Ranking method for Z-numbers based on centroid-point

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Abstract

Zadeh introduced the concept of Z-number to establish a method for computation with numbers that are not completely reliable, and it can portray the fuzziness and reliability of the information at the same time. Ranking of Z-numbers is an important aspect, especially in decision making. In this paper, a new ranking method for Z-numbers will be proposed. By converting Z-number into a fuzzy number, and then the centroid-point method and decision rules are utilized to rank the obtained fuzzy numbers. A numerical example is provided to demonstrate the validity of the proposed method. However, converting Z-number into a fuzzy number can lead to loss of original Z-information.

Keywords: *Z-number, Fuzzy number, Ranking, Decision making.*

1. Introduction

In 1965, Zadeh introduced the concept of fuzzy sets, to model vague phenomena (Zadeh, 1965). Fuzzy numbers are fuzzy subsets of the set of real numbers satisfying some additional conditions. Fuzzy numbers allow us to model non-probabilistic uncertainties efficiently. When operating with fuzzy numbers, the result relies on the shape of the membership functions of these numbers. Less regular membership functions lead to more complicated calculations. Moreover, fuzzy numbers with a simpler shape of membership functions commonly have more intuitive interpretation.

Imperfect information naturally related to real-world phenomena is often characterized by two main aspects (Aliev *et al*, 2015a). From the one side, values of interest are often not measured but are estimated based on the perception and knowledge of a human being. In such cases, natural language-based estimations are used which can be formally described by fuzzy numbers. From the other side, it should be noted that estimations are subjective, and therefore, are not fully reliable (Zadeh, 2011a).

In (Zadeh, 2011a) Zadeh introduced the concept of Z-number as an adequate formal construct for a description of real-world information. The concept of Z -numbers relates to the issue of the reliability of the information. A Z-number is an ordered pair of fuzzy numbers (A, B). A Z-number is associated with a real-valued uncertain variable X, with the first component, A, playing the role of a fuzzy restriction ($R(X)$) on the values which X can take, written as X is A, where A is a fuzzy number. The second component B is a measure of reliability (certainty) of the first component (Zadeh, 2011a). The concept of a Z-number is intended to establish a method for computation with numbers that are not completely reliable (Zadeh, 2011b).

The concept of ranking fuzzy numbers is crucial in decision-making problems. It enables decision-makers to adequately present their subjective judgment under conditions that are ambiguous, imprecise, vague, and uncertain. However, Z-number is more applicable than fuzzy numbers in the fields of decision making, risk assessment, and others. Decision making under Z-number based information requires a ranking of the Z-numbers method.

Kang *et al.* (2012) proposed an approach of dealing with Z-numbers which normally appear in the areas of system control, regression analysis, decision making, and others. The method is based on converting a Z-number into a fuzzy using the fuzzy expectation of a fuzzy number. Because of it less computational difficulties, many researchers utilized this method to proposed ranking models for Z-number such as in (Bakar & Gegov, 2015), (Glukhoded & Smetanin, 2016), (Mohamad *et al*, 2017). Also, direct computation over discrete or continuous Z-number is utilized as well (Ezadi & Allahviranloo, 2017), (Qiu *et al*, 2018) (Aliev *et al*, 2016b) (Gong *et al*, 2020). Summarized and comparative discussions of these models were presented in (Abdullahi *et al*, 2020). The centroid-point of a plane figure is the arithmetic mean (average) position of all the points in the shape. The definition extends to any object in n-dimensional space.

This paper proposed a new ranking method for Z-numbers, which consists of three stages. The first stage is converting Z-number into a fuzzy number, the second stage is calculating the centroid point of the obtained fuzzy numbers using the formulae proposed in (Wang *et al*, 2006), and the last stage is applying the decision rule introduced in (Abdullah & Jamal, 2010) based on the centroid-point of fuzzy numbers. The remainder of the paper is organized as follows: Section 2 discusses theoretical preliminaries for this study. Section 3 has the stages involved in the proposed ranking method. Section 4 presents a numerical example that shows the feasibility of the proposed method and Section 5 presents a conclusion.

2. Preliminaries

In this section, the basic preliminaries are presented.

Definition 1 (Zadeh, 1965) Fuzzy Sets

Let X be a collection of points (objects), with a generic element of X denoted by x. A fuzzy set F in X is characterized by a membership function $\mu_F : X \longrightarrow [0,1]$, with the value $\mu_F(x)$ representing the grade of membership of x in F.

