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The nonabelian tensor square of a crystallographic group with quaternion point group of order eight

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Abstract. A crystallographic group is a discrete subgroup of the set of isometries of Euclidean space where the quotient space is compact. A torsion free crystallographic group, or also known as a Bieberbach group has the symmetry structure that will reveal its algebraic properties. One of the algebraic properties is its nonabelian tensor square. The nonabelian tensor square is a special case of the nonabelian tensor product where the product is defined if the two groups act on each other in a compatible way and their action is taken to be conjugation. Meanwhile, Bieberbach group with quaternion point group of order eight is a polycyclic group. In this paper, by using the polycyclic method, the computation of the nonabelian tensor square of this group will be shown.

1. Introduction

A crystallographic group is a description on the symmetrical pattern of a crystal. It is a symmetry group which has configuration in space. When a crystallographic group is torsion free where all elements are of infinite order except its identity then it is called a Bieberbach group. It is an extension of a free abelian group of finite rank by a finite point group. Research on homological invariants has been increasing in number since it is related to the study of the properties of the crystal using mathematical approach. One of homological invariants is the nonabelian tensor square of the group. The nonabelian tensor square is requisite in determining the other properties of the group. The study of the nonabelian tensor square was first introduced by Brown and Loday [1]. He introduced the nonabelian tensor product, $G \otimes H$, for two groups G and H. The nonabelian tensor square is a special case of the nonabelian tensor product in which $G = H$ and the action is conjugation. In 1987, the nonabelian tensor squares of all nonabelian groups of order up to 30 are computed [2]. Kappe *et al.* [3] determined the nonabelian tensor squares of 2-generator 2-groups of class 2 in 1999. Rashid *et al.* [4] computed the nonabelian tensor square of groups of order $8q$ while Zainal *et al.* [5] determined the nonabelian tensor squares of some groups of p-power order. In this research, the crystallographic groups being considered are taken from Crystallographic, Algorithms and Table (CARAT) package [6]. By using the technique developed by Blyth and Morse [7], these groups are transformed from matrix representation to polycyclic presentation before their nonabelian tensor squares are computed. It is crucial to perform the consistency check for those polycyclic presentations so that we can proceed to find the homological invariants of the groups. Recently, Tan *et al.* [8] used this method to find the nonabelian tensor square of crystallographic groups with symmetric point group of order six. Therefore, in this research, the nonabelian tensor square of a crystallographic group with quaternion point group of order eight will be explicated.

1.1. Some Preliminaries

In this section, some basic definitions and preparatory results that are used in the computations of the nonabelian tensor square are presented.

Definition 1 *[9] The nonabelian tensor square of a group G, denoted as $G \otimes G$, is generated by the symbols $g \otimes h$ for all $g, h \in G$ subject to the relations*

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \text{ and } g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$$

for all $g, g', h, h' \in G$, where ${}^{h}g = hgh^{-1}$.

Definition 2 *[10] Let G be a group with presentation $\langle G|R\rangle$ and let G^{φ} be an isomorphic copy of G via the mapping $\varphi : g \to g^{\varphi}$ for all $g \in G$. The group $\nu(G)$ is defined to be*

$$\nu(G) = \langle G, G^{\varphi} | R, R^{\varphi}, {}^{x}[g, h^{\varphi}] = [{}^{x}g, ({}^{x}h)^{\varphi}] = {}^{x^{\varphi}}[g, h^{\varphi}], \forall x, g, h \in G\rangle.$$

Definition 3 *[11] Let F_n be a free group on generators $g_i, \ldots, g_n$ and R be a set of relations of group F_n. The relations of a polycyclic presentation ${}^{F_n}/_{R}$ have the form:*

$$\begin{aligned} g_i^{e_i} &= g_{i+1}^{x_{i,i+1}} \cdots g_n^{x_{i,n}} && \text{for } i \in I, \\ g_j^{-1} g_i g_j &= g_{j+1}^{y_{i,j,j+1}} \cdots g_n^{y_{i,j,n}} && \text{for } j < i, \\ g_j g_i g_j^{-1} &= g_{j+1}^{z_{i,j,j+1}} \cdots g_n^{z_{i,j,n}} && \text{for } j < i \text{ and } j \notin I \end{aligned}$$

for some $I \subseteq \{1, \ldots n\}$, certain exponents $e_i \in \mathbb{N}$ for $i \in I$ and $x_{i,j}, y_{i,j,k}, z_{i,j,k} \in \mathbb{Z}$ for all i, j and k.

