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Properties of Frequency Weighted Balanced Truncation Techniques

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***Abstract*— In this paper, we derive interesting conditions under which the frequency weighted balanced truncation techniques: Enns' technique, Lin and Chiu's technique, Wang et al's technique as well as Varga and Anderson's technique are equivalent.**

I. INTRODUCTION

The concept of approximating a linear system into a more manageable order has been a constant fascination for many years [1], [8], [9]. Enns [2] in particular has initiated a method for reducing a stable high order model with frequency weightings based on balanced truncation technique [1]. In Enns' method, when using input or output weighting, the reduced order system will yield stable reduced order model. However, when both weightings are present, the stability of the reduced order system is not guaranteed. Lin and Chiu [3] has since proposed a different method to guarantee stability even when both weightings are present under certain assumptions i.e. using strictly proper functions and no occurence of pole-zero cancellations when forming the augmented systems. Wang et al [5] has also solved the stability problem of Enns' for two-sided case by introducing fictitious input and output matrices.

The drawbacks in Lin and Chiu's technique are then rectified by Sreeram [4] and Varga and Anderson [7], where Sreeram et al generalized [3] to include proper weights while Varga and Anderson's technique still guarantees stability even when pole-zero cancellations occur. In addition, Varga and Anderson [7] modified Wang et al's technique by reducing the Gramian's distance to Enns' choice i.e. the sizes of $[P_W - P_E]$ and $[Q_W - Q_E]$ (refer to section II.E).

In this paper, we derive some conditions on the equations of different frequency weighting model reduction techniques for both continuous and discrete-time systems.

II. PRELIMINARIES

This section covers the frequency weighting techniques of Enns', Lin and Chiu's, Wang et al's as well as Varga and Anderson's. Some properties on inner functions are presented which will be utilized in obtaining the main result.

Consider $G(\lambda) = C(\lambda I - A)^{-1} B + D = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ the transfer function of a stable original system where $\lambda = s$ is the Laplace-transform variable in the case of continuous-time system or $\lambda = z$ is the Z-transform variable in the case of discrete-time system, and $\{A, B, C, D\}$ is a minimal realization. Similarly, let

$$W_i(\lambda) = C_v(\lambda I - A_v)^{-1} B_v + D_v = \left[\begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array}\right]$$

$$W_o(\lambda) = C_w(\lambda I - A_w)^{-1} B_w + D_w = \left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array}\right]$$

be the transfer functions of stable input and output weights with the following minimal realizations: $\{A_v, B_v, C_v, D_v\}$ and $\{A_w, B_w, C_w, D_w\}$ respectively. Assuming that there are no pole-zero cancellations between weights and the original system, the minimal realization of the augmented system $G(\lambda)W_i(\lambda)$ and $W_o(\lambda)G(\lambda)$ are given by

$$G(\lambda)W_i(\lambda) = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] = \left[\begin{array}{cc|c} A & BC_v & BD_v \\ 0 & A_v & B_v \\ \hline C & CD_v & DD_v \end{array}\right] \quad (1)$$

$$W_o(\lambda)G(\lambda) = \left[\begin{array}{c|c} \bar{A}_o & \bar{B}_o \\ \hline \bar{C}_o & \bar{D}_o \end{array}\right] = \left[\begin{array}{cc|c} A_w & B_wC & B_wD \\ 0 & A & B \\ \hline C_w & D_wC & D_wD \end{array}\right] \quad (2)$$

Let

$$\bar{P} = \begin{bmatrix} P & P_{12} \\ P_{12}^T & P_v \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} Q_w & Q_{12} \\ Q_{12}^T & Q \end{bmatrix} \quad (3)$$

be the solutions of the following pair of Lyapunov equations for continuous time system (cs) and discrete time system (ds)

$$(cs)\begin{cases} \bar{A}_i\bar{P} + \bar{P}\bar{A}_i^T + \bar{B}_i\bar{B}_i^T = 0 \\ \bar{A}_o^T\bar{Q} + \bar{Q}\bar{A}_o + \bar{C}_o^T\bar{C}_o = 0 \end{cases} \quad (4)$$

$$(ds)\begin{cases} \bar{A}_i\bar{P}\bar{A}_i^T - \bar{P} + \bar{B}_i\bar{B}_i^T = 0 \\ \bar{A}_o^T\bar{Q}\bar{A}_o - \bar{Q} + \bar{C}_o^T\bar{C}_o = 0 \end{cases} \quad (5)$$

Similarly, the minimal realization of the augmented system $W_o(\lambda)G(\lambda)W_i(\lambda)$ is given by

$$\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{ccc|c} A_w & B_wC & B_wDCv & B_wDD_v \\ 0 & A & BC_v & BD_v \\ 0 & 0 & A_v & B_v \\ \hline C_w & D_wC & D_wDD_v & D_wD \end{array}\right] \quad (6)$$

