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First order polarization tensor approximation using multivariate polynomial interpolation method via least square minimization technique

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Abstract. This paper proposes a new numerical approach useful in dealing with nearly singular integrals, specifically, the integral of the first order polarization tensor (PT). Polarization tensor represents the integral equations in an asymptotic series, and it can also define the boundary value problem of a partial differential equation (PDE). Since PT has been widely used and implemented in many engineering areas, particularly electric and magnetic field areas, it is crucial to estimate the first order PT solutions accurately. In this regard, the computation of PT for different geometry types is basically from the quadratic interpolation and the multivariate polynomial fitting using the least square method. The numerical calculation of the integral of the singular integral operator, K_B^* which is one of the primary integral processes before we obtained the solution of PT uses the multivariate polynomial fitting. This paper aims to provide an accurate numerical solution for first order PT for different geometry types, particularly sphere and ellipsoid geometry. The numerical results of the proposed method are shown together with the comparison of its analytical solutions. From the results obtained, the numerical solution of first order PT shows higher accuracy and higher convergence as the number of surface elements increases. The numerical and the analytical solution of first order PT for a sphere is discussed and represented in graphical form. The utilization of two different software types throughout this study is Netgen Mesh Generator and MATLAB to aid the numerical computation process. The simulation and the numerical examples verify the effectiveness and efficiency of the proposed method.

1. Introduction

Polarization tensor (PT) has vast application potentials in electrical, electromagnetic, and magnetic areas. In the study of the characterization of conducting objects, the conducting object separation is achieved using PT. The researchers separate cluttering items in the field region by differentiating between object properties using PT [1–4]. Since the conductivity of an object and its region is different hence, it is easier for the researcher to discriminate them by implementing the concept of PT. Besides, Ammari and Kang [5] uses PT to improve electrical imaging for medical and industrial purposes. In this regard, PT must be accurately computed to reduce incorrect discrimination or false separation of objects. Additionally, in this recent years, PT is adopted in a walk-through metal detector study where Marsh et al. [6, 7] identify the object location through magnetic polarization tensor. Polarization tensor (PT) is



originated from the study of Polya, where he describes PT as the virtual mass of a reciprocating object (with no conductivity) in a fluid [8]. The generalization of this classical concept of PT has then been introduced by Ammari and Kang [5], where they named it to be Generalized Polarization Tensor (GPT). The leading term of the asymptotic expansion representing the GPT is the first order PT. This paper focuses on the computation of the first order PT by using the method discussed later.

There are various numerical methods that the previous researcher has used to calculate the tensor of polarization. Capdebosq et al. [9] use a semi-algebraic method to show that, for an object with lower conductivity, the computation of PT will give higher convergence in its numerical results than a high conductivity object. However, this method does not apply to all cases of PT. It is restricted and specifically developed for the problem of two-dimensional PT. The researcher has developed program code in MATLAB to ease other researchers to find PT for a two-dimensional problem. From this study, Khairuddin and Lionheart [10] extend it to a three-dimensional problem. The researchers adopted a Boundary Element Method (BEM) method to develop a new operating scheme, which is BEM++. It was used to compute three-dimensional cases of PT. A similar BEM approach has been adopted by Lu et al. [11], where the researchers study magnetic polarization tensor (MPT) for cylindrical samples. Additionally, linear element integration is used by Kharuddin [11], and from this study, Sukri et al. [12] widen the scope of numerical integration in the computation of PT by evaluating the PT using quadratic element integration. Our study is an extension of the research by Sukri et al. [12], where a slightly different technique is used, that is, multivariate polynomial interpolation with the help of the least square technique.

