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Unsteady boundary layer flow in the region of the stagnation point on a stretching sheet

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Abstract

Unsteady two-dimensional stagnation point flow of an incompressible viscous fluid over a flat deformable sheet is studied when the flow is started impulsively from rest and the sheet is suddenly stretched in its own plane with a velocity proportional to the distance from the stagnation point. After a similarity transformation, the unsteady boundary layer equation is solved numerically using the Keller-box method for the whole transient from $\tau = 0$ to the steady state $\tau \to \infty$. Also, a complete analysis is made of the governing equation at $\tau = 0$, the initial unsteady flow, at large times $\tau = \infty$, the steady state flow, and a series solution valid at small times τ ($\ll$1). It is found that there is a smooth transition from the initial unsteady state flow (small time solution) to the final steady state flow (large time solution).
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1. Introduction

The production of sheeting material arises in a number of industrial manufacturing processes and includes both metal and polymer sheets. In the manufacture of the latter, the material is in a molten phase when thrust through an extrusion die and then cools and solidifies some distance away from the die before arriving at the collecting stage. The tangential velocity imported by the

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Nomenclature

a	constant proportional to free stream velocity far away from the sheet
b	proportionality constant of the velocity of the stretching sheet
C_f	skin friction coefficient
f	reduced stream function
Re_x	local Reynolds number
t	time
u	velocity component along the sheet
u_e	free stream velocity
u_w	velocity of the sheet
v	velocity component normal to the sheet
x	coordinate along the sheet
y	coordinate normal to the sheet
Greek letters	
η	pseudo-similarity variable
$\bar{\eta}$	transformed variable
λ	parameter
ν	kinematic viscosity
ξ	non-dimensional transformed variable
τ	non-dimensional time
ρ	density
Subscript	
w	conditions at the sheet
Superscript	
$'$	derivation with respect to η or $\bar{\eta}$

sheet induces motion in the surrounding fluid, which alters the convection cooling of the sheet. Similar situations prevail during the manufacture of plastic and rubber sheets where it is often necessary to blow a gaseous medium through the not-yet solidified material, and where the stretching force may be varying with time. Another example that belongs to this class of problems is the cooling of a large metallic plate in a bath, which may be an electrolyte. In this case the fluid flow is induced due to shrinking of the plate. Glass blowing, continuous casting, and spinning of fibers also involve the flow due to a stretching surface. Due to the much higher viscosity of the fluid near the sheet, one can assume that the fluid is affected by the sheet but not vice versa. Thus the fluid dynamic problem can be idealized to the case of a fluid disturbed by a tangentially moving boundary. Experiments show that the velocity of the boundary is approximately proportional to the distance from the orifice (Vleggaar [1]). The quality of the resulting sheeting material, as well as the cost of production, is affected by the speed of collection and the heat transfer rate, and knowledge of the flow properties of the ambient fluid is clearly desirable.

The steady state solution of this problem belongs to an important class of exact solutions of the Navier–Stokes equations (Wang [2,3]). Since the tangential velocities are proportional to the distance from the orifice, the Navier–Stokes equations reduce to an ordinary differential equation through a similarity transformation. The two-dimensional steady flow due to stretching of a sheet is particularly interesting because there is a closed form solution, which has been obtained by Crane [4]. Brady and Acrivos [5] investigated the similarity exact solutions of the steady flow inside a stretching channel and inside a stretching cylinder. The steady two-dimensional boundary layer flow, due to non-uniform stretching surfaces, was discussed by Kuiken [6], Banks [7], Banks and Zaturska [8], and Magyari and Keller [9]. On the other hand, the steady two-dimensional stagnation point flow of an incompressible viscous fluid towards a stretching plate, has been studied by Chiam [10], while Mahapatra and Gupta [11] studied the heat transfer situation. It is worth mentioning that existence, uniqueness and multiple similarity solutions for the steady pressure gradient driven flows over a stretching surface in a viscous and incompressible fluid of the Blasius and Falkner-Skan type flows, have been established by Klemp and Acrivos [12,13], Hussaini et al. [14], McLeod and Rajagopal [15], and Riley and Weidman [16]. The paper by McLeod and Rajagopal [15] has then been used by Troy et al. [17] to investigate uniqueness of the steady flow of an incompressible second-order fluid past a stretching sheet.

