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Seidel energy for Cayley graphs associated to dihedral group

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Abstract. The studies on the energy of a graph have been actively studied since 1978 and there have been various types of energy of graphs proposed and studied by mathematicians all over the world. One of the many types of energy being studied is Seidel energy, where it is defined as the summation of the absolute values of the eigenvalues of the Seidel matrix of a graph. This research combines the study on three important branches in mathematics, i.e. energy of graph in linear algebra, Cayley graph in graph theory and dihedral group in group theory. The aim of this research is to present some values of Seidel energy of Cayley graphs associated to dihedral groups with respect to certain subsets of the group, namely the subsets of order one and the subsets of order two; $\{a, a^{n-1}\}$ for $n \geq 3$ and $\{a^2, a^{n-2}\}$ for $n \geq 5$. The results are proven by using some properties of special graphs such as complete graph, cycle graph and complete bipartite graph, including some relations between the eigenvalues of a graph, with the Seidel eigenvalues of a graph.

1. Introduction

The study on Cayley graph was first initiated by Arthur Cayley in 1878 to explain the idea of abstract groups described by a set of generators [1]. It is defined as a graph with the elements of a group G as the vertices and there is an edge joining the vertices g and h in the group G if and only if there is $s \in S$, where S is a subset of G, such that the product of s and g is equal to h. The subset S of G does not include the identity element of the group G and it holds the inverse-closed property where every element of the subset has an inverse under the operation that is also an element of the subset. The Cayley graph of G with respect to the subset S is often denoted as $Cay(G; S)$ [2].

Through the years, the study on Cayley graph has developed and became a significant branch in algebraic graph theory. In 1988, Babai and Seress [3] have presented some results on the diameter of Cayley graphs of symmetric groups including the alternating groups. Not long after, in 1993, Lakshmivarahan *et al.* [4] have analysed the symmetries in the interconnection networks of a variety of Cayley graphs of permutation groups. The types of symmetry analyzed consist of vertex and edge transitivity, distance regularity and distance transitivity.

In 2000, Friedman [5] has shown in his study that among all sets of $n - 1$ transpositions which generate the symmetric groups S_n, the Cayley graph associated to the set $S =$

$\{(1, n), (2, n), \ldots, (n - 1, n)\}$ has the highest eigenvalue. Two years later, Li [6] has compiled and presented a survey on the findings, the problems and methods developed in finding the isomorphism for Cayley graphs which have effectively been used to solve the problems regarding finite vertex-transitive graphs.

In addition to that, in 2008, Konstantinova [7] came out with another survey which covered the historical changes of some problems on Cayley graphs such as the Hamiltonicity or diameter problems. The author also included various uses of Cayley graphs in solving combinatorial, graph-theoretical and also applied problems.

Later in 2012, Adiga and Ariamanesh [8] have determined the number of undirected Cayley graphs of symmetric groups S_n and alternating groups A_n up to isomorphism, as presented in the following proposition.

Proposition 1. *[8]   Let S_n be symmetric groups. Then $Cay(S_n, \{(ij)\}) \ni i, j \in \{1, 2, \ldots, n\}$ are all isomorphic to each other. i.e up to isomorphism, there is exactly one Cayley graph on symmetric groups S_n of valency 1.*

There have also been several works on the Cayley graphs especially for dihedral groups. In 2006, Wang and Xu [9] worked on the non-normal one-regular and 4-valent Cayley graphs of dihedral groups while in the same year, Kwak and Oh [10] have classified the 4-valent and 6-valent one regular normal Cayley graphs on dihedral groups whose vertex stabilizers in $Aut(\Gamma)$ are cyclic. A k-regular graph is a regular graph with vertices of degree k or also known as a k-valent graph. Meanwhile, vertex stabilizers is an element which fix a vertex of a graph. Besides, Jungreis *et al.* [11] have studied the hamiltonian of Cayley graphs on groups of low order where a hamiltonian path is a path in an undirected or directed graph that visits each vertex exactly once.

In addition, in 2008, Kwak *et al.* [12] have explored on one-regular Cayley graphs on dihedral groups of any prescribed valency. Kim *et al.* [13] have also studied on the Cayley graphs of dihedral groups on the classification of p-valent regular Cayley graphs. They came out with the following theorem and proposition.