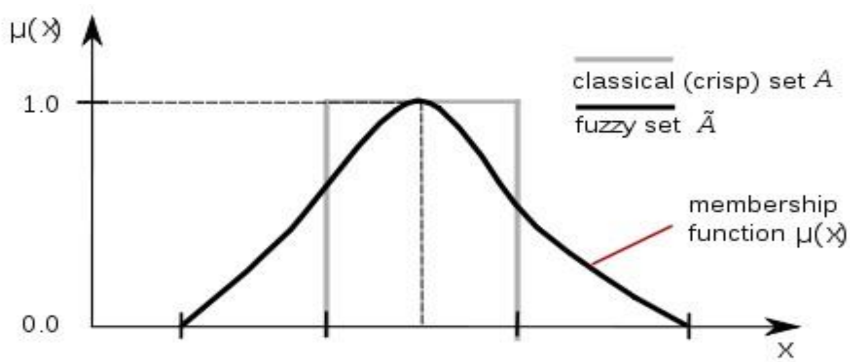

Figure. 1: Membership function of a fuzzy set (Aliev *et al*, 2015b)

Fuzzy sets are always *mappings* from a universe of objects into [0,1]. Conversely, every function $\mu_F : X \longrightarrow [0,1]$ may be considered as a fuzzy set. In general, the range of membership functions may be extended to an arbitrary bounded interval.

Definition 2 (Aliev et al, 2016a) A continuous fuzzy number is a fuzzy subset A of the real line R with membership function $\mu_A : R \longrightarrow [0,1]$ which possesses the following properties:

(1) A is a normal fuzzy set.

(2) A is a convex fuzzy set.

(3) α-cut A^α is a closed interval for every $\alpha \in (0,1]$.

(4) The support of A, supp(A) is bounded

A continuous fuzzy number A with membership function defined as:

$$\mu_A = \begin{cases} (x-a)/(b-a) & if\ a \le x < b \\ 1 & if\ b \le x \le c \\ (d-x)/(d-c) & if\ c < x \le d \\ 0 & otherwise \end{cases} \tag{1}$$

Is referred to as a trapezoidal fuzzy number and is denoted as(a, b, c, d). A special case is a triangular fuzzy number (TFN) A with membership function defined as:

$$\mu_A = \begin{cases} (x-a)/(b-a) & if\ a \le x < b \\ 1 & b = x \\ (c-x)/(c-b) & if\ c < x \le b \\ 0 & otherwise \end{cases} \tag{2}$$

and is denoted as (a, b, c).

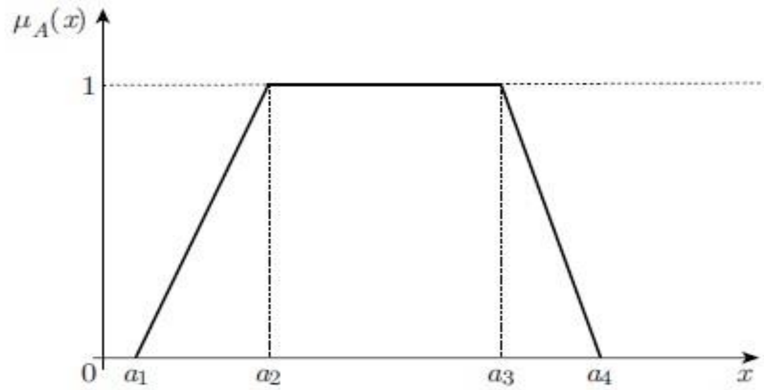

Figure 2. Membership function of a Trapezoidal fuzzy number (Aliev *et al*, 2015b)

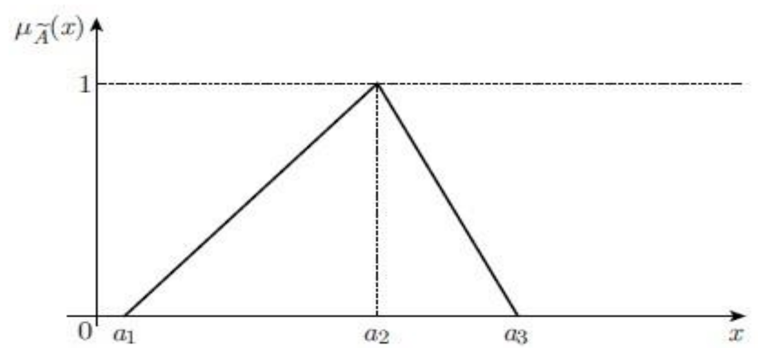

Figure 3. Membership function of a Triangular fuzzy number (Aliev *et al*, 2015b)

Definition 3 (Kang *et al*, 2012) Fuzzy Expectation
Let a fuzzy set A be defined on a universe X, which is given as $A = \{\langle x, \mu_A(x)\rangle | x \in [0,1]\}$, where $\mu_A : X \rightarrow [0,1]$ is the membership function of A. The membership value $\mu_A(x)$represents the degree of belongingness of x $\in$ X in A. The fuzzy expectation of a fuzzy set is denoted as:

$$E_A(x) = \int x\mu_A(x)\,dx \qquad (3)$$

3. Ranking method for Z-numbers

The proposed method consists of three stages, the first stage is converting Z-number into a fuzzy number based on fuzzy expectation (Kang *et al*, 2012). The second stage is calculating the centroid-point of fuzzy numbers using the formulae introduced by Wang *et al.* (2006). The last stage is applying a ranking rule or decision rule which is based on the centroid-point of a fuzzy number, this rule was introduced by (Abdullah and Jamal, 2010).

