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On Eigenfunctions of The Integral Operator with Neumann Kernel

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Abstract. In this paper we consider an integral operator **N** acting in the space $L_2[0, 2\pi]$ with a generalized Neumann kernel $N(s, t)$. We find all eigenfunctions of **N** corresponding to the number ± 1. We give some applications to solve Riemann-Hilbert boundary value problems, and construction of some conformal mappings.

INTRODUCTION

The Riemann-Hilbert (RH) problems on simply connected regions [6] and multiply connected regions [7] are reduced to Fredholm integral equations of the second kind with a generalized Neumann kernel. It has been shown that the problem of conformal mapping, Dirichlet problem, Neumann problem and mixed Dirichlet-Neumann problem can all be treated as RH problems and solved efficiently using integral equations with the generalized Neumann kernel [1, 3, 4, 5].

The index κ of the function A, contained in the generalized Neumann kernel plays an important role to solve RH problem. The RH problem is absolutely soluble if $\kappa \leq 0$ and if $\kappa > 0$, RH problem is soluble if and only if the solution satisfies $2\kappa - 1$ conditions of solvability for any arbitrary contour (See Theorem on p. 222 of [2]).

In this paper we consider an integral operator **N** acting in the space $L_2[0, 2\pi]$ with a generalized Neumann kernel $N(s, t) = \mathrm{Re}(\frac{A(s)}{A(t)} \frac{\eta'(t)}{\eta(t) - \eta(s)})$, where $L_2[0, 2\pi]$ is the Hilbert space of 2π periodic real functions, square-integrable over the interval $[0, 2\pi]$, A is the 2π-periodic complex function on $[0, 2\pi]$ that satisfies the Hölder condition and $\eta(t)$ is twice differentiable function with $\dot{\eta}(t) \neq 0$. Note that if $\kappa \leq 0$ ($\kappa > 0$), then the number 1 (-1) is an eigenvalue with multiplicity $n(1) = \max\{0, -2k + 1\}$ ($n(-1) = \max\{0, 2k - 1\}$) [6]. In the paper we find all of eigenfunctions of **N** corresponding to the number ± 1 and using the properties of eigenfunctions we solve Riemann-Hilbert boundary value problems, and give a method of construction of some class of conformal mappings.

STATEMENT OF THE MAIN RESULT

Let Ω be a bounded simply connected region in the complex plane $\mathbb{C}$ whose boundary Γ is a smooth Jordan curve and parameterized by a 2π-periodic complex function $\eta(\cdot)$ that is twice continuously differentiable on $\mathbb{R}$ with $\dot{\eta}(s) \neq 0$ for all $s \in [0, 2\pi]$. Let γ be a Hölder continuous real function on Γ and let A be a complex function that satisfies on Γ the Hölder condition and $A(s) := A(\eta(s))$ be 2π-periodic on $[0, 2\pi]$ with $A \neq 0$. We suppose that the curve Γ has counterclockwise orientation.

Riemann-Hilbert problems: Interior problem: Determine a function f analytic in Ω, continuous on the closure $\overline{\Omega}$, such that the boundary values f^+ satisfy on Γ

$$\mathrm{Re}[A f^+] = \gamma. \tag{1}$$

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Exterior problem: Determine a function f analytic in $\mathbb{C}\setminus\Omega$, continuous on the closure $\overline{\mathbb{C}\setminus\Omega}$ with $f(\infty)=0$, such that the boundary values f^- satisfy on Γ

$$\mathrm{Re}[Af^-]=\gamma.$$

Define the real kernels $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ as real and imaginary parts [6]

$$M(s,t)=\mathrm{Re}\left[\frac{1}{\pi}\frac{A(s)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right],\qquad N(s,t)=\mathrm{Im}\left[\frac{1}{\pi}\frac{A(s)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right].$$

The function $N(\cdot,\cdot)$ is called the generalized Neumann kernel formed with A and η [6]. For $A=1$ it reduces to the well-known Neumann kernel. Note that the function $N(\cdot,\cdot)$ is continuous in $[0,2\pi]\times[0,2\pi]$.

The following lemma was proven in [8] (See Theorem 3.1).