J. Wallis [13] has introduced the interpolation terms where this interpolation is used to approximate the numerical results with minimum error obtained. In a data set containing points, interpolation calculates functions where its graphs can go through each dataset point [14]. From this, the minimum error for the data set is obtained. Many researchers have developed this study and applied the interpolation method in many numerical examples [15, 16]. In a later year, multivariate polynomial interpolation (polynomial consists of two variables) is instigated by extending the study of univariate interpolation, which consists of one variable. Aforementioned, univariate polynomial interpolation is an interpolation of a function containing one variable. This type of interpolation is classified as a classical and old method in applied mathematics. Hence, multivariate polynomial interpolation is a suitable technique that can be implemented to solve the first order PT problem. Multivariate polynomial interpolation is a widespread technique that has been implemented in many different areas involving engineering area since it offers the possibility of approximating the scattered data on a closed surface [17]. For this paper, Newton's polynomial (multivariate) is applied to obtain the polynomial function. The representation of Newton's polynomial in two variables, which are x and y is stated as

$$P(x, y) = a_{00} + a_{01}y + a_{10}x + a_{02}y^2 + a_{11}xy + a_{20}x^2 + \dots + a_{mn}x^m y^n, \tag{1}$$

where a is the polynomial coefficient with degree up to m for variable x and degree n for the variable y and

$$P(x, y) = \sum_{m,n \geq 0}^{m+n=i} a_{mn} x_i^m y_i^n. \tag{2}$$

From this polynomial coefficient, we will combine it with the quadratic interpolation method involving discretizing an object into several meshes. The discretization of the objects is achieved by using developed software by Shoberl [18], which is Netgen Mesh Generator software. Here, different types of objects will be discretized, and each of the triangular meshes will have six vertex containing local coordinates of x, y and z . First order PT for an object that has been discretized is computed, and convergence of the numerical results is observed. The detailed mathematical formulation for the quadratic interpolation method and the least square technique is explained in detail afterward.

This paper is organized as follows. In the first section of this paper, the first order PT background is presented in mathematical representation. The following section discussed the method used to compute

the first order PT, a multivariate polynomial interpolation, and the least square minimization. Then, a few numerical examples involving basic shape is demonstrated. The accuracy in terms of convergence of the numerical results using the proposed method is illustrated in the graphical form. Not even that, the numerical approximation of first order PT when there are changes in the degree of the multivariate polynomial is presented in a graphical illustration. Lastly, the numerical result obtained is discussed and summarized. Few recommendations for future research are stated in the last section of this paper.

2. First order polarization tensor

This section is divided into two parts: the mathematical formulation of first order PT and the analytical solution of PT involving sphere and ellipsoid geometry.

2.1. Mathematical Formulation

The mathematical background of the first order PT has been derived by Ammari and Kang [5], where the researchers presented the first order PT in integral equation over the boundary of the domain B , which yields to

$$M_{ij}(k, B) = \int_{\partial B} y^j \phi_i(y) d\sigma(y), \tag{3}$$

with multi indices i and j . The conductivity of the object B is referred to as k while y^j is the element of the object B . Here, $\phi_i(y)$ can be represented as a linear system of equation containing singular integral operator K_B^* and the outward normal vector, V_x which is

$$\phi_i(y) = (\lambda I - K_B^*)^{-1} (V_x \cdot \nabla x^i)(y). \tag{4}$$

λ is defined as $\lambda = (k + 1) / (2k - 2)$ while the singular integral operator K_B^* is defined in terms of Cauchy Principle Value integral as

$$K_B^* \phi(x) = \frac{1}{4\pi} P.V. \int_{\partial B} \frac{\langle x - y, V_x \rangle}{|x - y|^3} \phi(y) d\sigma(y), \tag{5}$$

where the notation of $P.V.$ is used to represent that the integral is Cauchy Principle Value while $x - y$ is the distance between element x and element y . The next part presented the analytical solution of first order PT.

2.2. Analytical Solution of First Order Polarization Tensor

In this section, the analytical solution for the first order PT is reviewed. Ammari and Kang [5] have done the derivation of the analytical solution for first order PT for a specific geometry, sphere, and ellipsoid. Thus, the analytical solution is very crucial in order to verify whether the proposed method can be applied to the problem of PT. From this analytical solution, the result of its convergence will be computed, and the error analysis will be observed. Milton [19] initiates the derivation of the analytical solution by considering an ellipsoid with an equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ provided that it has semi principle axes of a, b and c . Then, Ammari and Kang [5] rewrite the derivation made by Milton in [19] and defined the analytical solution in matrix form containing conductivity, k as