Let

$$\hat{P} = \begin{bmatrix} P_w & P_{12} & P_{13} \\ P_{12}^T & P & P_{23} \\ P_{13}^T & P_{23}^T & P_v \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} Q_w & Q_{12} & Q_{13} \\ Q_{12}^T & Q & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_v \end{bmatrix} \quad (7)$$

be the solutions of the appropriate pair of Lyapunov equations

$$(cs)\begin{cases} \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = 0 \\ \hat{A}^T\hat{Q} + \hat{Q}\hat{A} + \hat{C}^T\hat{C} = 0 \end{cases} \quad (8)$$

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$$(ds)\begin{cases}\hat{A}\hat{P}\hat{A}^T-\hat{P}+\hat{B}\hat{B}^T=0\\ \hat{A}^T\hat{Q}\hat{A}-\hat{Q}+\hat{C}^T\hat{C}=0\end{cases} \quad (9)$$

A. Enns' Technique

Enns' technique [2] is utilized by firstly expanding the (1,1) and (2,2) block of (4) and (5) for controllability and observability Gramian respectively. This will yield the following pair of equations for $-P_E$ and $-Q_E$ respectively.

Continuous-time System

$$\begin{aligned}AP-PA^T &= -BC_vP_{12}-P_{12}^TC_v^TB^T-BD_vD_v^TB^T\\ &\triangleq -P_E \quad (10)\\ A^TQ-QA &= -Q_{12}B_wC-C^TB_w^TQ_{12}^T-C^TD_w^TD_wC\\ &\triangleq -Q_E \quad (11)\end{aligned}$$

Discrete-time System

$$\begin{aligned}APA^T-P &= -BC_vP_{12}^TA^T-AP_{12}C_v^TB^T\\ &\quad -BC_vP_vC_v^TB^T-BD_vD_v^TB^T\\ &\triangleq -P_E \quad (12)\\ A^TQA-Q &= -C^TB_w^TQ_{12}^TA-A^TQ_{12}B_wC\\ &\quad -C^TB_w^TQB_wC-C^TD_w^TD_wC\\ &\triangleq -Q_E \quad (13)\end{aligned}$$

Similar expressions are given for the (2,2) block of (8) and (9). The matrices P and Q in equations (10)-(13) are frequency weighted controllability and observability Gramians respectively. Simultaneously diagonalizing the frequency weighted controllability and observability Gramians yields

$$T^{-1}PT^{-T}=T^TQT=diag(\sigma_1,\sigma_2,\cdots,\sigma_r,\sigma_r+1,\cdots,\sigma_n) \quad (14)$$

where $\sigma_1>\sigma_2>\cdots>\sigma_n>0$. The original system is then transformed using the similarity transformation T and partitioned as shown below:

$$\left[\begin{array}{c|c}T^{-1}AT & T^{-1}B\\ \hline CT & D\end{array}\right]=\left[\begin{array}{cc|c}A_r & A_{12} & B_r\\ A_{21} & A_{22} & B_2\\ \hline C_r & C_2 & D\end{array}\right]$$

and the dimension of A_r is equal to the dimension of $diag(\sigma_1,\sigma_2,\cdots,\sigma_r)$.

Applying Enns' method, the reduced order model G_r is then given by $G_r(\lambda)=\left[\begin{array}{c|c}A_r & B_r\\ \hline C_r & D\end{array}\right]$. Essentially, Enns' method is based on diagonalizing simultaneously the solutions of Lyapunov equations as given in equations (10) and (11) for the continuous case and equations (12) and (13) for the discrete case. However, Enns' method cannot guarantee the stability of reduced order models as P_E and Q_E may be indefinite.

B. Lin and Chiu's Technique

Lin and Chiu's technique [3] differs from Enns' technique as it simultaneously diagonalizes the new Gramians P_{LC} and Q_{LC} instead of diagonalizing P and Q as given below satisfying the one-sided frequency weighting system

$$\begin{aligned}P_{LC}&=P-P_{12}P_v^{-1}P_{12}^T\\ Q_{LC}&=Q-Q_{12}^TQ_w^{-1}Q_{12}\end{aligned} \quad (15)$$