Lemma 1 *(a) The kernel $N(\cdot,\cdot)$ is continuous with*

$$N(t,t)=\frac{1}{\pi}\mathrm{Im}\left[\frac{1}{2}\frac{\ddot{\eta}(t)}{\dot{\eta}(t)}-\frac{\dot{A}(t)}{A(t)}\right].$$

(b) When $s,t\in[0,2\pi]$

$$M(s,t)=-\frac{1}{\pi}\cot\frac{s-t}{2}+M_1(s,t),$$

with a continuous kernel M_1 which take on the diagonal the values

$$M_1(t,t)=\frac{1}{\pi}\left(\frac{1}{2}\mathrm{Re}\left[\frac{\ddot{\eta}(t)}{\dot{\eta}(t)}\right]-\mathrm{Re}\left[\frac{\dot{A}(t)}{A(t)}\right]\right).$$

Define the Fredholm integral operator $\mathbf{N}$ with kernel $N(\cdot,\cdot)$ as [6]

$$(\mathbf{N}\gamma)(s)=\int_0^{2\pi}N(s,t)\gamma(t)dt,$$

and the singular operator $\mathbf{M}$ with kernel $M(\cdot,\cdot)$ as

$$(\mathbf{M}\gamma)(s)=\mathrm{p.v.}\int_0^{2\pi}M(s,t)\gamma(t)dt,$$

where p.v. $\int$ denotes a Cauchy principal value integral.

The index of the function A is defined as the winding number of A with respect to 0

$$\kappa=\mathrm{ind}(A)=\frac{1}{2\pi}\arg A,$$

i.e. the change of the argument of A over one period divided by 2π. If $\kappa\le 0\big(\kappa>0\big)$, then the number $1\big(-1\big)$ is an eigenvalue of N. Note that the multiplicities $n(1)$ and $n(-1)$ of the eigenvalues, respectively, 1 and -1 of N are connected with ind(A) as [6]

$$n(1)=\max\{0,\,-2\kappa+1\},\quad n(-1)=\max\{0,\,2\kappa-1\}. \tag{2}$$

Let $L_2[0,2\pi]$ be the Hilbert space of 2π periodic real functions, square-integrable over the interval $[0,2\pi]$. Furthermore, we denote by $\mathcal{H}_\pm\subset L_2[0,2\pi]$ a subspace of the eigenfunctions of $\mathbf{N}$ corresponding to the eigenvalue ±1. Note that if $\pm\kappa>0$, then dimensional of $\mathcal{H}_\pm$ is $n(\pm1)>0$.

With Hölder continuous real function γ on Γ, we define the function

$$\Phi(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\gamma}{A}\frac{d\zeta}{\zeta-z},\quad z\in\Omega. \tag{3}$$

The main result is the following.

Theorem 1 *Let the number $\pm\kappa > 0$ and $\gamma \in \mathcal{H}_{\pm}$. Then*

$$A\Phi^{\pm}(z) = \pm\gamma, \quad z \in \Gamma. \tag{4}$$

Moreover, in partially Φ^{+} $\left(\Phi^{-}\right)$ is a solution of the interior (exterior) RH problem (1)*, respectively.*

Further we consider a special case of A, i.e. $A(s) = [\eta(s)]^n$, where n is integer number. In this case $\operatorname{ind}(A) = n$ and $N(s,t)$ has the form

$$N(s,t) = \frac{1}{\pi}\operatorname{Im}\left[\frac{[\eta(s)]^n}{[\eta(t)]^n}\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right]. \tag{5}$$

Theorem 2 *(i) Let $A(s) = [\eta(s)]^n$ and n be positive integer number. Then the eigenfunctions of the integral operator* **N** *corresponding to* -1 *are*

$$1, \quad \operatorname{Re}[\eta(s)]^k, \quad \operatorname{Im}[\eta(s)]^k, \qquad k = 1,\dots,n-1. \tag{6}$$

(ii) Let $A(s) = [\eta(s)]^n$ and n be non-positive integer number. Then the eigenfunctions of the integral operator **N** *corresponding to* 1 *are*

$$1, \quad \operatorname{Re}[\eta(s)]^{-k}, \quad \operatorname{Im}[\eta(s)]^{-k}, \qquad k = 1,\dots,-n. \tag{7}$$

PROOF OF THE MAIN RESULTS

For a Hölder continuous function h on Γ, the function Ψ defined by

$$\Psi(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{h(w)}{w-z}dw.$$

The boundary values Ψ^{+} and Ψ^{-} of $\Psi^{\pm}$, respectively, from inside and from outside can be calculated by Sokhotcky formulas

$$\Psi^{\pm}(\zeta) = \pm\frac{1}{2}h(\zeta) + \text{p.v.}\frac{1}{2\pi i}\int_{\Gamma}\frac{h(w)}{w-\zeta}dw$$

for $\zeta \in \Gamma$. Both boundary functions $\Psi^{\pm}(\cdot)$ are Holder continuous on Γ.