However, the unsteady two-dimensional boundary layer flow due to a suddenly stretched plane surface in a viscous fluid has received much less attention. To our best knowledge, only Pop and Na [18], and Wang et al. [19] have considered this problem. The case of the three-dimensional unsteady flow with heat and mass transfer over a continuous stretching surface has been studied by Lakshmisha et al. [20]. The case of unsteady flow due to a stretching surface in a rotating fluid, where the unsteadiness is caused by the suddenly stretched surface has been recently considered by Nazar et al. [21]. Therefore, the aim of this paper is to study the unsteady two-dimensional stagnation point flow of a viscous and incompressible fluid towards a stretching surface. It is assumed that both the sheet and the velocity distribution in the potential flow start impulsively from rest. As time approaches infinity, the unsteady (transient) solution should approach the steady state solution reported by Mahapatra and Gupta [11]. Although strictly impulsive motion does not occur in practice (infinite stresses at time zero), it is nevertheless an excellent approximation of sudden changes in the boundary conditions. Due to nonlinear convection, there are no analytical solutions for impulsive start, except for parallel or concentric flows, where the nonlinear are identically zero. Most solutions are found by a variety of numerical schemes. For a very short time interval after impulsive start, nonlinear effects are secondary to diffusion and a perturbation method may be used to obtain an analytic solution. Numerous authors used small-time expansion to solve impulsive starting problems (see Pop and Na [18]). The solution, as the method suggests, is valid only for small times. As time τ approaches infinity, the solution, which is a power series in τ, becomes infinite.

As we have mentioned above, the objective of this paper is to present a detailed study of the development of the two-dimensional boundary layer flow of a viscous and incompressible fluid in the region of the stagnation point on a stretching sheet. The unsteadiness in the flow field is caused by impulsively creating motion in the free stream and at the same time suddenly stretching the surface. The governing continuity and momentum equations are transformed using semi-similar coordinates originated by Williams and Rhyne [22], and very recently used by Seshadri et al. [23] to study the unsteady mixed convection boundary layer flow of a viscous and incompressible fluid near the stagnation point on a vertical surface. Following Mahapatra and Gupta [11], the boundary layer structure of the present problem is found to depend on the parameter λ, which defines the ratio of the velocity of the stretching surface to that of the frictionless potential flow in the neighborhood of the stagnation point. A complete analysis is made of the transformed boundary layer equation for a wide range of values of the parameter λ. A closed form solution of this equation has been shown to exist at $\tau = 0$ (initial unsteady flow), as $\tau \to \infty$ (the steady state flow) and for small times τ ($\ll 1$). It is shown that for the steady state flow (large times), the analytical behavior of the solution approaches Crane's solution for small values of λ ($\ll 1$) and Hiemenz's solution for large values of λ ($\gg 1$), respectively. A numerical solution of the transformed equation using the Keller-box method was then obtained for the whole transient regime. The results show that there is a smooth transition from the small time solution to the large time solution. Particular cases of the present results are compared with those of Mahapatra and Gupta [11], and the agreement is very good.

2. Governing equations

Consider the unsteady flow of a viscous and incompressible fluid near the stagnation point of a flat sheet coinciding with the plane $y = 0$, the flow being confined to $y > 0$. Velocity is zero for $t < 0$. For $t > 0$, the sheet is suddenly stretched such that the local tangential velocity is $u_w = bx$, where b is a positive constant, keeping the origin fixed, as shown schematically in Fig. 1. It is also assumed that for $t > 0$, the velocity distribution in the potential flow (free stream velocity), given by $u_e = ax$, where a is a positive constant, starts impulsively in motion from rest.