$$M(k, B) = (k - 1) |B| \begin{bmatrix} \frac{1}{(1 - d_1) + kd_1} & 0 & 0 \\ 0 & \frac{1}{(1 - d_2) + kd_2} & 0 \\ 0 & 0 & \frac{1}{(1 - d_3) + kd_3} \end{bmatrix}, \tag{6}$$

in which the volume of B is denoted as $|B|$. Meanwhile, d_1, d_2 and d_3 are a constants that can be obtained by solving the following integral, which are

$$\begin{aligned}
 d_1 &= \frac{bc}{a^2} \int_1^{+\infty} \frac{dt}{t^2 \sqrt{t^2 - 1 + \frac{b^2}{a^2}} \sqrt{t^2 - 1 + \frac{c^2}{a^2}}}, \\
 d_2 &= \frac{bc}{a^2} \int_1^{+\infty} \frac{dt}{\left(t^2 - 1 + \frac{b^2}{a^2}\right)^{\frac{3}{2}} \sqrt{t^2 - 1 + \frac{c^2}{a^2}}}, \\
 d_3 &= \frac{bc}{a^2} \int_1^{+\infty} \frac{dt}{\sqrt{t^2 - 1 + \frac{b^2}{a^2}} \left(t^2 - 1 + \frac{c^2}{a^2}\right)^{\frac{3}{2}}}.
 \end{aligned}
 \tag{7}$$

Equation (7) will be equated to $\frac{1}{3}$ if the semi principal axes are similar to each other. Then, from this, equation (6) will eventually give the analytical solution of first order PT for a sphere, which can be represented as

$$M(k, B) = (k - 1)|B| \begin{bmatrix} \frac{3}{2+k} & 0 & 0 \\ 0 & \frac{3}{2+k} & 0 \\ 0 & 0 & \frac{3}{2+k} \end{bmatrix}.
 \tag{8}$$

Since the analytical solution has been reviewed, then, the numerical method used to compute first order PT is explained in detail in the next section.

3. Quadratic element interpolation

Initially, we start with the Cauchy Principle Value integral, where this integral is transformed from a global coordinates system containing x, y and z into local coordinates of ξ and η . Therefore, equation (5) yield to

$$K_B^* \phi(x) = \frac{1}{4\pi} P.V. \int_{\partial B} \frac{\langle x - y, V_x \rangle}{|x - y|^3} \phi(y) \cdot G(\xi, \eta) \cdot d\xi d\eta(y),
 \tag{9}$$

in which $G(\xi, \eta)$ is equated to the absolute value of the Jacobian determinant $\sqrt{\det(J^T \cdot J)}$ where this Jacobian is represented as in equation (10)

$$|J(\xi, \eta)| = \begin{bmatrix} (4\xi - 1)x_1 + (-4\tau + 1)x_3 + 4\eta x_4 & (4\xi - 1)y_1 + (-4\tau + 1)y_3 + 4\eta y_4 \\ -4\eta x_5 + (4 - 8\xi - 4\eta)x_6 & -4\eta y_5 + (4 - 8\xi - 4\eta)y_6 \\ (4\eta - 1)x_2 + (-4\tau + 1)x_3 + 4\xi x_4 & (4\eta - 1)y_2 + (-4\tau + 1)y_3 + 4\xi y_4 \\ +(4 - 4\xi - 8\eta)x_5 - 4\xi x_6 & +(4 - 4\xi - 8\eta)y_5 - 4\xi y_6 \end{bmatrix},
 \tag{10}$$

where $\tau = 1 - \xi - \eta$. This paper will use two types of Gaussian quadrature as shown in Table 1.

Table 1. Gaussian quadrature type (GQT) with its weight and points.