We define the function

$$\Phi(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{\gamma + i\mu}{A}\frac{d\zeta}{\zeta - z}, \quad z \in \Omega, \tag{8}$$

where γ and μ are Hölder continuous real functions on Γ.

Application of Sokhotcky formula and straightforward calculations give the following lemma [6]

Lemma 2 *The boundary values of the function Φ defined in (8) can be represented by*

$$2\operatorname{Re}[A(s)\Phi^{\pm}(\eta(s))] = \pm\gamma + \mathbf{N}\gamma + \mathbf{M}\mu, \tag{9}$$

$$2\operatorname{Im}[A(s)\Phi^{\pm}(\eta(s))] = \pm\mu + \mathbf{N}\mu - \mathbf{M}\gamma. \tag{10}$$

Using the properties of Cauchy's integral formula and Sokhotcky formula was proven the following lemma in [6]

Lemma 3 *The operators* **N**, **M** *and identity operator* **I** *are connected by the following relation:*

$$\mathbf{I} = \mathbf{N}^2 - \mathbf{M}^2,$$

$$\mathbf{NM} + \mathbf{MN} = 0.$$

Proof of Theorem 1. Let $\kappa > 0$. Then $n(-1) > 0$ and $n(1) = 0$. Therefore by (2) $\mathcal{H}_- \neq \emptyset$ and the equation

$$(\mathbf{I} - \mathbf{N})f = 0 \tag{11}$$

has only trivial solution. Let $\gamma \in \mathcal{H}_-$. Then

$$(\mathbf{I} + \mathbf{N})\gamma = 0.$$

Multiplying the last equality by $\mathbf{M}$ taking into account Lemma 3 we have

$$\mathbf{M}\gamma - \mathbf{N}\mathbf{M}\gamma = 0.$$

Since the equation (11) has only trivial solution, last equation gives $\mathbf{M}\gamma = 0$. Setting $\mu = 0$ from (9, 10) we have

$$\mathrm{Re}[A(s)\Phi^-(\eta(s))] = -\gamma, \qquad \mathrm{Im}[A(s)\Phi^-(\eta(s))] = 0.$$

Hence we have $A\Phi^-(z) = -\gamma$.

The proof of the equation

$$A\Phi^+(z) = \gamma$$

can be proven similarly.

Theorem 1 is proven.

Proof of Theorem 2. Let $f(w)$ be a Hölder continuous function on Γ and $f(s) := f(\eta(s))$.

(i) Let n be a positive integer number. Then -1 is an eigenvalue of $\mathbf{N}$ and its multiplicity is equal to $n(-1) = 2n - 1$. We consider a function

$$\mathbf{N}f - i\mathbf{M}f = \frac{1}{i}[\mathbf{M} + i\mathbf{N}]f = \text{p.v.}\frac{1}{\pi i}\int_0^{2\pi} \frac{[\eta(s)]^n}{[\eta(t)]^n}\frac{f(t)\dot{\eta}(t)}{\eta(t) - \eta(s)} \tag{12}$$

and present as

$$\mathbf{N}f - i\mathbf{M}f = \text{p.v.}\frac{1}{\pi i}\int_0^{2\pi} \frac{f(t)\dot{\eta}(t)}{\eta(t) - \eta(s)} - \text{p.v.}\frac{1}{\pi i}\int_0^{2\pi} \left(1 - \left[\frac{\eta(s)}{\eta(t)}\right]^n\right)\frac{f(t)}{1 - \frac{\eta(s)}{\eta(t)}}\frac{\dot{\eta}(t)dt}{\eta(t)}$$

or

$$\mathbf{N}f - i\mathbf{M}f = \text{p.v.}\frac{1}{\pi i}\int_0^{2\pi} \frac{f(t)\dot{\eta}(t)}{\eta(t) - \eta(s)} - \frac{1}{\pi i}\sum_{r=0}^{n-1}\int_0^{2\pi}\left[\frac{\eta(s)}{\eta(t)}\right]^r \frac{f(t)\,\dot{\eta}(t)dt}{\eta(t)}. \tag{13}$$