The unsteady two-dimensional Navier–Stokes equations can be written under the boundary layer approximation as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

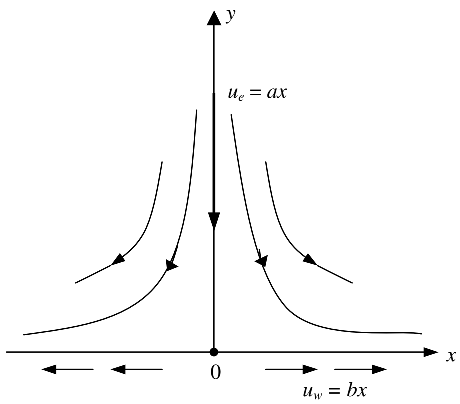

Fig. 1. Physical model and coordinate system.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = a^2 x + v\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

subject to initial and boundary conditions

$$\begin{aligned} &t < 0: \quad u = v = 0, \\ &t \geqslant 0: \quad u = bx, \quad v = 0 \text{ at } y = 0 \qquad u \to ax \quad \text{as} \quad y \to \infty, \end{aligned} \tag{3}$$

where u and v denote the velocity components along x and y axes, respectively, with x being measured along the sheet and y measured normal to it, and ν is the kinematic viscosity.

To solve Eqs. (1) and (2) for $t \geqslant 0$, it is convenient to choose a new time scale ξ so that the region of time integration may become finite. Thus, following Williams and Rhyne [22], we introduce the variables

$$\begin{aligned} &\eta = (b/\nu)^{1/2}\xi^{-1/2}y, \quad u = bxf'(\xi,\eta), \quad v = -(b\nu)^{1/2}\xi^{1/2}f(\xi,\eta), \\ &\xi = 1 - \exp(-\tau), \quad \tau = bt. \end{aligned} \tag{4}$$

Substituting these variables into Eq. (2) we get

$$f''' + \frac{1}{2}(1-\xi)\eta f'' + \xi(\lambda^2 + ff'' - f'^2) = \xi(1-\xi)\frac{\partial f'}{\partial \xi}, \tag{5}$$

which must be solved over the range of ξ with $0 \leqslant \xi \leqslant 1$, and subject to the boundary conditions

$$f(\xi,0) = 0, \quad f'(\xi,0) = 1, \quad f'(\xi,\eta) \to \lambda \quad \text{as} \quad \eta \to \infty, \tag{6}$$

where $\lambda = a/b$ is a positive constant and primes denote partial differentiation with respect to η.

The wall shear stress τ_w can be related to the non-dimensional skin friction coefficient C_f according to

$$C_f = \frac{\tau_w}{\rho(bx)^2} = \frac{\nu}{(bx)^2}\left(\frac{\partial u}{\partial y}\right)_{y=0}, \tag{7}$$

where ρ is the density of the fluid. Using variables (4), we obtain

$$C_f Re_x^{1/2} = \xi^{-1/2} f''(\xi,0), \tag{8}$$

where $Re_x = (bx)x/\nu$ is the local Reynolds number.

3. Solution

The governing partial differential equation (5) and the boundary conditions (6) permit separate reductions to ordinary differential systems governing the velocity profiles in the initial unsteady flow at $\xi = 0$ ($\tau = 0$), and the final steady state flow at large times ($\tau \to \infty$) given by $\xi = 1$. The equation for the initial unsteady flow admits a closed form analytical solution, while the equation for the steady state flow cannot be solved explicitly. This case is studied using series solutions for both λ small ($\lambda \ll 1$) and λ large ($\lambda \gg 1$). A numerical solution of Eq. (5) subjected to the boundary conditions (6) is then obtained for $0 \leqslant \xi \leqslant 1$ using the Keller-box method described in the book by Cebeci and Bradshaw [24].