3.1 Converting Z-number to Fuzzy number

As pointed out earlier a method of converting Z-number into fuzzy number based on fuzzy expectation is introduced, which is presented as: Let $A = \{\langle x, \mu_A(x)\rangle | x \in [0,1]\}$, and $R = \{\langle x, \mu_R(x)\rangle | x \in [0,1]\}$, where A and R are fuzzy numbers. The method consists of two-stage as follows:

1. Convert the second component (reliability) into a crisp number using Eq (4)

$$\alpha = \frac{\int x\mu_R(x)dx}{\int \mu_R(x)dx} \qquad (4)$$

   where the integral sign represents an algebraic integration.

2. Add the weight of the second component (reliability) to the first component (restriction). The weighted Z-number can be represented in as

$$Z^{\alpha} = \{\langle x, \mu_{A^{\alpha}}(x)\rangle | \mu_{A^{\alpha}}(x) = \alpha\mu_{A^{\alpha}}(x), x \in [0,1]\}, \qquad (5)$$

3.2 Calculate the centroid-point of fuzzy number

The centroid-point of a fuzzy number A corresponded to a value x on the horizontal axis and a value y on the vertical axis. The centroid-point $(x(A), y(A))$ of a fuzzy number A was defined as

$$COGP(\tilde{A}) = (\bar{x}_0(\tilde{A}), \bar{y}_0(\tilde{A})) \qquad (6)$$

where

$$\begin{cases} \bar{x}_0(\tilde{A}) = \dfrac{\int_b^b x f_{\tilde{A}}^{-1}(x)dx + \int_b^c x dx + \int_c^d x g_{\tilde{A}}^{-1}(x)dx}{\int_b^b f_{\tilde{A}}^{-1}(x)dx + \int_b^c dx + \int_c^d g_{\tilde{A}}^{-1}(x)dx} \\ \bar{y}_0(\tilde{A}) = \dfrac{\int_0^1 y f_{\tilde{A}}(y)dy + \int_0^1 y g_{\tilde{A}}(y)dy}{\int_0^1 f_{\tilde{A}}(y)dy + \int_0^1 g_{\tilde{A}}(y)dy} \end{cases} \qquad (7)$$

for triangular fuzzy numbers. In the case of trapezoidal fuzzy numbers, the above formula takes the form of:

$$\bar{x}_0(\tilde{A}) = \frac{1}{3}[a + b + c + d - \frac{cd - ab}{(c+d)-(a+b)}] \qquad (8)$$

$$\bar{y}_0(\tilde{A}) = \frac{1}{3}[1 + \frac{c-b}{(c+d)-(a+b)}] \qquad (9)$$

3.3 Apply the Ranking Rule

Based on a centroid point, two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are compared using the following rules (Abdullah and Jamal, 2010):

$$\begin{cases} If\ \bar{x}_0(\tilde{A}) > \bar{x}_0(\tilde{B}),\ Then\ \tilde{A} > \tilde{B}\,. \\ If\ \bar{x}_0(\tilde{A}) < \bar{x}_0(\tilde{B}),\ Then\ \tilde{A} < \tilde{B}\,. \\ If\ \bar{x}_0(\tilde{A}) = \bar{x}_0(\tilde{B}),\ Then \\ If\ \bar{y}_0(\tilde{A}) > \bar{y}_0(\tilde{B}),\ Then\ \tilde{A} > \tilde{B}\,. \\ Else\ If\ \bar{y}_0(\tilde{A}) < \bar{y}_0(\tilde{B}),\ Then\ \tilde{A} < \tilde{B}\,. \\ Else\ \tilde{A} = \tilde{B}. \end{cases} \quad (10)$$

4. Numerical example

This section presents a numerical example, which shows the feasibility of the proposed method. Let two Z-numbers be given as:

$$Z_1 = (A_1, R_1) \qquad (11)$$

and

$$Z_2 = (A_2, R_2) \qquad (12)$$

where $A_1 = (0.9,1,1,1)$, $R_1 = (0.8,0.9,0.9,1)$, and $A_2 = (0.7,0.8,0.9,1)$, $R_2 = (0.8,0.9,0.9,1)$. Even though the reliability parts of the given Z-numbers are equal, but it does not mean that the two Z-numbers are equal. Therefore, we can apply the proposed ranking method to determine which one is greater than the other.