Changing of variables in the integrals under summation in (13) we have

$$\mathbf{N}f - i\mathbf{M}f = \text{p.v.}\frac{1}{\pi i}\int_0^{2\pi} \frac{f(w)dw}{w - \eta(s)} - \frac{1}{\pi i}\sum_{r=0}^{n-1}[\eta(s)]^r \int_\Gamma \frac{f(w)\,dw}{w^{r+1}}. \tag{14}$$

We set

$$f_k(s) = [\eta(s)]^k, \qquad k = 0, 1, \cdots, n-1.$$

Note that the first integral in (14) is a Cauchy principal value integral. Since $f_k(w) = w^k$ is analytic in Ω, by Sokhotcky formula we get

$$\text{p.v.}\frac{1}{\pi i}\int_0^{2\pi} \frac{f_k(t)\dot{\eta}(t)}{\eta(t) - \eta(s)} = [\eta(s)]^k. \tag{15}$$

Taking into account $[\eta(s)]^k = w^k, \quad w \in \Gamma$ by Cauchy integral we have

$$\frac{1}{\pi i}\sum_{r=0}^{n-1}[\eta(s)]^r \int_\Gamma \frac{f_k(w)\,dw}{w^{r+1}} = 2[\eta(s)]^k. \tag{16}$$

Hence, from (14) as $f = f_k$ we get

$$\mathbf{N}f_k - i\mathbf{M}f_k = -f_k, \qquad k = 0, 1, \cdots, n-1. \tag{17}$$

Now multiplying (17) to $\mathbf{N} - i\mathbf{M}$ and using Lemma 3 we have

$$i[\mathbf{NM} + \mathbf{MN}]f_k + [\mathbf{N}^2 - \mathbf{M}^2]f_k = -[\mathbf{N} + i\mathbf{M}]f_k = f_k - i\mathbf{M}f_k$$

or

$$f_k = f_k - i\mathbf{M}f_k.$$

Hence $\mathbf{M}f_k = 0$. Then (17) equals

$$\mathbf{N}\,\mathrm{Re}f_k = -\mathrm{Re}f_k, \quad \mathbf{N}\,\mathrm{Im}f_k = -\mathrm{Im}f_k \qquad k = 0, 1, \cdots, n-1. \tag{18}$$

So, the operator $\mathbf{N}$ has $2n - 1$ linearly indipendent eigenfunctions.

(ii) Let $-n$ be a nonnegative integer number. Then by (2) the number 1 is an eigenvalue with multiplicity $n(1) = 2n + 1$.

The case $n = 0$. In this case

$$\mathbf{N}f - i\mathbf{M}f = \text{p.v.}\frac{1}{\pi i}\int_0^{2\pi} \frac{f(t)\dot{\eta}(t)}{\eta(t) - \eta(s)}dt.$$

We set $g_0(t) = 1$. One can check that

$$\mathbf{N}g_0 - i\mathbf{M}g_0 = g_0. \tag{19}$$

The case $-n > 0$. We represent the equation (12) as

$$\mathbf{N}f - i\mathbf{M}f = -\text{p.v.}\frac{1}{\pi i}\int_0^{2\pi} \frac{f(t)\dot{\eta}(t)}{\eta(t) - \eta(s)} + \text{p.v.}\frac{1}{\pi i}\int_0^{2\pi}\left(1 - \left[\frac{\eta(t)}{\eta(s)}\right]^{-n}\right)\frac{f(t)}{1 - \frac{\eta(t)}{\eta(s)}}\frac{\dot{\eta}(t)dt}{\eta(s)}$$

or

$$\mathbf{N}f - i\mathbf{M}f = -\text{p.v.}\frac{1}{\pi i}\int_0^{2\pi} \frac{f(t)\dot{\eta}(t)}{\eta(t) - \eta(s)} + \frac{1}{\pi i}\sum_{r=0}^{-n}\int_0^{2\pi}\left[\frac{\eta(t)}{\eta(s)}\right]^r\frac{f(t)\,\dot{\eta}(t)dt}{\eta(s)}. \tag{20}$$

Changing of variables in the integrals under summation in (20) we have

$$\mathbf{N}f - i\mathbf{M}f = -\text{p.v.}\frac{1}{\pi i}\int_0^{2\pi} \frac{f(t)\dot{\eta}(t)}{\eta(t) - \eta(s)} + \frac{1}{\pi i}\sum_{r=0}^{-n}\frac{1}{[\eta(s)]^{r+1}}\int_\Gamma w^r f(w)\,dw.$$

We set

$$g_k(s) = \frac{1}{[\eta(s)]^{k+1}}, \qquad k = 0, 1, \cdots, -n-1.$$

Then

$$(\mathbf{N} - i\mathbf{M})g_k = g_k. \tag{21}$$

Using the Lemma 3 and analyzing as above we can get

$$\mathbf{N}\,\mathrm{Re}g_k = \mathrm{Re}g_k, \quad \mathbf{N}\,\mathrm{Im}g_k = \mathrm{Im}g_k \qquad k = 0, 1, \cdots, -n-1.$$

Theorem 2 is proven.