3.1. Initial unsteady solution at $\xi = 0$

The initial solution profile at $\xi = 0$, corresponding to $\tau = 0$, can be obtained by taking $f(0, \eta) = g(\eta)$ in Eq. (5), where $g(\eta)$ satisfies the ordinary differential equation

$$g''' + \frac{1}{2}\eta g'' = 0, \tag{9}$$

subject to

$$g(0) = 0, \quad g'(0) = 1, \quad g' \to \lambda \quad \text{as} \quad \eta \to \infty. \tag{10}$$

The closed form solution of these equations is given by

$$g(\eta) = \lambda\eta + (1 - \lambda)\eta\,\mathrm{erfc}\left(\frac{\eta}{2}\right) + \frac{2}{\sqrt{\pi}}(1 - \lambda)\left(1 - \mathrm{e}^{-\eta^2/4}\right), \tag{11}$$

where erfc$(\cdot)$ is the complementary error function defined as

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty \mathrm{e}^{-s^2}\,\mathrm{d}s. \tag{12}$$

The non-dimensional velocity profiles at $\xi = 0$, given explicitly as

$$g'(\eta) = \lambda + (1 - \lambda)\mathrm{erfc}\left(\frac{\eta}{2}\right) \tag{13}$$

can then be used to obtain

$$g''(0) = -\frac{1 - \lambda}{\sqrt{\pi}}. \tag{14}$$

3.2. Steady state solution at $\xi = 1$

The flow at $\xi = 1$, corresponding to $\tau \to \infty$, is steady and hence $f(1, \eta) = h(\eta)$, where $h(\eta)$ satisfies the following ordinary differential equation

$$h''' + hh'' - h'^2 + \lambda^2 = 0, \tag{15}$$

subject to the boundary conditions

$$h(0) = 0, \quad h'(0) = 1, \quad h'(\eta) \to \lambda \quad \text{as} \quad \eta \to \infty. \tag{16}$$

For $\lambda = 0$, Eqs. (15) and (16) reduce to those found by Crane [4] with the closed form solution

$$h(\eta) = 1 - e^{-\eta}. \tag{17}$$

It is also possible to obtain an approximate solution of Eqs. (15) and (16) for small values of λ ($\ll 1$). In this case, we seek a power series solution of Eq. (15) of the form

$$h(\eta) = h_0(\eta) + h_1(\eta)\lambda + h_2(\eta)\lambda^2 + \text{h.o.t.} \tag{18}$$

Substituting (18) into Eq. (15) and the boundary conditions (16) leads to the following three sets of ordinary differential equations defining the first three functions h_0, h_1 and h_2 namely,

$$h_0''' + h_0 h_0'' - h_0'^2 = 0 \quad h_0(0) = 0, \quad h_0'(0) = 1, \quad h_0'(\infty) = 0; \tag{19}$$

$$h_1''' + h_0 h_1'' - 2h_0' h_1' + h_0'' h_1 = 0 \quad h_1(0) = h_1'(0) = 0, \quad h_1'(\infty) = 0; \tag{20}$$

$$h_2''' + h_0 h_2'' - 2h_0' h_2' + h_0'' h_2 + h_1 h_1'' - h_1'^2 + 1 = 0 \quad h_2(0) = h_2'(0) = 0, \quad h_2'(\infty) = 0. \tag{21}$$

The solution $h_0(\eta)$ of the system (19) is exactly the $\lambda = 0$ solution given by expression (17). Eqs. (20) and (21) are solved numerically, so that we can obtain the skin friction coefficient for the final steady state flow ($\xi = 1$) as

$$C_f Re_x^{1/2} = -1 + 0.1840\lambda + 1.3831\lambda^2 + \text{h.o.t} \tag{22}$$

for $\lambda \ll 1$.