The new Gramians now satisfy the following pair of Lyapunov equations

$$(cs)\begin{cases}AP_{LC}+P_{LC}A^T+B_{LC}B_{LC}^T=0\\ A^TQ_{LC}-Q_{LC}A+C_{LC}^TC_{LC}=0\end{cases}$$

$$(ds)\begin{cases}AP_{LC}A^T-P_{LC}+B_{LC}B_{LC}^T=0\\ A^TQ_{LC}A-Q_{LC}+C_{LC}^TC_{LC}=0\end{cases}$$

where B_{LC} and C_{LC} are given as [4]

$$cs\begin{cases}B_{LC}=BD_v-P_{12}P_v^{-1}B_v\\ C_{LC}=D_wC-C_wQ_v^{-1}Q_{12}\end{cases}$$

$$(ds)\begin{cases}B_{LC}=\begin{bmatrix}\left(AP_{12}P_v^{-1}+BC_v-P_{12}P_v^{-1}A_v\right)P_v^{1/2}\\ BD_v-P_{12}P_v^{-1}B_v\end{bmatrix}^T\\ C_{LC}=\begin{bmatrix}Q_w^{1/2}\left(Q_w^{-1}Q_{12}A+B_wC-A_wQ_w^{-1}Q_{12}\right)\\ D_wC-C_wQ_w^{-1}Q_{12}\end{bmatrix}\end{cases}$$

Assuming that there are no pole-zero cancellations between the weights and the original system, the realization $\{A,B_{LC},C_{LC}\}$ is minimal and Lin and Chiu's technique yields stable models for two-sided frequency weighting system.

C. Varga and Anderson's modification on Lin and Chiu's Technique

In controller reduction applications, since the weights are of the form $(I+G(\lambda)K(\lambda))^{-1}$ and $(I+G(\lambda)K(\lambda))^{-1}G(\lambda)$ where K is the controller for the plant $G(\lambda)$, Lin and Chiu's requirement of no pole/zero cancellation between the weights and the controller will not be satisfied.

To overcome this drawback, Varga and Anderson [7] proposed on diagonalizing simultaneously the Gramians $P_{V_{LC}}$ and $Q_{V_{LC}}$ as shown below: $T^TQ_{V_{LC}}T=T^{-1}P_{V_{LC}}T^{-T}=diag(\sigma_1,\sigma_2,\ldots,\sigma_n)$

$$\begin{aligned}P_{V_{LC}}&=P-\alpha_c^2P_{12}P_v^{-1}P_{12}^T \quad (16)\\ Q_{V_{LC}}&=Q-\alpha_o^2Q_{12}^TQ_w^{-1}Q_{12} \quad (17)\end{aligned}$$

and $0\le\alpha_c\le1$, $0\le\alpha_o\le1$ where $\sigma_i\ge\sigma_{i+1}$, $i=1,2,\cdots,n-1$ and $\sigma_r\ge\sigma_{r+1}$. Reduced order models are then obtained by transforming and partitioning the original system. When $\alpha_c=\alpha_o=0$, it can be seen that this method is equal to Enns' technique. When $\alpha_c=\alpha_o=1$, this method is equal to Lin and Chiu's technique with guaranteed stability. Since the stability is guaranteed, the same is expected to be true in sub-unitary neighborhood of $\alpha_c=1$ and $\alpha_o=1$ even though pole-zero cancellations occur.

D. Wang et al's Technique

Stability of model are achieved in Wang et al's technique [5] by making the matrices P_E and Q_E positive (semi) definite. In this technique, new controllability (P_W) and observability (Q_W) Gramians are diagonalized as obtained from the solution of the following pair of Lyapunov equations:

$$(cs)\begin{cases}AP_W+P_WA^T+B_WB_W^T=0\\ A^TQ_W+Q_WA+C_W^TC_W=0\end{cases} \quad (18)$$

$$(ds)\begin{cases}AP_WA^T-P_W+B_WB_W^T=0\\ A^TQ_WA-Q_W+C_W^TC_W=0\end{cases} \quad (19)$$

The matrices B_W and C_W in the above Lyapunov equations are fictitious input and output matrices which are determined from $B_W = U|S_W|^{1/2}$ and $C_W = |R_W|^{1/2} V^T$ where U, S_W, R_W and V^T are obtained from the singular value decomposition of matrices, $P_E = US_W U^T$ and $Q_E = VR_W V^T$. Since

$$P_E \leq B_W B_W^T \geq 0, Q_E \leq C_W^T C_W \geq 0 \qquad (20)$$

and $\{A, B_W, C_W\}$ is minimal, stability of the reduced order model in case of two-sided frequency weighting is guaranteed.