Theorem 1. *[12] Let p be an odd prime number. Then, any p-valent one-regular Cayley graphs on a dihedral groups is isomorphic to one of $Cay(D_n : l, p)$ for $(n, l, p) \in O$ except at most finitely many ones in which l is positive integers less than n and O be the set of triples (n, l, k) satisfying the following two conditions:*
(i) For any $r, s, t, u(0 \le r, s, t, u \le k - 1), l[r] + l[s] = l[t] + l[u] (mod\, n)$ if and only if either $(r, s) = (t, u)$ or $(r, s) = (u, t)$.
(ii) For any sequence of numbers $i_0, i_1, i_2, i_3, i_4, i_5 \in \{0, 1, 2, \cdots, k - 1\}$ such that $i_j \ne i_j + 1$ and $i_5 \ne i_0, l[i_0] + l[i_2] + l[i_4] = l[i_1] + l[i_3] + l[i_5] (mod\, n)$ if and only if the numbers i_0, i_1 and i_2 are all distinct and $i_0 = i_3, i_1 = i_4$ and $i_2 = i_5$.

Proposition 2. *[13] Every prime valent one-regular Cayley graphs on a dihedral group is normal.*

Meanwhile, the study on the energy of general simple graphs was inspired from the Hückel Molecular Orbital Theory (HMO) proposed in 1930s. In the early years, the Hückel Molecular Orbital Theory has been used by chemists to approximate the energies related with $\pi-$electron orbitals in conjugated hydrocarbon. In 1956, the fact that the method is actually applying a first degree polynomial of the adjacency matrix of a graph was first realized by Günthard and Primas [14]. Later in 1978, the energy of a simple graph has been defined by Gutman [15].

The ordinary energy of a graph Γ is defined as the summation of all positive values of the eigenvalues of the adjacency matrix of the graph. The adjacency matrix $A(\Gamma)$ of the graph Γ is a square matrix of size $n \times n$, whose ij-entry is equal to 1 if the vertices v_i and v_j are adjacent and is equal to zero otherwise. The characteristic polynomial of the adjacency matrix,

i.e, $\det(\lambda I_n - A(\Gamma))$, where I_n is the unit matrix of order n is said to be the characteristic polynomial of the graph Γ and often denoted by $f(\Gamma, x)$. Since the eigenvalues of a graph Γ are defined as the eigenvalues of its adjacency matrix $A(\Gamma)$, so they are just the roots of the equation $f(\Gamma, x) = 0$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$.

One of the energy which is closely related by their adjacency matrix is the Seidel energy. The Seidel energy $SE(\Gamma)$ of a graph Γ is defined as the sum of the absolute values of the eigenvalues of the Seidel matrix of Γ, denoted as $SE(\Gamma) = \sum_{i=1}^{n} |\theta_i|$ where θ_i are the eigenvalues of the Seidel matrix of the graph Γ. The Seidel matrix of a simple graph with n vertices and m edges, denoted by $S(\Gamma) = (s_{ij})$ is a real square symmetric matrix of order n defined as $s_{ij} = -1$ if v_i and v_j are adjacent, $s_{ij} = 1$ if v_i and v_j are not adjacent and 0 if $i = j$.[16]

In order to obtain the Seidel energy of a graph, we need to extract the eigenvalues of the Seidel matrix which we referred as the Seidel eigenvalues. The eigenvalues can be obtained from the roots of the characteristic equation of the matrix. The set of all Seidel eigenvalues is also referred as the Seidel spectrum of a matrix.

In the year 2008, Zhou [17] has presented some findings on the the relations between the main eigenvalues and eigenvectors of an adjacency matrix of a graph which is used in finding the ordinary energy of the graph, with the Seidel matrix of the graph which is used in finding the Seidel energy of the graph. Some of the results emphasized by Zhou in [17] are as given in the following theorem and corollary.