A solution of the system (15) and (16), which is valid for large values of λ ($\gg 1$) can be also obtained by first introducing the function $\bar{h}(\bar{\eta})$ defined according to

$$h(\eta) = \lambda^{1/2}\bar{h}(\bar{\eta}), \quad \bar{\eta} = \lambda^{1/2}\eta. \tag{23}$$

Using this transformation, Eq. (15) becomes

$$\bar{h}''' + \bar{h}\bar{h}'' - \bar{h}'^2 + 1 = 0, \tag{24}$$

subject to the boundary conditions

$$\bar{h}(0) = 0, \quad \bar{h}'(0) = \lambda^{-1}, \quad \bar{h}'(\bar{\eta}) \to 1 \quad \text{as} \quad \bar{\eta} \to \infty, \tag{25}$$

where now primes denote differentiation with respect to $\bar{\eta}$.

We look for a solution of Eq. (24) of the form

$$\bar{h}(\bar{\eta}) = \bar{h}_0(\bar{\eta}) + \bar{h}_1(\bar{\eta})\lambda^{-1} + \bar{h}_2(\bar{\eta})\lambda^{-2} + \text{h.o.t}, \tag{26}$$

where the function $\bar{h}_0$ is given by

$$\bar{h}_0''' + \bar{h}_0\bar{h}_0'' - \bar{h}_0'^2 + 1 = 0 \quad \bar{h}_0(0) = \bar{h}_0'(0) = 0, \quad \bar{h}_0'(\bar{\eta}) \to 1 \quad \text{as} \quad \bar{\eta} \to \infty \tag{27}$$

and these equations describe the steady state flow near the forward stagnation point of a fixed semi-infinite wall first studied by Hiemenz [25]. From (27), we get $h_0''(0) = 1.232627$ (see Lok et al. [26]).

The equations for the functions $\bar{h}_1$ and $\bar{h}_2$ are

$$\bar{h}_1''' + \bar{h}_0\bar{h}_1'' - 2\bar{h}_0'\bar{h}_1' + \bar{h}_0''h_1 = 0 \quad \bar{h}_1(0) = 0, \quad \bar{h}_1'(0) = 1, \quad \bar{h}_1'(\bar{\eta}) \to 0 \quad \text{as} \quad \bar{\eta} \to \infty; \tag{28}$$

$$\bar{h}_2''' + \bar{h}_0\bar{h}_2'' - 2\bar{h}_0'\bar{h}_2' + \bar{h}_0''h_2 + \bar{h}_1\bar{h}_1'' - \bar{h}_1'^2 = 0 \quad \bar{h}_2(0) = \bar{h}_2'(0) = 0, \quad \bar{h}_2'(\bar{\eta}) \to 0 \quad \text{as} \quad \bar{\eta} \to \infty \tag{29}$$

and these equations were solved numerically. Thus, the skin friction coefficient is given by

$$C_f Re_x^{1/2} = \lambda^{3/2}[1.2326 - 0.8236\lambda^{-1} - 0.4915\lambda^{-2} + \text{h.o.t.}] \tag{30}$$

for $\lambda \gg 1$.

3.3. Small ξ and τ time solutions

On the other hand, an approximate solution of Eq. (5) subjected to the boundary conditions (6) which is valid in the region $\xi \ll 1$, equivalent to small time $\tau \ll 1$ solution, can be expressed as

$$f(\xi, \eta) = \bar{f}_0(\eta) + \bar{f}_1(\eta)\xi + \bar{f}_2(\eta)\xi^2 + \text{h.o.t.}, \tag{31}$$

where $\bar{f}_0$ coincides with Eqs. (9) and (10), while $\bar{f}_1$ and $\bar{f}_2$ are given by the following two sets of ordinary differential equations

$$\bar{f}_1''' + \frac{1}{2}\eta\bar{f}_1'' - \bar{f}_1' = \frac{1}{2}\eta\bar{f}_0'' - \bar{f}_0\bar{f}_0'' + \bar{f}_0'^2 - \lambda^2 \quad \bar{f}_1(0) = \bar{f}_1'(0) = 0, \quad \bar{f}_1'(\eta) \to 0 \quad \text{as} \quad \eta \to \infty; \tag{32}$$

$$\bar{f}_2''' + \frac{1}{2}\eta\bar{f}_2'' - 2\bar{f}_2' = \frac{1}{2}\eta\bar{f}_1'' - \bar{f}_1\bar{f}_0'' + 2\bar{f}_0'\bar{f}_1' - \bar{f}_1' \quad \bar{f}_2(0) = \bar{f}_2'(0) = 0, \quad \bar{f}_2'(\eta) \to 0 \quad \text{as} \quad \eta \to \infty. \tag{33}$$