Definition 4 *[11] Let G be a group generated by $g_1, \ldots, g_n$ and the consistency relations in G can be evaluated in the polycyclic presentation of G using the collection from the left as in the following:*

$$\begin{aligned} g_k(g_j g_i) &= (g_k g_j) g_i && \text{for } k > j > i, \\ (g_j^{e_j}) g_i &= g_j^{e_j - 1}(g_j g_i) && \text{for } j > i, j \in I, \\ g_j(g_i^{e_i}) &= (g_j g_i) g_i^{e_i - 1} && \text{for } j > i, i \in I, \\ (g_i^{e_i}) g_i &= g_i (g_i^{e_i}) && \text{for } i \in I, \\ g_j &= (g_j g_i^{-1}) g_i && \text{for } j > i, i \notin I \end{aligned}$$

for some $I \subseteq \{1, \ldots, n\}$, $e^i \in \mathbb{N}$. Then, G is said to be given by a consistent polycyclic presentation.

Next, some properties of the group $\nu(G)$ are presented. Besides, the relations of the homological functors of a group G with the subgroups of $\nu(G)$ such as $[G, G^{\varphi}]$ is also shown.

Theorem 1 *[7] If G is polycyclic then $\nu(G)$ is polycyclic.*

Theorem 2 *[10] Let G be a group. The map $\sigma : G \otimes G \to [G, G^{\varphi}] \lhd \nu(G)$ defined by $\sigma(g \otimes h) = [g, h^{\varphi}]$ for all g, h in G is an isomorphism.*

Theorem 3 *[7] Let G be a polycyclic group with a polycyclic generating sequence $g_1,\dots,g_k$. Then $[G,G^\varphi]$, a subgroup of $\nu(G)$, is given by*

$$[G,G^\varphi]=\langle [g_i,g_i^\varphi],[g_i^\delta,(g_j^\varphi)^\varepsilon],[g_i,g_j^\varphi][g_j,g_i^\varphi]\rangle$$

for $1\le i<j\le k$, where

$$\varepsilon=\begin{cases}1 & \textit{if } |\mathfrak{g}_i|<\infty;\\ \pm1 & \textit{if } |\mathfrak{g}_i|=\infty\end{cases}$$

and

$$\delta=\begin{cases}1 & \textit{if } |\mathfrak{g}_j|<\infty;\\ \pm1 & \textit{if } |\mathfrak{g}_j|=\infty.\end{cases}$$

Theorem 4 *[12] Let $Q(6)$ be a Bieberbach group of dimension 6 with quaternion point group of order eight, then its polycyclic presentations is established as:*

$$\begin{aligned}Q(6)=\langle a,b,c,l_1,l_2,l_3,l_4,l_5,l_6 | &a^2=cl_6, b^2=cl_5l_6^{-1}, b^a=bcl_5^{-2}l_6^2, c^2=l_5l_6^{-1},\\ &c^a=cl_5l_6^{-1}, c^b=c, l_1^a=l_4^{-1}, l_1^b=l_3^{-1}, l_1^c=l_1^{-1}, l_2^a=l_3,\\ &l_2^b=l_4^{-1}, l_2^c=l_2^{-1}, l_3^a=l_2^{-1}, l_3^b=l_1, l_3^c=l_3^{-1}, l_4^a=l_1,\\ &l_4^b=l_2, l_4^c=l_4^{-1}, l_5^a=l_6, l_5^b=l_5, l_5^c=l_5, l_6^a=l_5, l_6^b=l_6, l_6^c=l_6,\\ &l_j^{l_i}=l_j, l_j^{l_i^{-1}}=l_j \text{ for } j>i,\ 1\leqslant i,\ j\leqslant 6\rangle.\end{aligned}$$

This presentation is proven to be consistent.