Theorem 2. *[17] Let $\theta_1, \theta_2, \cdots, \theta_l$ be the main eigenvalues of the Seidel matrix $S(\Gamma)$ and $\alpha_1, \alpha_2, \cdots, \alpha_l$ be the associated orthonormal eigenvectors. Let E be the $l \times l$ matrix whose (i,j)-entry is $e^T \alpha_i e^T \alpha_j$ and $M = \frac{1}{2}(E - I - \text{diag}(\theta_1, \theta_2, \cdots, \theta_l))$. Then the eigenvalues of M are precisely the main eigenvalues of $S(\Gamma)$. Furthermore, if $b = (b_1, b_2, \cdots, b_l)^T$ is an eigenvector that corresponding to an eigenvalue μ of M, then $\sum_{j=1}^{l} b_j \alpha_j$ is an eigenvector of $A(\Gamma)$ corresponding to μ.*

Corollary 1. *[17] Let $\lambda_1, \lambda_2, \cdots, \lambda_l$ and $\theta_1, \theta_2, \cdots, \theta_l$ are all main eigenvalues of $A(\Gamma)$ and $S(\Gamma)$, respectively. Then*

$$\sum_{i=1}^{l}(2\lambda_i + \theta_i) = n - l.$$

Next, in 2012, Haemers [18] has obtained the upper and lower bounds for $SE(\Gamma)$, characterized the equality for the upper bound, and formulated a conjecture for the lower bound for $SE(\Gamma)$. Haemers conjectured that the Seidel energy of any graph with n vertices is at least $2n - 2$, the Seidel energy of the complete graph with n vertices.

In 2014, Nageswari and Sarasija [19] have presented their findings on the sharp upper and lower bounds for the Seidel energy of connected and disconnected graphs. Their findings are given in the following:

Proposition 3. *[19] The Seidel eigenvalues $s_1, s_2, \cdots, s_n$ of the Seidel matrix of the graph Γ satisfy the following relations: $\sum_{i=1}^{n} s_i = 0$; $\sum_{i=1}^{n} s_i^2 = n(n-1)$.*

Theorem 3. *[19] If Γ is a connected graph with n vertices and m edges then the inequality $SE(\Gamma) \leq |n - 1 - \frac{4m}{n}| + \sqrt{(n-1)(n^2 - n - (n - 1 - \frac{4m}{n})^2)}$ holds.*

Theorem 4. *[19] If Γ_1 and Γ_2 are two components of a disconnected graph Γ with vertices n_1 and n_2 respectively then the Seidel energy of Γ, $SE(\Gamma)$ has the following inequality, $SE(\Gamma_1) + SE(\Gamma_2) \leq SE(\Gamma) \leq SE(\Gamma_1) + SE(\Gamma_2) + 2\sqrt{n_1 n_2}$.*

Corollary 2. *[19] If Γ_1 and Γ_2 are two components of a disconnected graph Γ with equal number of vertices say n then the Seidel energy of Γ has the following inequality, $SE(\Gamma_1) + SE(\Gamma_2) \leq SE(\Gamma) \leq SE(\Gamma_1) + SE(\Gamma_2) + 2(n)$.*

In the following year, Ramane *et al.* [20] have shown that if Γ is regular of order n and of degree $r \geq 3$, then for each $k \geq 2$, the Seidel energy of the k-th iterated line graph of Γ depends solely on n and r. Their findings have enable the construction of pairs of non-cospectral, Seidel equienergetic graphs of the same order.

2. Preliminaries Results

Several forms of the ordinary and Seidel spectrum of special graphs such as complete graph, cycle graph and complete bipartite graph have been found by Brouwer and Haemers [22] as stated in the following propositions.

Proposition 4. *[22] Consider the undirected n-cycle graph C_n. The spectrum of C_n consists of the numbers $\{2cos(\frac{2\pi j}{n})\}; j = \{0, 1, \ldots, n-1\}$.*

Proposition 5. *[22] Let G be a complete graph K_n on n vertices. Its Seidel spectrum is $\{(1-n)^1, (1)^{n-1}\}$.*

Proposition 6. *[22] Let G be a complete bipartite graph $K_{m,n-m}$ on n vertices. Its Seidel spectrum is $\{(1-n)^1, (1)^{n-1}\}$.*

In 2018 and 2019, Fadzil *et al.* in [23] and [24] have proved the energy of the Cayley graph with respect to the few subsets associated to the dihedral group of order $2n$ as presented in the following propositions, lemmas and theorems.