The analytical solution of Eq. (32) is given by

$$f_1'(\eta) = \left[\frac{(1-\lambda)(1+3\lambda)}{2} - \frac{2(1-\lambda)^2}{3\pi}\right]\left[\left(1+\frac{1}{2}\eta^2\right)\mathrm{erfc}(\eta/2) - \frac{1}{\sqrt{\pi}}\eta \mathrm{e}^{-\eta^2/4}\right]$$
$$- \frac{1}{2}(1-\lambda)^2\left(1-\frac{1}{2}\eta^2\right)\mathrm{erfc}^2(\eta/2) - (1-\lambda)\left[2\lambda + \frac{3}{2\sqrt{\pi}}(1-\lambda)\eta \mathrm{e}^{-\eta^2/4}\right]\mathrm{erfc}(\eta/2)$$
$$+ \frac{2}{\pi}(1-\lambda)^2\mathrm{e}^{-\eta^2/2} - \frac{1-\lambda}{\sqrt{\pi}}\left[\frac{1}{2}\left(\frac{1}{2}-\lambda\right)\eta + \frac{4}{3\pi}(1-\lambda)\right]\mathrm{e}^{-\eta^2/4}, \tag{34}$$

where

$$f_1''(0) = \frac{1-\lambda}{\sqrt{\pi}}\left[-\frac{7}{4} + \frac{4(1-\lambda)}{3\pi}\right] \tag{35}$$

and Eq. (33) can be easily solved numerically. Thus, the skin friction coefficient can be expressed as

$$C_f Re_x^{1/2} = \frac{1-\lambda}{\sqrt{\pi}}\left[-\xi^{-1/2} + \left(-\frac{7}{4} + \frac{4(1-\lambda)}{3\pi}\right)\xi^{1/2} + \text{h.o.t.}\right] \tag{36}$$

for $\xi \ll 1$. Using the approximate polynomial relationship

$$\xi = \tau - \frac{1}{2}\tau^2 + \frac{1}{6}\tau^3 + \text{h.o.t.} \tag{37}$$

between ξ and τ, which follows from the definition (4), we can easily obtain the evolution of the skin friction coefficient (36) for the initial unsteady flow in terms of small τ.

4. Results and discussion

Eq. (5) under the boundary conditions (6) is solved numerically for some values of the parameter λ using the Keller-box method described by Cebeci and Bradshaw [24]. In order to validate our results, we have compared the reduced skin friction $f''(\xi, 0)$ when $\xi = 1$ (final steady state flow) with those of Mahapatra and Gupta [11]. The results are found to be in excellent agreement. The comparison is shown in Table 1.

The evolution of the velocity profile $f'(\eta)$ at the initial unsteady flow ($\xi = 0$) is shown in Fig. 2 for some values of λ. It is seen that when $\lambda > 1$, the flow has a boundary layer structure. The thickness of the boundary layer decreases with the increase in λ. According to Mahapatra and Gupta [11], it can be explained as follows: For fixed value of b, corresponding to the stretching of the surface, increase in a in relation to b (such that $\lambda = a/b > 1$) implies increase in straining motion near the stagnation region resulting in increased acceleration of the external stream, and this leads to thinning of the boundary layer with increase in λ. Further, it is seen from Fig. 2 that

Table 1
Values of $f''(\xi, 0)$ for $\xi = 1$ (final steady state flow) and different values of λ