E. Varga and Anderson's modification on Wang et al's Technique

Varga and Anderson's [7] modification to Wang et al's [5] technique is aimed at reducing the Gramian's distance to Enns choice i.e. sizes of $[P_W - P_E]$ and $[Q_W - Q_E]$. This is done by simultaneously diagonalizing the Gramians P_{V_W} and Q_{V_W} as shown below:

$$T^T Q_{V_W} T = T^{-1} P_{V_W} T^{-T} = diag(\sigma_1, \sigma_2, \ldots, \sigma_n) \qquad (21)$$

where the pair of Lyapunov equations are given as

$$(cs)\begin{cases} AP_{V_W} + P_{V_W}A^T + B_{V_W}B_{V_W}^T = 0 \\ A^T Q_{V_W} + Q_{V_W}A + C_{V_W}^T C_{V_W} = 0 \end{cases} \qquad (22)$$

$$(ds)\begin{cases} AP_{V_W}A^T - P_{V_W} + B_{V_W}B_{V_W}^T = 0 \\ A^T Q_{V_W}A - Q_{V_W} + C_{V_W}^T C_{V_W} = 0 \end{cases} \qquad (23)$$

and $\sigma_i \geq \sigma_{i+1}$, $i = 1, 2, \cdots, n-1$ and $\sigma_r > \sigma_{r+1}$. The new pseudo input and output matrices B_{V_W} and C_{V_W} are defined as $B_{V_W} = U_{V_{W1}} S_{V_{W1}}^{1/2}$ and $C_{V_W} = R_{V_{W1}}^{1/2} V_{V_{W1}}^T$ respectively and $U_{V_{W1}}$, $S_{V_{W1}}$, $R_{V_{W1}}$ and $V_{V_{W1}}$ are obtained from the orthogonal eigen decomposition of symmetric matrices

$$P_E = \begin{bmatrix} U_{V_{W1}} & U_{V_{W2}} \end{bmatrix} \begin{bmatrix} S_{V_{W1}} & 0 \\ 0 & S_{V_{W2}} \end{bmatrix} \begin{bmatrix} U_{V_{W1}}^T \\ U_{V_{W2}}^T \end{bmatrix}$$

$$Q_E = \begin{bmatrix} V_{V_{W1}} & V_{V_{W2}} \end{bmatrix} \begin{bmatrix} R_{V_{W1}} & 0 \\ 0 & R_{V_{W2}} \end{bmatrix} \begin{bmatrix} V_{V_{W1}}^T \\ V_{V_{W2}}^T \end{bmatrix}$$

where $\begin{bmatrix} S_{V_{W1}} & 0 \\ 0 & S_{V_{W2}} \end{bmatrix} = diag\{s_1, s_2, \cdots, s_n\}$, $\begin{bmatrix} R_{V_{W1}} & 0 \\ 0 & R_{V_{W2}} \end{bmatrix} = diag\{r_1, r_2, \cdots, r_n\}$ and $S_{V_{W1}} > 0$, $S_{V_{W2}} \leq 0$, $R_{V_{W1}} > 0$ and $R_{V_{W2}} \leq 0$. Reduced order model is then obtained by transforming and partitioning the original system. Since

$$P_E \leq B_{V_W}B_{V_W}^T \leq B_W B_W^T \geq 0$$
$$Q_E \leq C_{V_W}^T C_{V_W} \leq C_W^T C_W \geq 0$$

and $\{A, B_{V_W}, C_{V_W}\}$ is minimal, stability of the reduced order model for two-sided frequency weighting is guaranteed.

F. Inner Functions

Inner functions have norm preserving properties and are used extensively in H_∞ control design [8] and model order reduction [8]. A transfer matrix $N(\lambda)$ is called *inner* if $N(\lambda) \in \Re H_\infty$, stable and $N^\sim(\lambda)N(\lambda) = I$. While it is *co-inner* if $N(\lambda) \in \Re H_\infty$ and $N(\lambda)N^\sim(\lambda) = I$. Note that $N(\lambda)$ need not be square and if $N(\lambda) = \{A_N, B_N, C_N, D_N\}$ then $N^\sim(\lambda) = \{-A_N^T, C_N^T, -B_N^T, D_N^T\}$. A transfer function $N(\lambda) \in \Re L_\infty$ is called *all-pass* if $N(\lambda)$ is square i.e. a square *inner* function is all-pass.

Let X_c, Y_c, X_d, and Y_d satisfy following Lyapunov equations:

$$(cs)\begin{cases} A_N X_c + X_c A_N^T + B_N B_N^T = 0 \\ A_N^T Y_c + Y_c A_N + C_N^T C_N = 0 \end{cases} \qquad (24)$$

$$(ds)\begin{cases} A_N X_d A_N^T - X_d + B_N B_N^T = 0 \\ A_N^T Y_d A_N - Y - d + C_N^T C_N = 0 \end{cases} \qquad (25)$$

The following lemmas are presented for both continuous and discrete-time system.