Proposition 7. *[23] Let D_{2n} be a dihedral group of order $2n$ where $n \geq 3$ and n even. Let $S = \{a^{n/2}, b\}$ be a subset of D_{2n}. The Cayley graphs of D_{2n} with respect to the set S, $Cay(D_{2n}, \{a^{n/2}, b\})$, are the cycle graphs $\frac{n}{2}C_4$.*

Lemma 1. *[23] Let D_{2n} be a dihedral group of order $2n$ where $n \geq 3$ and n even. Let $S = \{a^{n/2}, b\}$ be a subset of D_{2n}. Thus, the spectrum of the $Cay(D_{2n}, \{a^{n/2}, b\})$, denoted as $Spec(Cay(D_{2n}, \{a^{n/2}, b\}))$ are 0 with multiplicity n and ± 2 with multiplicity to $n/2$.*

Theorem 5. *[23] Let D_{2n} be a dihedral group of order $2n$ where $n \geq 3$ and $S = \{a^{n/2}, b\}$ is a subset of D_{2n}. The energy of the Cayley graphs of D_{2n} with respect to the set S, $E(Cay(D_{2n}, \{a^{n/2}, b\})) = 2n$.*

Proposition 8. *[24] Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X = \{b, ab, \ldots, a^{n-1}b\}$ be the generating subset of D_{2n}. Then, the Cayley graph of D_{2n} with respect to the generating subset X denoted as $Cay(D_{2n}, \{b, ab, \ldots, a^{n-1}b\})$ is the complete bipartite graph $K_{n,n}$.*

Lemma 2. *[24] Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X = \{b, ab, \ldots, a^{n-1}b\}$ be the generating subset of D_{2n}. Therefore, the eigenvalues of $Cay(D_{2n}, \{b, ab, \ldots, a^{n-1}b\}) = K_{n,n}$ are 0 with multiplicity $2n - 2$ and $\pm n$ with multiplicity 1.*

Theorem 6. *[24] Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X = \{b, ab, \ldots, a^{n-1}b\}$ be the generating subset of D_{2n}. The energy of the Cayley graphs of D_{2n} with respect to the generating subset X, $\varepsilon(Cay(D_{2n}, \{b, ab, \ldots, a^{n-1}b\})) = 2n$.*

Recently, in 2020, Akbari *et al.* [21] have established the validity of Haemers conjecture that the Seidel energy of any graph of order n is at least $2n - 2$ and that up to Seidel equivalence, the equality holds for K_n.

This paper aims to present the Seidel energy of the Cayley graphs for dihedral group D_{2n} with respect to the subsets of order up to two. The methodology consists of finding the eigenvalues of the Cayley graphs and thus computing the Seidel energy of the respected Cayley graphs.

This paper is structured as follows: in first section, some introductions and previous studies on the topics are laid out. In second section, some main results on the Seidel energy of the

Cayley graphs of the dihedral group are presented in form of lemmas and theorems. Lastly, some conclusion of the results are summarized in the third section.

3. Main Results

In this section, the Seidel energy of the Cayley graphs associated to the dihedral group D_{2n}, with respect to subsets of order one and two are presented.

3.1. The Seidel Energy of the Cayley Graphs Associated to Dihedral Group With Respect to Subsets of Order One

In the following lemma and theorem, the Seidel eigenvalues and Seidel energy are computed and presented for the Cayley graph associated to the dihedral group with respect to subsets of order one.

Lemma 3. *Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X^{(1)}$ be a subset of order one of D_{2n}. Then, the Seidel eigenvalues of the Cayley graphs $Cay(D_{2n}, X^{(1)})$ are ± 1 with multiplicity n.*

Proof. Suppose D_{2n} is the dihedral group of order $2n$, where $n \geq 3$ and $X^{(1)}$ is a subset of order one of D_{2n}. The Cayley graph of D_{2n} with respect to subsets of order one, $Cay(D_{2n}, X^{(1)})$ is nK_2. From Proposition 5, since the Seidel spectrum of a complete graph K_n on n vertices is $\{(1 - n), (1)^{n-1}\}$, then the Seidel eigenvalues of $Cay(D_{2n}, X^{(1)}) = nK_2$ are $\theta_i = \pm 1$ with multiplicity n.

The Seidel energy of the Cayley graph associated to the dihedral group of order $2n$ with a subset of order one is found and presented in the following theorem.