	Weight, w_i		ξ		η		
GQT2	1/6	1/2	1/2	1/2	1/2	1/2	0
GQT3	1/6	2/3	1/6	1/6	1/6	2/3	1/6

Since this paper considered quadratic element interpolation, which contained six nodal points of each triangular meshes, therefore, this would eventually lead to the usage of six values of shape functions in terms of ξ and η as

$$\begin{aligned}
 N(1) &= \xi(2\xi - 1), & (11) \\
 N(2) &= \eta(2\eta - 1), \\
 N(3) &= \xi(2\xi - 1) + \eta(2\eta - 1) + 4\xi\eta + 1, \\
 N(4) &= 4\xi\eta, \\
 N(5) &= 4\eta\tau, \\
 N(6) &= 4\xi\tau.
 \end{aligned}$$

From equation (9), the normal outward vector V_x can be computed by using the formula as

$$V_x = \frac{\left\langle \sum_{i=1}^6 N_i x_i, \sum_{i=1}^6 N_i y_i, \sum_{i=1}^6 N_i z_i \right\rangle}{\left\| \left\langle \sum_{i=1}^6 N_i x_i, \sum_{i=1}^6 N_i y_i, \sum_{i=1}^6 N_i z_i \right\rangle \right\|}. \tag{12}$$

After obtaining the result of the Cauchy Principle Value integral, we substitute this solution into equation (4), and it will eventually lead to a linear system of the equation where the size of the matrix system depends on the number of Gaussian quadrature points and weight that is used. To find the solution of that linear system, we apply the inverse matrix method, where we find the inverse of $(\lambda I - K_B^*)$ and multiplied it with the normal outward vector of an element x .

From the solution of the linear system of equation, $\phi(y)$ we then substitute it into the next integral that we need to solve, which is the integral of first order PT in equation (3). The equation in (3) is firstly is transformed into global coordinates containing ξ and η as

$$M_{ij}(k, B) = \int_{\partial B} y^j \phi(y) \cdot G(\xi, \eta) d\xi d\eta. \tag{13}$$

We are using the same formula in the computation of the Jacobian as aforementioned will then lead to the numerical solution of first order PT. For the next section, the least square minimization technique is explained in detail before the program code is developed using MATLAB software.

4. Least square minimization technique

The least square method is frequently used to estimate the best solution to a particular data set in [20]. The least square fit is rather different and frequently more effective than curve fitting. Rather than trying to fit the points exactly, finding a polynomial of low degree (often first or second) that closely fits the points is required. Suppose we have a set of values of one variable, x_1, x_2, \dots, x_n where we specify its function to be a set of functions with k parameters C_1, C_2, \dots, C_k . This function can be denoted as

$$P(x) = f(x; C_1, C_2, \dots, C_k). \tag{14}$$

The function C_k shall minimize the sum of squares of errors,

$$\begin{aligned} \text{minimize, } S &= \sum_{i=1}^n e_i^2, \\ &= \sum_{i=1}^n (f(x_i) - P(x_i))^2. \end{aligned} \tag{15}$$

Since in this paper, we consider integral with multivariate polynomial, which involved ξ and η , hence, the set values of two variables is in the form of $(\xi_1, \eta_1), (\xi_2, \eta_2), \dots, (\xi_n, \eta_n)$. The function that is considered here is

$$P(\xi, \eta) = f((\xi, \eta); C_1, C_2, \dots, C_n) \tag{16}$$

where

$$\begin{aligned} n=0 &\rightarrow C_{00} \xi^0 \eta^0 = C_{00}, \\ n=1 &\rightarrow C_{10} \xi^1 \eta^0 + C_{01} \xi^0 \eta^1 = C_{10} \xi + C_{01} \eta, \\ &\vdots \\ n=k+l &\rightarrow C_{kl} \xi^k \eta^l = C_n \xi^k \eta^l \end{aligned} \tag{17}$$

From equation (17), we summarize it in the form of summation, and it will then yield to

$$P(\xi, \eta) = \sum_{k,l=1}^{n=k+l-2} C_n \xi^k \eta^l \tag{18}$$

In order to minimize the sum of squares of errors, we eventually obtained

$$S = \sum_{i=1}^n (f(\xi_i, \eta_i) - P(\xi_i, \eta_i))^2 \tag{19}$$

where the function of $f(\xi_i, \eta_i)$ obtained from the integral containing Cauchy Principle Value in equation (9), which is

$$f(\xi_i, \eta_i) = \frac{\langle x_i - y_j, V_{x_i} \rangle}{|x_i - y_j|^3} \tag{20}$$

The next section provides the numerical solution for first order PT for sphere by using our proposed method. The convergence of the numerical solution is achieved by increasing the number of surface elements, and the results are represented in graphical form for easy interpretation.

