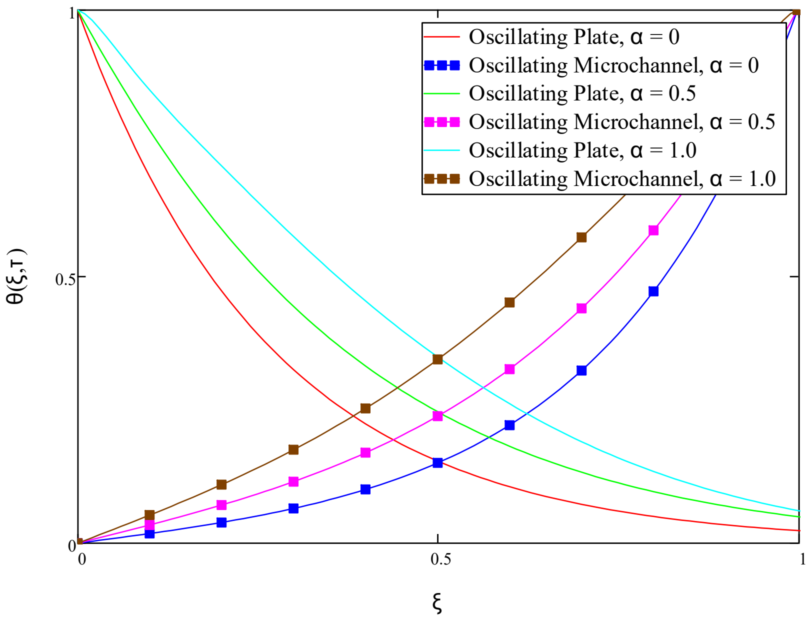

Fig. 1. Temperature profile

Fig. 2 illustrates the velocity profile results for both geometries. In comparison to the oscillating microchannel, the oscillating plate has a higher velocity. Since the plate has an infinite length, its tendency to oscillate with a higher velocity is elevated as compared to the microchannel who have a finite length. For both geometries, as the fractional parameter α, increase, the velocity profile increases as well. It shows that fluid velocity es enhanced by increasing fractional parameter, α. The thickness of thermal and momentum boundary layers increases along with the increasing of fractional parameter, α, values which causes the velocity profile to rise as well.

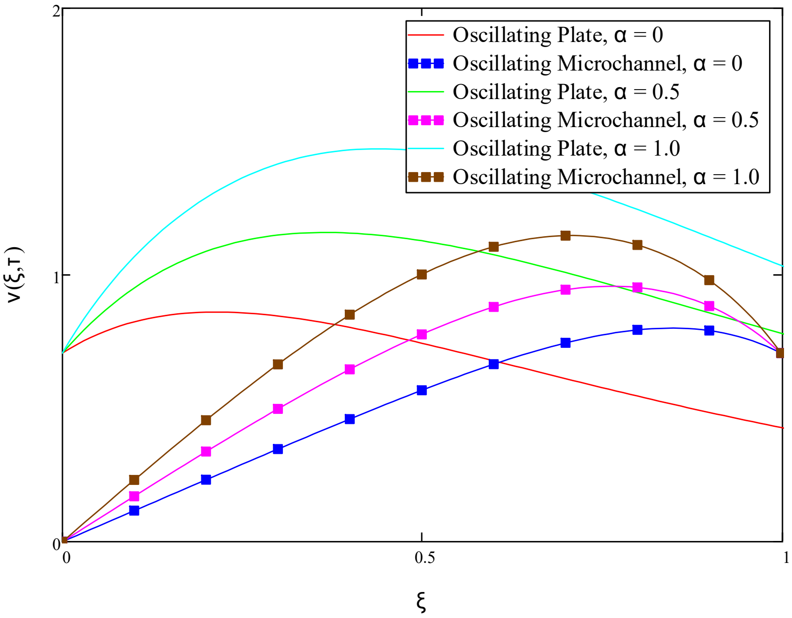

Fig. 2. Velocity profile

4. Conclusions

The Caputo fractional derivative is used to model the convective flow of Casson fluid via an oscillating motion of plates and microchannel in this article. The exact solutions for both geometries were derived using the Laplace transform method, which is then utilized to develop the velocity and temperature profiles. However, since it is not possible to compute the inversion of exact equations for the oscillating microchannel, the inverse Laplace transform is numerically calculated using the Zakian's explicit formula. The oscillating plate has a higher velocity compared to oscillating microchannel. Meanwhile, the temperature profile for both geometries has an opposite pattern of curve. Where the temperature profile for oscillating plate decreases exponentially, whereas for oscillating microchannel, it increases exponentially. The fractional parameter's influence is displayed for both profiles. Both profiles increased as the fractional parameter increased.

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