Theorem 7. *Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X^{(1)}$ be a subset of order one of D_{2n}. Then, the Seidel energy of the Cayley graphs $Cay(D_{2n}, X^{(1)})$ is $2n$.*

Proof. Suppose D_{2n} is the dihedral group of order $2n$, where $n \geq 3$ and $X^{(1)}$ is a subset of order one of D_{2n}. $Cay(D_{2n}, X^{(1)}) = nK_2$ is the Cayley graph of D_{2n} with respect to the subset of order one. Clearly, the Seidel eigenvalues of $Cay(D_{2n}, X^{(1)})$ are $\theta_i = \pm 1$ with multiplicity n, then, the Seidel energy of the $Cay(D_{2n}, X^{(1)})$ denoted as $SE(Cay(D_{2n}, X^{(1)})) = \sum_{i=1}^{n} |\theta_i| = n(|1| + |-1|) = 2n$.

An example is provided to illustrate the previous theorem.

Example 1. *Let D_{10} be the dihedral group of order 10, where $D_{10} = \langle a, b | a^5 = b^2 = 1, bab = a^{-1} \rangle$ and $X^{(1)} = \{b\}$ be the subset of D_{10} of order one.*

Then, the Cayley graph $Cay(D_{10}, \{b\})$ is the complete graph $5K_2$. This gives the Seidel spectrum of the Cayley graph $Cay(D_{10}, \{b\})$ as $\{(1 - 5), (1)^{5-1}\}$. Therefore, its easy to see that the eigenvalues of the graph is ± 1 with multiplicity 5 since $n = 5$. Therefore, the Seidel energy of the $Cay(D_{10}, \{b\})$ denoted as $SE(Cay(D_{10}, \{b\})) = \sum_{i=1}^{5} |\theta_i| = 5(|1| + |-1|) = 10$.

3.2. The Seidel Energy of the Cayley Graphs Associated to Dihedral Group With Respect to Subsets of Order Two

This subsection presented the results on the Seidel energy of the Cayley graph associated to the dihedral group of order $2n$ with respect to the subsets of order two in the form of theorems.

Theorem 8. *Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X^{(2)} = \{a, a^{n-1}\}$ be a subset of order two of D_{2n}. Then, the Cayley graphs of D_{2n} with respect to the subset $X^{(2)}$, denoted by $Cay(D_{2n}, \{a, a^{n-1}\})$ is two cycle graphs of length n, denoted as $2C_n$.*

Proof. Suppose D_{2n} is the dihedral group of order $2n$, where $n \geq 3$ and $X^{(2)} = \{a, a^{n-1}\}$ is a subset of order two of D_{2n}. Then, by using the definition of Cayley graph, it is shown that the followings are two cycles of length n for $Cay(D_{2n}, \{a, a^{n-1}\})$.

$$1 - a - a^2 - \cdots - a^{n-1} - 1 \quad (1)$$
$$b - ab - a^2 b - \cdots - a^{n-1} b - b \quad (2)$$

Since $(a^{i+1})(a^i)^{-1} = a^{i+1-i} = a \in X^{(2)}$ and $(a^{n-1})(1)^{-1} = a^{n-1} \in X^{(2)}$, then a^i is adjacent to a^{i+1} for $0 \leq i \leq n-2$ and a^{n-1} is adjacent to 1. Thus, it is proven that (1) is a cycle.

Next, since $(a^{i+1} b)(a^i b)^{-1} = a^{i+1} b b^{-1} a^{-i} = a^{i+1} a^{-i} = a \in X^{(2)}$ and $(a^{n-1} b)(b)^{-1} = a^{n-1} b b^{-1} = a^{n-1} \in X^{(2)}$, then $a^i b$ is adjacent to $a^{i+1} b$ for $0 \leq i \leq n-2$ and $a^{n-1} b$ is adjacent to b. Thus, it is proven that (2) is a cycle.

Since all Cayley graphs of D_{2n} with respect to the subset $X^{(2)} = \{a, a^{n-1}\}$ are 2-regular graphs, then there is no edge between both cycles. Therefore, $Cay(D_{2n}, \{a, a^{n-1}\})$ is two cycle graphs of length n, denoted as $2C_n$.