2. Main Results

In this section, the main result of this research which is the nonabelian tensor square of a crystallographic group with quaternion point group of order eight is presented as follows:

Theorem 5 *Let $Q(6)$ be a Bieberbach group of dimension 6 with quaternion point group of order eight, then its nonabelian tensor square is established as:*

$$\begin{aligned}Q(6)\otimes Q(6)=\langle &a\otimes a, b\otimes b, l_1\otimes l_1, a\otimes b, a\otimes c, a\otimes l_1, a\otimes l_2, b\otimes l_1, b\otimes l_2,\\ &(a\otimes b)(b\otimes a),(a\otimes c)(c\otimes a),(a\otimes l_1)(l_1\otimes a),(a\otimes l_2)(l_2\otimes a),\\ &(a\otimes l_6)(l_6\otimes a),(b\otimes l_1)(l_1\otimes b),(b\otimes l_2)(l_2\otimes b)\rangle.\end{aligned}$$

Proof : By Theorem 2, $G\otimes G\cong[G,G^\varphi]$, therefore we can find $G\otimes G$ by finding $[G,G^\varphi]$. By Theorem 3, $[G,G^\varphi]=\langle [g_i,g_i^\varphi],[g_i^\delta,(g_j^\varphi)^\varepsilon],[g_i,g_j^\varphi][g_j,g_i^\varphi]\rangle$.
First, we have,

$$\langle [g_i,g_i^\varphi]\rangle=\{[a,a^\varphi],[b,b^\varphi],[c,c^\varphi],[l_1,l_1^\varphi],[l_2,l_2^\varphi],[l_3,l_3^\varphi],[l_4,l_4^\varphi],[l_5,l_5^\varphi],[l_6,l_6^\varphi]\}.$$

Next, we have,

$$\begin{aligned}\langle [{g_i}^\varepsilon,(g_j^\varphi)^\partial]\rangle=\{&[[a^{\pm1},(b^\varphi)^{\pm1}],[[a^{\pm1},(c^\varphi)^{\pm1}],[[a^{\pm1},(l_1^\varphi)^{\pm1}],[[a^{\pm1},(l_2^\varphi)^{\pm1}],[[a^{\pm1},(l_3^\varphi)^{\pm1}],\\ &[[a^{\pm1},(l_4^\varphi)^{\pm1}],[[a^{\pm1},(l_5^\varphi)^{\pm1}],[[a^{\pm1},(l_6^\varphi)^{\pm1}],[[b^{\pm1},(c^\varphi)^{\pm1}],[[b^{\pm1},(l_1^\varphi)^{\pm1}],\\ &[[b^{\pm1},(l_2^\varphi)^{\pm1}],[[b^{\pm1},(l_3^\varphi)^{\pm1}],[[b^{\pm1},(l_4^\varphi)^{\pm1}],[[b^{\pm1},(l_5^\varphi)^{\pm1}],[[b^{\pm1},(l_6^\varphi)^{\pm1}],\\ &[[c^{\pm1},(l_1^\varphi)^{\pm1}],[[c^{\pm1},(l_2^\varphi)^{\pm1}],[[c^{\pm1},(l_3^\varphi)^{\pm1}],[[c^{\pm1},(l_4^\varphi)^{\pm1}],[[c^{\pm1},(l_5^\varphi)^{\pm1}],\\ &[[c^{\pm1},(l_6^\varphi)^{\pm1}],[[l_1^{\pm1},(l_2^\varphi)^{\pm1}],[[l_1^{\pm1},(l_3^\varphi)^{\pm1}],[[l_1^{\pm1},(l_4^\varphi)^{\pm1}],[[l_1^{\pm1},(l_5^\varphi)^{\pm1}],\\ &[[l_1^{\pm1},(l_6^\varphi)^{\pm1}],[[l_2^{\pm1},(l_3^\varphi)^{\pm1}],[[l_2^{\pm1},(l_4^\varphi)^{\pm1}],[[l_2^{\pm1},(l_5^\varphi)^{\pm1}],[[l_2^{\pm1},(l_6^\varphi)^{\pm1}],\\ &[[l_3^{\pm1},(l_4^\varphi)^{\pm1}],[[l_3^{\pm1},(l_5^\varphi)^{\pm1}],[[l_3^{\pm1},(l_6^\varphi)^{\pm1}],[[l_4^{\pm1},(l_5^\varphi)^{\pm1}],[[l_4^{\pm1},(l_6^\varphi)^{\pm1}],\\ &[[l_5^{\pm1},(l_6^\varphi)^{\pm1}]\}.\end{aligned}$$