5. The convergence of quadratic interpolation for multivariate polynomial via least square technique

The main objectives of this paper are to verify whether the proposed method can be used to compute the first order PT accurately for specific types of geometry. For this purpose, smooth geometries with the provided analytical solution for first order PT, which are a sphere with radius, 0.01 are used to validate that the numerical results are accurate. To demonstrate the proposed method, we generated several numbers of meshes for chosen geometry until finest meshes. The estimation of the error is based on the relative error formula, which is

$$\text{relative error} = \frac{\|\overline{M} - M_A\|}{\|M_A\|}, \tag{21}$$

where \overline{M} is the numerical solution obtained by using the proposed method while M_A is the analytical solution. We will estimate the relative error for the numerical solution until it achieved a very low error and from this, the convergence results will be analyzed and represented in graphical form. For the next section, the numerical solutions for the first order PT for sphere are illustrated and discussed.

6. Result and discussion

6.1. First Order Polarization Tensor for Sphere

We start the computation of the first order PT for a sphere, $x^2 + y^2 + z^2 = 0.01^2$ at conductivity $k = 1.5$ by using the analytical formula in (8), our previous method in [12], and the recent method, which combines the method in [12] with the multivariate polynomial. The sphere is firstly being triangularized by Netgen Mesh Generator software. The information obtained from Netgen, which includes the number of surface elements and each coordinate of the mesh, is then exported to MATLAB, and the first order PT for the sphere is computed. By fixing the total number of mesh, $N = 44, 72$, up to 118 surface elements, the relative error for different degrees, Gaussian quadrature types (gqt), and the number of elements (nel) is plotted. Figure 1 plots the relative errors of the computed first order PT for the sphere using a multivariate polynomial fitting with the least square minimization technique under mesh refinement. As noticed, the numerical results of PT are accurately computed at degree 3 for Gaussian quadrature type (gqt) is 3. Gaussian quadrature type (gqt) 3 with degree (deg) 3 and number of element (nel) 3 showed the lowest relative error compared to the other numerical results of PT. The highest degree of the multivariate polynomial used to compute the relative error for first order PT is three. For the smooth geometry, the analytical solution is available; therefore, we compute its error using the previously mentioned formula in equation (20).

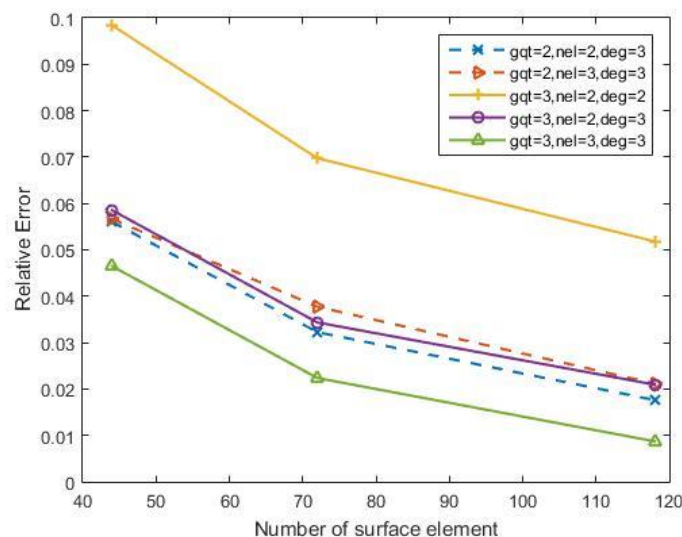


Figure 1. The relative errors for the first order PT for a sphere with different Gaussian quadrature types with a degree, $\text{deg} = 2$ and 3 , $N = 44, 72$ and 118 with $k = 1.5$ and radius $r = 0.01$.

The graph obtained in Figure 1 shows that Gaussian quadrature type 3 with the number of element three and degree 3 depicted the lowest relative error in its numerical results. Hence, by using this type of Gaussian quadrature, we plotted the values of elements for the main diagonal of first order PT. The comparison between the result of the analytical formula of PT, interpolation with multivariate polynomial (MP) usage, and interpolation without multivariate polynomial (nMP) are represented in Figures 2,3, and 4.