Firstly, the reliability parts of the given Z-numbers are converted into a crisp number. For Z_1 we have:

$$\alpha_1 = \frac{\int x\mu_R(x)dx}{\int \mu_R(x)dx} = 0.9$$

Secondly, add the weight of reliability to the restriction part

$$Z_1^{\alpha_1} = (0.7, 0.8, 0.9, 1, ; 0.9)$$

Finally, the weighted Z-number is converted into regular fuzzy number as

$$Z_1^{\alpha_1} = (\sqrt{0.9} \times 0.9, \sqrt{0.9} \times 1, \sqrt{0.9} \times 1, \sqrt{0.9} \times 1,$$
$$= (0.8,0.9,0.9,0.9).$$

Similarly, for Z_2 which is obtained as:

Firstly, the reliability part of the given Z-number is converted into a crisp number using Eq (4)

$$\alpha_2 = \frac{\int x\mu_R(x)dx}{\int \mu_R(x)dx} = 0.9$$

Secondly, add the weight of reliability to the restriction part

$$Z_2^{\alpha_2} = (0.9,1,1,1, ; 0.9)$$

Finally, the weighted Z-number is converted into regular fuzzy number as

$$Z_2^{\alpha_2} = (\sqrt{0.9} \times 0.7, \sqrt{0.9} \times 0.8, \sqrt{0.9} \times 0.9, \sqrt{0.9} \times 1,$$
$$= (0.66, 0.76, 0.85, 0.95).$$

Therefore, the obtained fuzzy numbers are:

$$Z_1^{\alpha_1} = (0.8,0.9,0.9,0.9)$$

and

$$Z_2^{\alpha_2} = (0.66, 0.76, 0.85, 0.95).$$

Next, the centroid-point of fuzzy numbers is calculated using Eq (8) and (9). For $Z_1^{\alpha_1}$ is obtain as:

$$\bar{x}_0\left(Z_1^{\alpha_1}\right) = \frac{1}{3}[0.8 + 0.9 + 0.9 + 0.9 - \frac{0.9 \times 0.9 - 0.8 \times 0.9}{(0.9 + 0.9) - (0.8 + 0.9)}]$$

$$\bar{x}_0\left(Z_1^{\alpha_1}\right) = 0.8666.$$

$$\bar{y}_0\left(Z_1^{\alpha_1}\right) = \frac{1}{3}[1 + \frac{0.9 - 0.9}{(0.9 + 0.9) - (0.8 + 0.9)}]$$

$$\bar{y}_0\left(Z_1^{\alpha_1}\right) = 0.3333.$$

Similarly, for $Z_2^{\alpha_2}$ which is obtained as:

$$\bar{x}_0\left(Z_2^{\alpha_2}\right) = \frac{1}{3}[0.66 + 0.76 + 0.85 + 0.95 - \frac{0.85 \times 0.95 - 0.66 \times 0.76}{(0.85 + 0.95) - (0.66 + 0.76)}]$$

$$\bar{x}_0\left(Z_2^{\alpha_2}\right) = 0.8050.$$

$$\bar{y}_0\left(Z_2^{\alpha_2}\right) = \frac{1}{3}[1 + \frac{0.85 - 0.76}{(0.85 + 0.95) - (0.66 + 0.76)}]$$

$$\bar{y}_0\left(Z_2^{\alpha_2}\right) = 0.4123.$$

Finally, the decision rule in Eq (10) are used to rank $Z_1^{\alpha_1}$ and $Z_2^{\alpha_2}$ and the result is obtained as follows:

$$\bar{x}_0\left(Z_1^{\alpha_1}\right) > \bar{x}_0\left(Z_2^{\alpha_2}\right).$$

Hence,$Z_1^{\alpha_1} > Z_2^{\alpha_2}$ even though the reliability parts of the original Z-numbers are equal.

5. Conclusion

A ranking model for fuzzy numbers is crucial in decision-making and risk assessment problems. It allows the decision makers to present surjective perceptions under situations that are uncertain or vague. However, Z-number is more suitable than fuzzy numbers in the fields of decision making, and risk assessment, since it can accommodate both fuzziness and reliability at the same time. Decision making under Z-number based information needs a ranking method for Z-numbers. This paper proposed a ranking method for Z-numbers which consists of three stages. The first stage is converting Z-numbers into fuzzy numbers, the second stage is calculating the centroid-point of the obtained fuzzy numbers using the in Eq (8) and (9), and the last stage is applying the decision rules in Eq (10) based on the centroid-point fuzzy numbers. A numerical example was provided to illustrate the validity of the suggested approach. However, it is acknowledged that converting Z-number into a fuzzy number can result in the loss of original information.

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