Lastly, we have,

$$\begin{aligned}\langle [g_i, g_j^{\varphi}][g_j, g_i^{\varphi}]\rangle = \{&[a, b^{\varphi}][b, a^{\varphi}], [a, c^{\varphi}][c, a^{\varphi}], [a, l_1^{\varphi}][l_1, a^{\varphi}], [a, l_2^{\varphi}][l_2, a^{\varphi}], [a, l_3^{\varphi}][l_3, a^{\varphi}],\\
&[a, l_5^{\varphi}][l_5, a^{\varphi}], [a, l_6^{\varphi}][l_6, a^{\varphi}], [b, c^{\varphi}][c, b^{\varphi}], [b, l_1^{\varphi}][l_1, b^{\varphi}], [b, l_2^{\varphi}][l_2, b^{\varphi}],\\
&[b, l_3^{\varphi}][l_3, b^{\varphi}], [b, l_4^{\varphi}][l_4, b^{\varphi}], [b, l_5^{\varphi}][l_5, b^{\varphi}], [b, l_6^{\varphi}][l_6, b^{\varphi}], [c, l_1^{\varphi}][l_1, c^{\varphi}],\\
&[c, l_2^{\varphi}][l_2, c^{\varphi}], [c, l_3^{\varphi}][l_3, c^{\varphi}], [c, l_4^{\varphi}][l_4, c^{\varphi}], [c, l_5^{\varphi}][l_5, c^{\varphi}], [c, l_6^{\varphi}][l_6, c^{\varphi}],\\
&[l_1, l_2^{\varphi}][l_2, l_1^{\varphi}], [l_1, l_3^{\varphi}][l_3, l_1^{\varphi}], [l_1, l_4^{\varphi}][l_4, l_1^{\varphi}], [l_1, l_5^{\varphi}][l_5, l_1^{\varphi}],\\
&[l_1, l_6^{\varphi}][l_6, l_1^{\varphi}], [l_2, l_3^{\varphi}][l_3, l_2^{\varphi}], [l_2, l_4^{\varphi}][l_4, l_2^{\varphi}], [l_2, l_5^{\varphi}][l_5, l_2^{\varphi}],\\
&[l_2, l_6^{\varphi}][l_6, l_2^{\varphi}], [l_3, l_4^{\varphi}][l_4, l_3^{\varphi}], [l_3, l_5^{\varphi}][l_5, l_3^{\varphi}], [l_3, l_6^{\varphi}][l_6, l_3^{\varphi}],\\
&[l_4, l_5^{\varphi}][l_5, l_4^{\varphi}], [l_4, l_6^{\varphi}][l_6, l_4^{\varphi}], [l_5, l_6^{\varphi}][l_6, l_5^{\varphi}]\}.\end{aligned}$$