λ	Mahapatra and Gupta [11]	Present		
		Numerical Eq. (15)	Series small λ, Eq. (22)	Series large λ, Eq. (30)
0.01		−0.9980	−0.9980	
0.02		−0.9958	−0.9958	
0.05		−0.9876	−0.9873	
0.10	−0.9694	−0.9694	−0.9678	
0.20	−0.9181	−0.9181	−0.9080	
0.50	−0.6673	−0.6673	−0.5622	
2.00	2.0175	2.0176		1.9740
3.00	4.7293	4.7296		4.6945
5.00		11.7537		11.7197
10.00		36.2687		36.2191
20.00		106.5744		106.4562
50.00		430.6647		429.9061

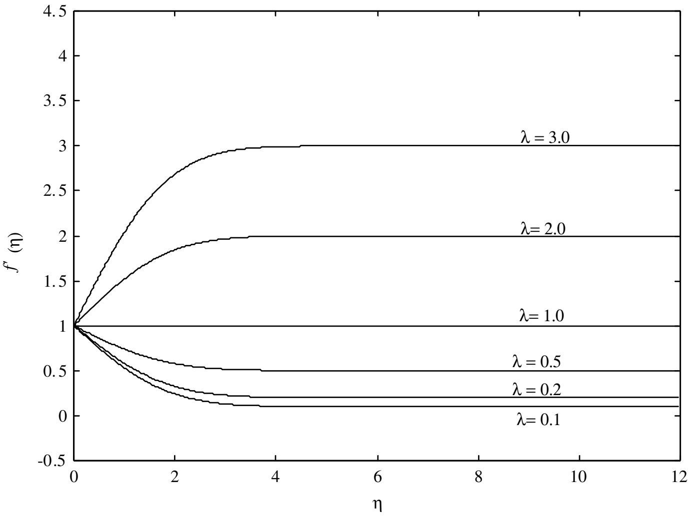

Fig. 2. Velocity profiles in the initial unsteady state flow ($\xi = 0$) for some values of λ.

when $\lambda < 1$, the flow has an inverted boundary layer structure. It results from the fact that when $\lambda < 1$, the stretching velocity bx of the surface exceeds the velocity ax of the external stream.

Fig. 3 illustrates the variation with λ of the reduced skin friction $f''(\xi, 0)$ for the steady state flow ($\xi = 1$) by solving numerically Eqs. (15) and (16), and from the small and large λ series solutions (22) and (30), respectively. It is seen that the three solutions are in excellent agreement.

Finally, Figs. 4 and 5 show the variation of the skin friction coefficient $C_f Re_x^{1/2}$ with ξ (Fig. 4) and with τ (Fig. 5), for some values of λ obtained by solving numerically Eqs. (5) and (6). The steady state solution at $\xi = 1$ given by Eq. (15) is also included in Fig. 4. This figure shows clearly

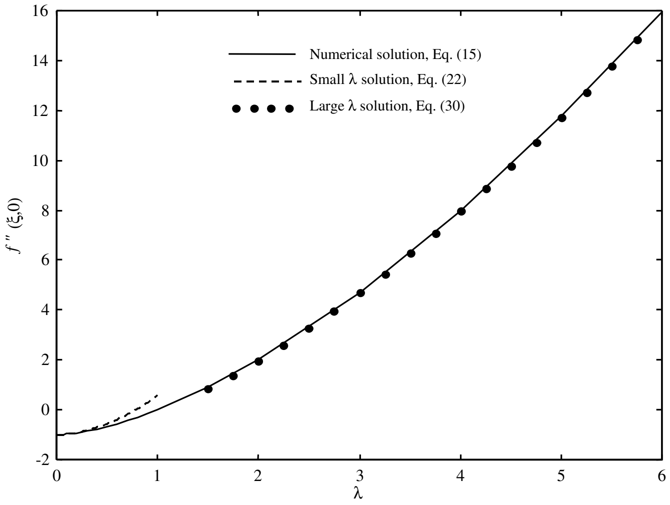

Fig. 3. Variation with λ of the reduced skin friction for the final steady state flow ($\xi = 1$).