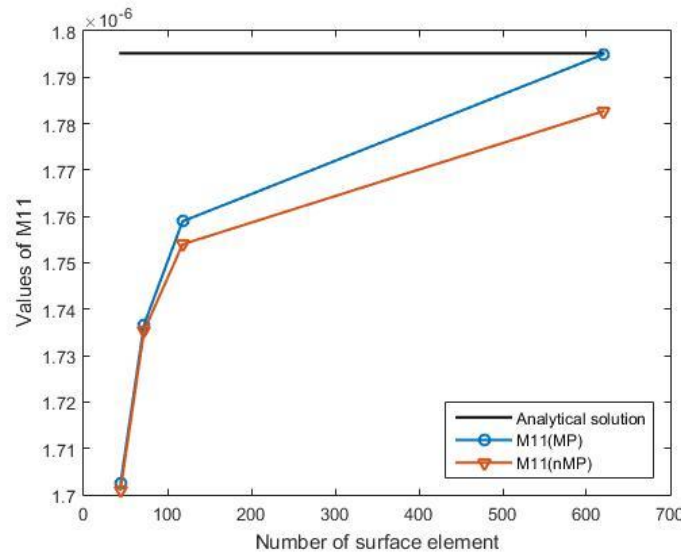


Figure 2. The main diagonal values, M_{11} for the first order PT for a sphere with different mesh $N = 44, 72, 118$ and 620 with $k = 1.5$ and radius $r = 0.01$.

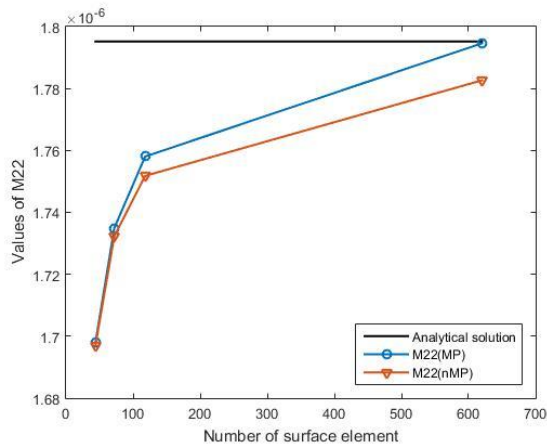


Figure 3. The main diagonal values, M_{22} for the first order PT for a sphere with different mesh $N = 44, 72, 118$ and 620 with $k = 1.5$ and radius $r = 0.01$.

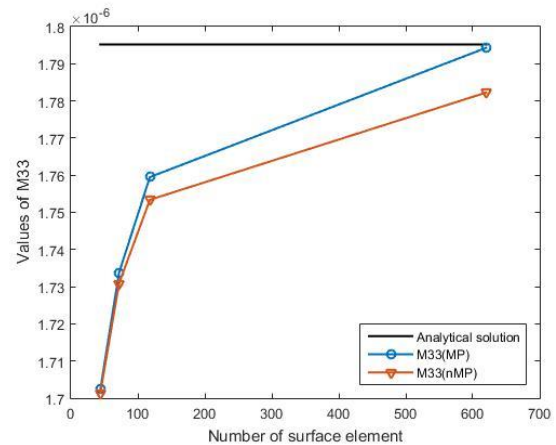


Figure 4. The main diagonal values, M_{33} for the first order PT for a sphere with different mesh $N = 44, 72, 118$ and 620 with $k = 1.5$ and radius $r = 0.01$.

The numerical approximation using multivariate polynomial gives accurate values for the main diagonal of first order PT compared to the numerical approximation without using the multivariate polynomial. As observed, at $N = 620$ triangles, the values of main diagonal PT using multivariate polynomial is equal to $0.179492648783043 \times 10^{-5}$ which is very close to the analytical solution which is $0.179519580205131 \times 10^{-5}$.