Continuous-time System

Lemma 1: [8], A stable transfer function, $N(\lambda)$ with minimal realization A_N, B_N, C_N, D_N and observability Gramian $Y_c = Y_c^T > 0$ is *inner* if and only if

$$D_N^T C_N + B_N^T Y_c = 0$$
$$D_N^T D_N = I$$

Lemma 2: [8], A stable transfer function $N(\lambda)$ with minimal realization A_N, B_N, C_N, D_N and controllability Gramian $X_c = X_c^T > 0$ is *co-inner* if and only if

$$D_N B_N^T + C_N X_c = 0$$
$$D_N D_N^T = I$$

Discrete-time System

Lemma 3: [8], A stable transfer function, $N(\lambda)$ with minimal realization A_N, B_N, C_N, D_N and observability Gramian $Y_d = Y_d^T > 0$ is *inner* if and only if

$$C_N^T D_N + A_N^T Y_d B_N = 0$$
$$D_N^T D_N + B_N^T Y_d B_N = I$$

Lemma 4: [8], A stable transfer function, $N(\lambda)$ with minimal realization A_N, B_N, C_N, D_N and controllability Gramian $X_d = X_d^T > 0$ is *co-inner* if and only if

$$D_N B_N^T + C_N X_d A_N^T = 0$$
$$D_N D_N^T + C_N X_d C_N^T = I$$

where D_N in Lemma 3 and Lemma 4 is assumed to be nonsingular.

III. MAIN RESULT

The main result that contributes to the foundation of this paper is presented in this section. This section explores the properties of the frequency weighted techniques presented above when using some special input and output weightings. This special functions are *inner* and *co-inner* functions as mentioned in the previous section.

Theorem 3.1:

1) If the input weight $W_i(z)$ is a *co-inner* function, then $P_{12} = 0$, which yields a block diagonalized $\bar{P}$ that satisfies the Lyapunov equations in (5) for controllability Gramian $\bar{P} = \begin{bmatrix} P & 0 \\ 0 & P_v \end{bmatrix}$.

2) If the output weight $W_o(z)$ is an *inner* function, then $Q_{12}=0$, which yields a block diagonalized $\bar{Q}$ that satisfies the Lyapunov equations in (5) for the observability Gramian. $\bar{Q}=\begin{bmatrix} Q_w & 0 \\ 0 & Q \end{bmatrix}$.

Proof: First of all, consider the Lyapunov equation of (5) for the controllability Gramian. Consider the expansion of the (1,2) block of this equation which is shown below

$$AP_{12}A_v^T - P_{12} = -BC_vP_vA_v^T + BD_vB_v^T \quad (26)$$

Then, expand the (2,2) block of the same Lyapunov equation to obtain

$$A_vP_vA_v^T - P_v + B_vB_v^T = 0 \quad (27)$$

When Lemma 4 is applied to (27), it is apparent that X_d equals to P_v. It can be clearly seen that $C_v = -D_vB_v^TA_v^{-T}P_v^{-1}$, hence making the RHS of (26) to simplify to zero and gives $P_{12}=0$.

Similarly, we can prove the second part of Theorem 3.1 using Lemma 3, (5) for the observability Gramian. The equivalent continuous time system can be seen in [6].

Theorem 3.2:

1) If the input weight $W_i(\lambda)$ is a *co-inner* , then $P_{13}=0$ and $P_{23}=0$, which yield an almost diagonalized $\hat{P}$ that satisfies the Lyapunov equations in (8) and (9) for controllability Gramian, $\hat{P}=\begin{bmatrix} P_w & P_{12} & 0 \\ P_{12}^T & P & 0 \\ 0 & 0 & P_v \end{bmatrix}$.
2) If the output weight $W_o(\lambda)$ is an *inner* , then $Q_{12}=0$ and $Q_{13}=0$, which yield an almost diagonalized $\hat{Q}$ that satisfies (8) and (9) for observability Gramian, $\hat{Q}=\begin{bmatrix} Q_w & 0 & 0 \\ 0 & Q & Q_{23} \\ 0 & Q_{23}^T & Q_v \end{bmatrix}$.

Proof: Proof of Theorem 3.2 is divided into two parts i.e. for the continuous-time system and the discrete-time system.

Continuous-Time System

First, consider the expansion of (2,3) and (1,3) block of (8) respectively as given below:

$$AP_{23} + P_{23}A_v^T = -BC_vP_v - BD_vB_v^T \quad (28)$$

$$A_wP_{13} + P_{13}A_v^T = -B_wCP_{23} - B_wDC_vP_v - B_wDD_vB_v^T \quad (29)$$

Then, the expansion of (3,3) block of (8) is similar to (27). When Lemma 2 is applied to (27), it is apparent that X_c equals to P_v. It can be clearly seen that $C_v = -D_vB_v^TP_v^{-1}$, hence giving the RHS of (28) to simplify to zero i.e. $P_{23}=0$. Substituting these into (29) will yield $P_{13}=0$ i.e. the RHS of (29) will also simplify to zero.