Theorem 9. *Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X^{(2)} = \{a, a^{n-1}\}$ be a subset of order two of D_{2n}. Then, the Seidel energy of the Cayley graphs $Cay(D_{2n}, \{a, a^{n-1}\})$ is $\sum_{i=1}^{n} |8 \cos(\frac{2\pi i}{n})|$.*

Proof. Suppose D_{2n} is the dihedral group of order $2n$, where $n \geq 3$ and $X^{(2)} = \{a, a^{n-1}\}$ is a subset of valency two of D_{2n}. By Theorem 8, the Cayley graph of D_{2n} with respect to the subset $\{a, a^{n-1}\}$, denoted as $Cay(D_{2n}, \{a, a^{n-1}\})$ is $2C_n$. In order to find the Seidel energy of the Cayley graph, Corollary 1 is used. Obviously, from Proposition 4, the eigenvalues of $Cay(D_{2n}, \{a, a^{n-1}\})$ are $\lambda_i = \{2 \cos(\frac{2\pi j}{n}); j = 0, 1, \ldots, n-1\}$ with multiplicity 2.

Since $Cay(D_{2n}, \{a, a^{n-1}\})$ is a 2-regular graph, its Seidel spectrum is determined by its adjacency spectrum as given in the following:
$$\sum_{i=1}^{l} (2\lambda_i + \theta_i) = n - l$$
where λ_i and θ_i are the eigenvalues of $l \times l$ adjacency matrix $A(\Gamma)$ and $l \times l$ Seidel matrix $A^*(\Gamma)$ respectively. Therefore, by rearranging the above equation gives the following equation:
$$\sum_{i=1}^{l} \theta_i = n - l - \sum_{i=1}^{l} 2\lambda_i.$$
Thus, the Seidel energy of $Cay(D_{2n}, \{a, a^{n-1}\})$ is the summation of its Seidel eigenvalues as given in the following:

$$SE(Cay(D_{2n}, \{a, a^{n-1}\})) = \sum_{i=1}^{l} \theta_i = n - l - \sum_{i=1}^{l} 2\lambda_i.$$

Therefore, the Seidel energy of the Cayley graphs $Cay(D_{2n}, \{a, a^{n-1}\})$ is as given in the following:

$$SE(Cay(D_{2n}, \{a, a^{n-1}\})) = \sum_{i=1}^{n} |2\lambda_i|$$
$$= \sum_{i=1}^{n} |-8 \cos(\frac{2\pi i}{n})|$$
$$= \sum_{i=1}^{n} |8 \cos(\frac{2\pi i}{n})|.$$

Next theorem presents the Seidel energy for the Cayley graph of the dihedral group with respect to the subset $\{a^2, a^{n-2}\}$ for the case n is odd.

Theorem 10. *Let D_{2n} be the dihedral group of order $2n$, where $n \geq 5$ and n is odd. Let $X^{(2)} = \{a^2, a^{n-2}\}$ be a subset of order two of D_{2n}. Then, the Cayley graphs of D_{2n} with respect to the subset $X^{(2)}$, denoted by $Cay(D_{2n}, \{a^2, a^{n-2}\})$ is two cycle graphs of length n, denoted as $2C_n$.*

Proof. Suppose D_{2n} is the dihedral group of order $2n$, where $n \geq 5$ and n is odd. Let $X^{(2)} = \{a^2, a^{n-2}\}$ be a subset of order two of D_{2n}. Then, by using the definition of Cayley graph, it is shown that the followings are two cycles of length n for $Cay(D_{2n}, \{a^2, a^{n-2}\})$.

$$1 - a^2 - a^4 - \cdots - a^{2k} - a - a^3 - a^5 - \cdots - a^{2k-1} - 1 \quad (3)$$
$$b - a^2b - a^4b - \cdots - a^{2k}b - ab - a^3b - a^5b - \cdots - a^{2k-1}b - b \quad (4)$$