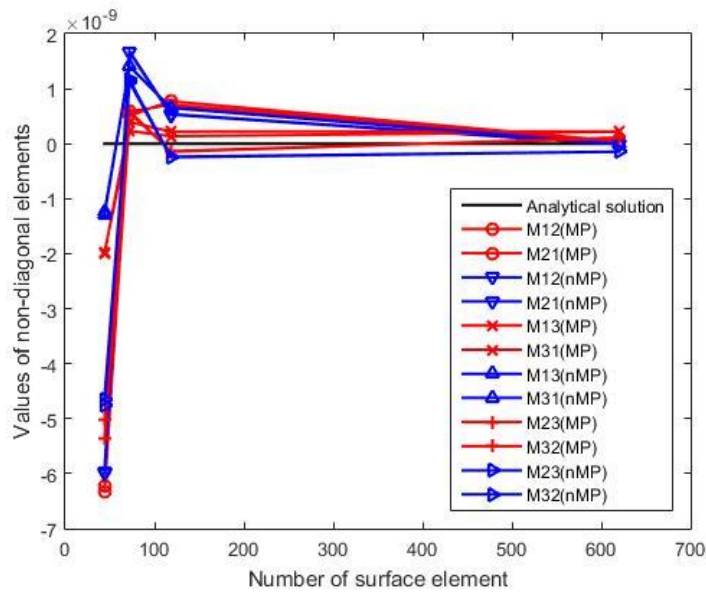


Figure 5. The non-diagonal values, $M_{12}, M_{13}, M_{21}, M_{23}, M_{31}$ and M_{32} for the first order PT for a sphere with different mesh $N = 44, 72, 118$ and 620 with $k = 1.5$ and radius $r = 0.01$.

The non-diagonal values of elements of the tensor of polarization are depicted in Figure 5. The numerical solution for both interpolations with and without the multivariate polynomial will eventually lead to zero, satisfying its analytical solution as the number of surface element increased.

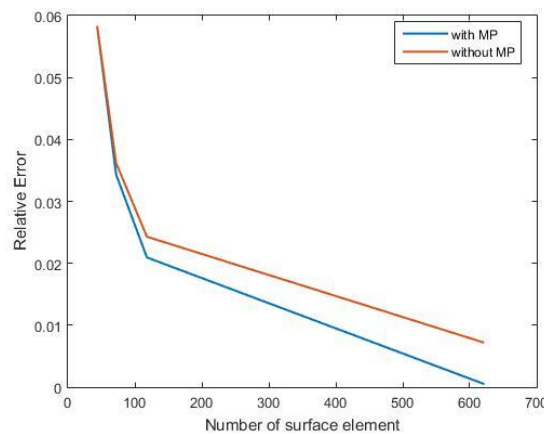


Figure 6. The relative errors for the first order PT for a sphere with different mesh $N = 44, 72, 118$ and 620 with $k = 1.5$ and radius $r = 0.01$.

Figure 6 shows that, as the total number of surface elements increased, the numerical result for first order PT for sphere will eventually lead to zero error. It shows that first order PT computation is accurately being evaluated if a smaller mesh size is used. Additional to that, the multivariate polynomial implementation can minimize the error as the numerical computation is evaluated. The next section will summarize the result obtained.

7. Conclusion

We proposed a method for computation of first order polarization tensor over chosen specific geometry. The method that we proposed uses least squares multivariate polynomial over quadratic interpolation in each geometry. The numerical result of first order PT obtained has a minimum error and highly dependent on the size of the geometry. Preferably, a higher number of meshes generated from free software, Netgen Mesh Generator, is required to have a highly converge and accurate numerical solution. The conductivity of an object is fixed to be close to 1 since the first order PT is sensitive to the changing of its conductivity. It is shown that the closer the conductivity values to 1, the higher the convergence of the numerical result obtained with the provided analytical solutions.

Due to the time limitation, the first order polarization tensor for only a few geometries has been tested to verify the efficacy of the proposed method. Since the analytical solution for first order PT is provided for only smooth geometries, which are sphere and ellipsoid, it will be used as a first benchmark for the numerical computation of PT. We intend to compute the first order PT for unknown analytical solutions such as cylinder, cube, or any non-smooth geometries in future work where the convergence of the solution will be studied.

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