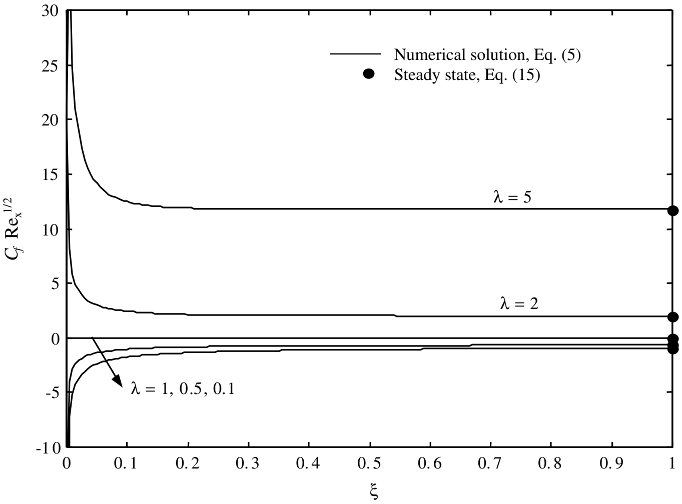

Fig. 4. Variation with ξ of the skin friction coefficient for some values of λ.

that the transition from the initial unsteady state flow to the final steady state flow takes place smoothly and without any singularity. Further, Fig. 5 indicates that the small time τ solution is in a very good agreement with the numerical solution for small values of λ, while for large values of λ this agreement is not good enough. However, the agreement between these two solutions can be improved if more terms in the series (36) are considered.

It is worth mentioning at this place that the present results are believed to be very accurate and consistent, which potentially, make them of importance to future theoretical studies of flow and heat transfer problems due to stretching surfaces.

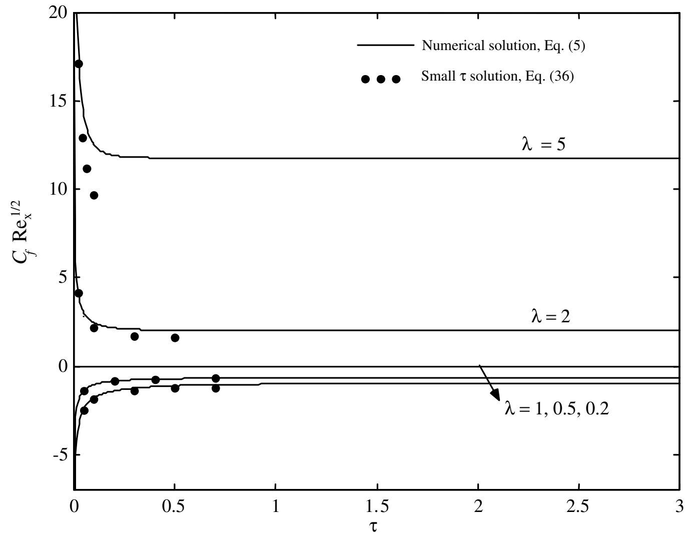

Fig. 5. Variation with τ of the skin friction coefficient for some values of λ.

5. Conclusions

Unsteady boundary layer flow in the stagnation point region on a stretching flat sheet, where the unsteadiness is caused by the impulsive motion of the free stream velocity and by the suddenly stretched surface, has been analyzed in detail. From an analytical investigation of the governing boundary layer equation, we have been able to deduce solutions for the non-dimensional velocity function and the skin friction coefficient in the initial unsteady state flow, the final steady state flow, and at small times. The numerical Keller-box solutions of the governing boundary layer equations were found to agree excellently with the analytical small and large time solutions, thus ensuring the validity of the results presented in this paper. A considerable advantage was found with the use of a transformed, finite time scale in which $\tau = \infty$ corresponds to $\xi = 1$, when the governing parabolic equation can be solved by means of smooth transition from the small time solution to the large time solution.

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