APPLICATIONS

1. **The exterior Riemann-Hilbert problem.** Let $A(s) = e^{ins}$ and C is the circle of the unit disc D. Then by Theorem 2 the number -1 is eigenvalue of the corresponding integral operator $\mathbf{N}$ and corresponding eigenfunctions are

$$1, \cos ks, \sin ks, k = 1, \ldots, n-1.$$

Hence $\mathcal{H}_-$ is a linear space of the functions

$$1, \cos ks, \sin ks, k = 1, \ldots, n-1.$$

Let the functions $\gamma(\cdot)$ and $\mu(\cdot)$ have the form

$$\gamma(t) = \sum_{k=0}^{n-1} a_k \cos kt + \sum_{k=1}^{n-1} b_k \cos kt, \quad \mu(t) = \sum_{k=0}^{n-1} c_k \cos kt + \sum_{k=1}^{n-1} d_k \cos kt,$$

$a_k, b_k, c_k, d_k, k = 1, \ldots, n-1$ are any real numbers. Then $\gamma, \mu \in \mathcal{H}_-$.

Let

$$\Psi(z) = \frac{1}{2\pi i} \int_C \frac{\gamma + i\mu}{A} \frac{d\zeta}{\zeta - z}, \qquad z \in \mathbb{C} \setminus D. \tag{22}$$

Then by the Theorem 1

$$A\psi^-(e^{ins}) = \gamma(s) + i\mu(s).$$

It shows that the function $F(z)$ is a solution of the exterior Riemann-Hilbert problem.

2. **Construction conformal mappings.**

To construct conformal mappings we need the following theorem which was proven in [9], p.37.

Theorem 3 *Suppose G and R are bounded simply connected domains enclosed by piecewise smooth, analytic closed curves Γ and L, respectively, and $w = f(z)$ is a function satisfying satisfying the following conditions:*

1) $w = f(z)$ is analytic inside G and continuous on $\overline{G} = G \cup \Gamma$,

2) $w = f(z)$ maps bijectively onto L.

Then $w = f(z)$ is univalent in G and conformally maps G onto R.

Let $A(s) = e^{-ins}$, $n > 0$. Then by Theorem 2 the number 1 is eigenvalue of the corresponding integral operator $\mathbf{N}$ with multilicity $2n + 1$ and corresponding eigenfunctions are

$$1, \cos ks, \sin ks, k = 1, \ldots, n.$$

Hence $\mathcal{H}_+$ is a linear space of these eigenfunctions.

Let Ω be a simply connected domain enclosed by curve Γ which has the equation

$$\gamma(t) = \frac{\alpha(t) + i\beta(t)}{A(t)},$$

where $\alpha, \beta \in \mathcal{H}_+$ such that $\dot{\gamma}(t) \neq 0$ for all $t \in [0, 2\pi]$. We set

$$F(z) = \frac{1}{2\pi i} \int_C \frac{\alpha + i\beta}{A} \frac{d\zeta}{\zeta - z}, \qquad z \in D.$$

The function $F(\cdot)$ is analytic in the disc D. By Theorem 1

$$F(z)\Big|_{z \in C} = \frac{\alpha + i\beta}{A}.$$

Then the image of $F(\cdot)$ on D is Ω. It follows from here that $F(\cdot)$ is continuous on $\overline{D}$ and is bijective on C. Therefore by Theorem 3 $F : D \to \Omega$ is the conformal mapping on D.

For example, if we choose $A(s) = e^{-is}$ and $\alpha(s) = 1 + a\cos s, \beta(s) = -ib\sin s, a, b > 0$. Then

$$\dot{\gamma}(t) = e^{is}[i(1 + a\cos s + b\sin s) - a\sin s + b\cos s] \neq 0$$

for all $a, b > 0$ and $t \in [0, 2\pi]$. Hence the corresponding function F maps C bijectively onto Γ parameterized by the function γ. Hence the function F is conformal mapping from the disc to the simply connected region Ω with boundary Γ.

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