Similarly, we can prove the second part of Theorem 3.2 using Lemma 1 and (8) for observability Gramian.

Discrete-Time System

First, consider the expansion (2,3) and (1,3) block of (9) for controllability Gramian respectively as given below :

$$AP_{23}A_v^T - P_{23} = -BC_vP_vA_v^T - BD_vB_v^T \quad (30)$$

$$A_wP_{13}A_v^T - P_{13} = -B_wCP_{23}A_v^T - B_wDC_vP_vA_v^T - B_wDD_vB_v^T \quad (31)$$

Similar to the continuous case, the (3,3) block expansion of (9) will yield equivalent result to (27). When Lemma 4 is applied to (30), it is apparent that X_d equals to P_v. It can be clearly seen that $C_v = -D_vB_v^TA_v^{-T}P_v^{-1}$, hence giving the RHS of the equation to simplify to zero i.e. $P_{23}=0$. Substituting these into (29) will yield $P_{13}=0$ and simplifies the equation to zero.

Similarly, we can prove the second part of Theorem 3.2 using Lemma 3 and (9) for observability Gramian.

Lemma 5: The frequency weighted Gramians diagonalized in the frequency weighted balanced truncation techniques satisfies the following relations:

$$P_{LC} \le P_{V_{LC}} \le P \le P_{V_W} \le P_W$$
$$Q_{LC} \le Q_{V_{LC}} \le Q \le Q_{V_W} \le Q_W$$

The proof follows immediately from *Lemma 5* of [4] and Theorem 3.1 of [5].

Remark 1: When $P_E \ge 0$ and $Q_E \ge 0$, $P = P_W = P_{V_W}$ and $Q = Q_W = Q_{V_W}$. However when P_E and Q_E are indefinite, $P < P_{V_W} < P_W$ and $Q < Q_{V_W} < Q_W$.

Theorem 3.3:

1) In the augmented system $G(\lambda)W_i(\lambda)$ and $W_o(\lambda)G(\lambda)$ cases, if the input weight $W_i(\lambda)$ is a *co-inner* function that satisfies $W_i^{\sim}(\lambda)W_i(\lambda) = I$ and the output weight $W_o(\lambda)$ is an *inner* function that satisfies $W_o(\lambda)W_o^{\sim}(\lambda) = I$ (or *co-inner* and *inner* respectively) then

$$P_{LC} = P = P_W = P_{V_{LC}} = P_{V_W} = P_{un}$$
$$Q_{LC} = Q = Q_W = Q_{V_{LC}} = Q_{V_W} = Q_{un}$$

where P_{un} and Q_{un} are the unweighted controllability and observability Gramians of the original system satisfying following Lyapunov equations respectively:

$$(cs)\begin{cases} AP_{uw} + P_{uw}A^T + BB^T = 0 \\ A^TQ_{uw} + Q_{uw}A + C^TC = 0 \end{cases} \quad (32)$$

$$(ds)\begin{cases} AP_{uw}A^T - P_{uw} + BB^T = 0 \\ A^TQ_{uw}A - Q_{uw} + C^TC = 0 \end{cases} \quad (33)$$

2) Similar result is obtained in case of the augmented system $W_o(\lambda)G(\lambda)W_i(\lambda)$.

Proof: Since the proofs for continuous and discrete systems are similar, the proof is given only for the discrete case.

1) For Lin and Chiu's as well as its modification by Varga and Anderson, applying Theorem 3.3, it is proven that the values of P_{12} and Q_{12} equals to zero. Substituting this into (15) will yield

$$P_{LC} = P - P_{12}P_v^{-1}P_{12}^T = P$$
$$Q_{LC} = Q - Q_{12}^TQ_w^{-1}Q_{12} = Q$$

and substituting P_{12} and Q_{12} equals to zero into equations (16) and (17) yields

$$P_{V_{LC}} = P - \alpha_c^2P_{12}P_v^{-1}P_{12}^T = P$$
$$Q_{V_{LC}} = Q - \alpha_o^2Q_{12}^TQ_w^{-1}Q_{12} = Q$$

Furthermore, substitution of P_{12} and Q_{12} for the one-sided frequency weighting system as well as $D_v D_v^T = I - C_v P_v C_v^T$ and $D_w^T D_w = I - B_w^T Q_w B_w$ from Lemma 4 and Lemma 3 respectively into (12) and (13) yields

$$APA^T - P + BB^T = 0, \quad A^T QA - Q + C^T C = 0$$

which are exactly the same as the original unweighted Lyapunov equation. Hence giving $P = P_{uw}$ and $Q = Q_{uw}$. The same result is obtained when substituting $P_{12} = 0$ and $Q_{12} = 0$ into (18), (19), (22) and (23) i.e.

$$P_W = P_{V_W} = P = P_{uw}, \quad Q_W = Q_{V_W} = Q = Q_{uw}$$

2) The proof is similar to the proof of part 1) above, and hence omitted here.