Thus, there are 189 elements of $[G, G^{\varphi}]$ based on Theorem 3, but they are not independent. Some of these elements are identities and some of those are products of power of the other elements. Hence, some of them can be eliminated. Some calculations of the elimination are shown in the following:

$$\begin{aligned}\text{i.}[l_3, l_3^{\varphi}] &= [l_3, l_3^{\varphi}]^b\\
&= [l_1, l_1^{\varphi}] \text{ by relations of } Q(6).\end{aligned}$$

$$\begin{aligned}\text{ii.}[l_4, l_1^{\varphi}] &= [l_4, l_1^{\varphi}]^a\\
&= [l_1, l_4^{-1\varphi}] \text{ by relations of } Q(6)\\
&= [l_1, l_4^{\varphi}]^{-1}.\end{aligned}$$

Therefore, $[l_1, l_4^{\varphi}][l_4, l_1^{\varphi}] = 1$.

$$\begin{aligned}\text{iii.}[b, l_4^{\varphi}][l_4, b^{\varphi}] &= ([b, l_4^{\varphi}][l_4, b^{\varphi}])^b\\
&= [b, l_4^{\varphi}]^b[l_4, b^{\varphi}]^b\\
&= [b^b, (l_4^b)^{\varphi}][l_4^b, (b^b)^{\varphi}]\\
&= [b, l_2^{\varphi}][l_2, b^{\varphi}] \text{ by relations of } Q(6).\end{aligned}$$

$$\begin{aligned}\text{iv.}[a, c^{-\varphi}] &= [c^{-1}, [a, c]][a, c^{\varphi}]^{-1}\\
&= [c^{-1}, (l_6^{-1}l_5)^{\varphi}]\\
&= [c^{-1}, l_5^{\varphi}][c^{-1}, l_6^{-\varphi}][[c^{-1}, l_6^{-1}], l_5^{\varphi}][a, c^{\varphi}]^{-1}\\
&= [a, c^{\varphi}]^{-1} \text{ since } c \text{ commute with } l_5 \text{ and } l_6.\end{aligned}$$

$$\begin{aligned}\text{v.}[a, l_6^{\varphi}] &= [a, c^{-1}a^2] \text{ by relations of } Q(6)\\
&= [a, a^{\varphi}][a, (c^{-1}a)^{\varphi}][[a, c^{-1}a], a^{\varphi}]\\
&= [a, a^{\varphi}][a, a^{\varphi}][a, (c^{-1})^{\varphi}][[a, (c^{-1})], a^{\varphi}], [l_6^{-1}l_5, a^{\varphi}]\\
&= [a, a^{\varphi}]^2[a, (c^{-1})^{\varphi}][l_5^{-1}l_6, a^{\varphi}][l_6^{-1}l_5, a^{\varphi}]\\
&= [a, a^{\varphi}]^2[a, (c^{-1})^{\varphi}][l_5^{-1}, a^{\varphi}][[l_5^{-1}, a], l_6^{\varphi}][l_6, a^{\varphi}]\\
&\quad [l_6^{-1}l_5, a^{\varphi}]\\
&= [a, a^{\varphi}]^2[a, (c^{-1})^{\varphi}][l_5^{-1}, a^{\varphi}][[l_5l_6^{-1}, l_6^{\varphi}][l_6, a^{\varphi}]\\
&\quad [l_6^{-1}l_5, a^{\varphi}]\\
&= [a, a^{\varphi}]^2[a, (c^{-1})^{\varphi}][l_5^{-1}, a^{\varphi}][l_5, l_6^{\varphi}][[l_5, l_6], l_6^{-\varphi}]\\
&\quad [l_6^{-1}, l_6^{\varphi}][l_6, a^{\varphi}][l_6^{-1}l_5, a^{\varphi}] \qquad = [a, a^{\varphi}]^2[a, (c^{-1})^{\varphi}][l_5^{-1}, l_6^{\varphi}][l_5^{-1}, l_5^{\varphi}][l_5, a^{\varphi}]^{-1}\\
&\quad [l_5, l_6^{\varphi}][l_6^{-1}, l_6^{\varphi}][l_6, a^{\varphi}][l_6^{-1}l_5, a^{\varphi}]\\
&= [a, a^{\varphi}]^2[a, (c^{-1})^{\varphi}][l_5^{-1}, l_5^{\varphi}]\\
&\quad [l_6^{-1}, l_6^{\varphi}][l_6^{-1}l_5, a^{\varphi}] \text{ by relations of } Q(6)\\
&= \ldots\ldots\\
&= [a, a^{\varphi}]^2[a, c^{\varphi}]^{-1}\end{aligned}$$