Remark 2 : For both one-sided and two sided frequency weighting cases, When the input weight is *inner* and the output weight is *co-inner* the structure of the Gramians obtained is the same as when the input and output weights are *co-inner* and *inner* respectively.

Remark 3 : If the input weight, $W_i(\lambda)$ is *co-inner* and the output weight, W_o is *inner* , then Enns' technique [2], Lin and Chiu's technique [4], Wang et al's technique [5] and Varga and Anderson's technique [7] are all equivalent to the unweighted balanced truncation technique [1].

IV. NUMERICAL EXAMPLE

A. Continuous-Time system

Consider

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} B = \begin{bmatrix} 0 & 5 \\ 1/2 & -3/2 \\ 1 & -5 \\ -1/2 & 1/6 \end{bmatrix} C^T = \begin{bmatrix} 1 & 4/15 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

in [4] for continuous-time system where the input $\{A_v, B_v, C_v, D_v\}$ and output $\{A_w, B_w, C_w, D_w\}$ weights are *co-inner* and *inner* functions respectively as given below

$$\begin{aligned} A_v &= \begin{bmatrix} -4.1 & 0 \\ 0 & -4.5 \end{bmatrix} B_v = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ C_v &= \begin{bmatrix} -2.7333 & 0 \\ 0 & -3 \end{bmatrix} \quad D_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ A_w &= \begin{bmatrix} -4.1 & 0 \\ 0 & -4.5 \end{bmatrix} B_w = \begin{bmatrix} -5.4667 & 0 \\ 0 & -6 \end{bmatrix} \\ C_w &= \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} \quad D_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

The balanced augmented system are given in (34) and (35). Clearly, they satisfy Theorem 3.2 for both the controllability $\hat{P}$ and observability $\hat{Q}$ Gramians.

Frequency weighted Gramians for this example obtained using Enns', Lin and Chiu's, Wang et al's as well as Varga and Anderson's, are the same as the unweighted Gramians of the original system. This satisfies Theorem 3.3 (See the (2,2) blocks of (34) and (35)).

B. Discrete-Time system

Consider $K(s) = \frac{z^3}{z^4 + 1.1z^3 - 0.01z^2 - 0.275z - 0.06}$ from [4] for discrete-time system. The input $\{A_v, B_v, C_v, D_v\}$ and output $\{A_w, B_w, C_w, D_w\}$ weights are *co-inner* and *inner* function respectively as given below:

$$\begin{aligned} A_v &= \begin{bmatrix} -0.0329 & 0.9976 \\ -0.6995 & -0.0671 \end{bmatrix} B_v = \begin{bmatrix} 0.0617 \\ 0.7115 \end{bmatrix} \\ C_v &= \begin{bmatrix} 0.7139 & -0.0197 \end{bmatrix} \quad D_v = \begin{bmatrix} 0.7000 \end{bmatrix} \\ A_w &= \begin{bmatrix} -0.1208 & 0.7163 \\ -0.9807 & 0.0208 \end{bmatrix} B_w = \begin{bmatrix} -0.6872 \\ 0.1942 \end{bmatrix} \\ C_w &= \begin{bmatrix} 0.1534 & 0.6975 \end{bmatrix} \quad D_w = \begin{bmatrix} 0.7000 \end{bmatrix} \end{aligned}$$

The balanced augmented system obtained in (36) and (37) satisfies Theorem 3.1 while (38) and (39) satisfies Theorem 3.2 for both the controllability ($\bar{P}$, $\hat{P}$) and observability ($\bar{Q}$, $\hat{Q}$) Gramian respectively.

$$\bar{P} = \left[\begin{array}{cccc|cc} 5.6044 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.6695 & 0.0000 & 0.0000 & -0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.1071 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0048 & 0.0000 & -0.0000 \\ \hline 0.0000 & -0.0000 & 0.0000 & 0.0000 & 1.0000 & -0.0000 \\ -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 1.0000 \end{array}\right] \quad (36)$$

$$\bar{Q} = \left[\begin{array}{cc|cccc} 1.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -0.0000 & 1.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ \hline -0.0000 & -0.0000 & 5.6044 & -0.0000 & 0.0000 & 0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 0.6695 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.1071 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0048 \end{array}\right] \quad (37)$$

Frequency weighted Gramians for this example obtained using Enns', Lin and Chiu's, Wang et al's as well as Varga and Anderson's, are the same as the unweighted Gramians of the original system. This satisfies Theorem 3.3 (See the (2,2) blocks of (38) and (39)).

V. CONCLUSION

For one-sided frequency weighting case, when the input weight is a *co-inner* function and the output weight is an *inner* function, or vice versa, the commonly referred frequency weighted balanced truncation techniques, i.e. Enns', Lin and Chiu's, Wang et al's as well as Varga and Anderson's will give a diagonalized controllability and observability Gramians. These Gramians are equal to the Gramians of unweighted balanced truncation technique [1]. This is applicable to both continuous and discrete-time systems.

When *inner* and *co-inner* function are used in the two-sided frequency weighting case, the commonly referred frequency weighted balanced truncation techniques will yield an almost diagonalized controllability and observability Gramians. The (2,2) block of these Gramians are equal to the Gramians of unweighted balanced truncation technique [1] for both the continuous and discrete-time systems.

$$\hat{P} = \left[\begin{array}{cc|cccc|cc} 1.9111 & 0.3235 & 1.6022 & -0.3850 & 0.0281 & 0.0352 & -0.0000 & -0.0000 \\ 0.3235 & 0.0575 & 0.2533 & -0.0692 & 0.0129 & 0.0084 & -0.0000 & 0.0000 \\ \hline 1.6022 & 0.2533 & 1.9763 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 \\ -0.3850 & -0.0692 & -0.0000 & 0.2998 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ 0.0281 & 0.0129 & 0.0000 & -0.0000 & 0.0446 & 0.0000 & -0.0000 & 0.0000 \\ 0.0352 & 0.0084 & 0.0000 & -0.0000 & 0.0000 & 0.0170 & 0.0000 & 0.0000 \\ \hline -0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 & 1.0000 & 0 \\ -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0 & 1.0000 \end{array}\right] \quad (34)$$

$$\hat{Q} = \left[\begin{array}{cc|cccc|cc} 1.0000 & 0 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 \\ 0 & 1.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 \\ \hline -0.0000 & -0.0000 & 1.9763 & -0.0000 & -0.0000 & 0.0000 & 0.2896 & 1.5555 \\ 0.0000 & -0.0000 & -0.0000 & 0.2998 & 0.0000 & -0.0000 & -0.0091 & 0.3789 \\ 0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0446 & -0.0000 & 0.0014 & 0.0186 \\ -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0170 & -0.0040 & 0.0016 \\ \hline -0.0000 & -0.0000 & 0.2896 & -0.0091 & 0.0014 & -0.0040 & 0.0512 & 0.1960 \\ -0.0000 & -0.0000 & 1.5555 & 0.3789 & 0.0186 & 0.0016 & 0.1960 & 1.7707 \end{array}\right] \quad (35)$$

$$\hat{P} = \left[\begin{array}{cc|cccc|cc} 2.6092 & 1.2844 & 3.2287 & 0.5418 & 0.0856 & 0.0227 & 0.0000 & 0.0000 \\ 1.2844 & 1.2329 & 1.5789 & 0.5431 & -0.1075 & 0.0065 & -0.0000 & 0.0000 \\ \hline 3.2287 & 1.5789 & 5.6044 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 \\ 0.5418 & 0.5431 & -0.0000 & 0.6695 & 0.0000 & 0.0000 & -0.0000 & 0.0000 \\ 0.0856 & -0.1075 & 0.0000 & 0.0000 & 0.1071 & 0.0000 & 0.0000 & 0.0000 \\ 0.0227 & 0.0065 & 0.0000 & 0.0000 & 0.0000 & 0.0048 & -0.0000 & 0.0000 \\ \hline 0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 & 1.0000 & -0.0000 \\ 0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & 1.0000 \end{array}\right] \quad (38)$$

$$\hat{Q} = \left[\begin{array}{cc|cccc|cc} 1.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -0.0000 & 1.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \\ \hline 0.0000 & -0.0000 & 5.6044 & -0.0000 & 0.0000 & 0.0000 & -2.6106 & 2.4703 \\ -0.0000 & -0.0000 & -0.0000 & 0.6695 & 0.0000 & -0.0000 & 0.3551 & -0.6800 \\ -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.1071 & 0.0000 & -0.1137 & -0.0771 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0048 & -0.0197 & 0.0129 \\ \hline -0.0000 & -0.0000 & -2.6106 & 0.3551 & -0.1137 & -0.0197 & 1.7552 & -1.4476 \\ 0.0000 & 0.0000 & 2.4703 & -0.6800 & -0.0771 & 0.0129 & -1.4476 & 2.0868 \end{array}\right] \quad (39)$$

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