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Until at the end the elements can be reduced to 16. Thus,

$$\begin{aligned}[Q(6), Q(6)^{\varphi}] = &\langle [a, a^{\varphi}], [b, b^{\varphi}], [l_1, l_1{}^{\varphi}], [a, b^{\varphi}], [a, c^{\varphi}], [a, l_1{}^{\varphi}], [a, l_2{}^{\varphi}], [b, l_1{}^{\varphi}],\\ &[b, l_2{}^{\varphi}], [a, b^{\varphi}][b, a^{\varphi}], [a, c^{\varphi}][c, a^{\varphi}], [a, l_1{}^{\varphi}][l_1, a^{\varphi}],\\ &[a, l_2{}^{\varphi}][l_2, a^{\varphi}], [a, l_6{}^{\varphi}][l_6, a^{\varphi}], [b, l_1{}^{\varphi}][l_1, b^{\varphi}], [b, l_2{}^{\varphi}][l_2, b^{\varphi}]\rangle.\end{aligned}$$

Therefore, by Theorem 2,

$$Q\,(6) \otimes Q\,(6) \cong [Q(6), (Q(6)^{\varphi}].$$

Hence,

$$\begin{aligned}Q(6) \otimes Q(6) = &\langle a \otimes a, b \otimes b, l_1 \otimes l_1, a \otimes b, a \otimes c, a \otimes l_1, a \otimes l_2, b \otimes l_1, b \otimes l_2,\\ &(a \otimes b)(b \otimes a), (a \otimes c)(c \otimes a), (a \otimes l_1)(l_1 \otimes a), (a \otimes l_2)(l_2 \otimes a),\\ &(a \otimes l_6)(l_6 \otimes a), (b \otimes l_1)(l_1 \otimes b), (b \otimes l_2)(l_2 \otimes b)\rangle.\end{aligned}$$

□

3. Conclusion

In this research, the nonabelian tensor square of a crystallographic group with quaternion point group of order eight, $Q(6) \otimes Q(6)$ is computed.

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References

[1] Brown R and Loday J L 1987 *Topology* **26** pp 311–35
[2] Brown R, Johnson D L and Robertson E F 1987 *J. Algebra* **111** pp 177–202
[3] Kappe L C, Sarmin N H and Visscher M P 1999 *Glasg. Math. J.* **41** pp 417–30
[4] Rashid S, Sarmin N H, Erfanian A, Mohd Ali N M and Zainal R 2013 *Indag. Math.* **24** pp 581–588
[5] Zainal R, Mohd Ali N M, Sarmin N H and Rashid S 2013 *AIP Conf. Proc.* **1522** pp 1039–1044
[6] The CARAT homepage. (http://wwwb.math.rwth-aachen.de/carat/).
[7] Blyth R D and Morse R S 2009 *J. Algebra* **321** pp 2139–2148
[8] Tan Y T, Mohd Idrus N, Masri R, Wan Mohd Fauzi W N F, Sarmin N H and Mat Hassim H I 2016 *Jurnal Teknologi* **78**(1) pp 189–193
[9] Bacon M R and Kappe L C 2003 *Illinois J. Math.* **47** pp 49–62
[10] Rocco N R 1991 *Bol. Soc. Brasil. Mat. (N. S.)* **22**(1) pp 63–79
[11] Eick B and Nickel W 2008 *J. Algebra* **320**(1) pp 927–44
[12] Mohammad S A, Sarmin N H and Mat Hassim H I 2015 *Jurnal Teknologi* **77**(3) pp 151–156