First, it is proven that (3) is a cycle based on the followings.
(i) Since $(a^{2(i+1)})(a^{2i})^{-1} = a^{2i+2}a^{-2i} = a^2 \in X^{(2)}$, then a^{2i} is adjacent to $a^{2(i+1)}$.
(ii) Since $(a^{2k})(a)^{-1} = a^{2k-1} = a^{n-2} \in X^{(2)}$, then a^{2k} is adjacent to a.
(iii) Since $(a^{2j+3})(a^{2j+1})^{-1} = a^{2j+3-2j-1} = a^2 \in X^{(2)}$, then a^{2j+1} is adjacent to a^{2j+3}.
(iv) Since $(a^{2k-1})(1)^{-1} = a^{2k-1} = a^{n-2} \in X^{(2)}$, then a^{2k-1} is adjacent to 1.
Next, it is proven that (4) is a cycle based on the followings.
(i) Since $(a^{2(i+1)}b)(a^{2i}b)^{-1} = a^{2i+2}bb^{-1}a^{-2i} = a^2 \in X^{(2)}$, then $a^{2i}b$ is adjacent to $a^{2(i+1)}b$.
(ii) Since $(a^{2k}b)(ab)^{-1} = a^{2k}bb^{-1}a^{-1} = a^{2k-1} = a^{n-2} \in X^{(2)}$, then $a^{2k}b$ is adjacent to ab.
(iii) Since $(a^{2j+3}b)(a^{2j+1}b)^{-1} = a^{2j+3}bb^{-1}a^{-2j-1} = a^{2j+3-2j-1} = a^2 \in X^{(2)}$, then $a^{2j+1}b$ is adjacent to $a^{2j+3}b$.
(iv) Since $(a^{2k-1}b)b^{-1} = a^{2k-1} = a^{n-2} \in X^{(2)}$, then $a^{2k-1}b$ is adjacent to b.
Therefore, both (3) and (4) are cycles. Since all Cayley graphs of D_{2n} with respect to the subset $X^{(2)}$ are 2-regular graphs, then there is no edge between both cycles. Thus, the Cayley graphs of D_{2n} with respect to the subset $X^{(2)} = \{a^2, a^{n-2}\}$, where $n \geq 5$ and n is odd, is the two cycle graphs of length n, denoted as $2C_n$.

Theorem 11. *Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and n is odd. Let $X^{(2)} = \{a^2, a^{n-2}\}$ be a subset of valency two of D_{2n}. Then, the Seidel energy of the Cayley graphs $Cay(D_{2n}, \{a^2, a^{n-2}\})$ is $\sum_{i=1}^{n} |8 \cos(\frac{2\pi i}{n})|$.*

Proof. Suppose D_{2n} is the dihedral group of order $2n$, where $n \geq 3$ and n is odd. Let $X^{(2)} = \{a^2, a^{n-2}\}$ be a subset of valency two of D_{2n}. By Theorem 10, the Cayley graph of D_{2n} with respect to the subset $\{a^2, a^{n-2}\}$, denoted as $Cay(D_{2n}, \{a^2, a^{n-2}\})$ is $2C_n$. Its easy to see that the eigenvalues of $Cay(D_{2n}, \{a^2, a^{n-2}\})$ are $\lambda_i = \{2 \cos(\frac{2\pi j}{n}); j = 0, 1, \ldots, n-1\}$ with multiplicity 2. Similarly, as in the proof of Theorem 9, the Seidel energy of the Cayley graphs $Cay(D_{2n}, \{a^2, a^{n-2}\})$ where $n \geq 3$ and n is odd is as given in the following:

$$SE(Cay(D_{2n}, \{a^2, a^{n-2}\})) = \sum_{i=1}^{n} |-2\lambda_i|$$
$$= \sum_{i=1}^{n} |-2(4\cos(\frac{2\pi i}{n}))|$$
$$= \sum_{i=1}^{n} |8\cos(\frac{2\pi i}{n})|.$$

An example is provided to illustrate the previous theorem.

Example 2. *Let D_{10} be the dihedral group of order* 10*, where $D_{10} = \langle a, b | a^5 = b^2 = 1, bab = a^{-1} \rangle$ and $X^{(2)} = \{a^2, a^3\}$ be the subset of D_{10} of valency two. Then, clearly, the Cayley graph $Cay(D_{10}, \{a^2, a^3\})$ is the union of two cycle graphs of five vertices, $2C_5$. Then, the Seidel energy of $Cay(D_{10}, \{a^2, a^3\})$ is given as $\sum_{i=1}^{5} |8 \cos(\frac{2\pi i}{5})|$.*

The following theorem presents the Seidel energy for the Cayley graph of the dihedral group with respect to the subset $\{a^2, a^{n-2}\}$ for the case n is even.

Theorem 12. *Let D_{2n} be the dihedral group of order $2n$, where $n \geq 5$ and n is even. Let $X^{(2)} = \{a^2, a^{n-2}\}$ be a subset of order two of D_{2n}. Then, the Cayley graphs of D_{2n} with respect to the subset $X^{(2)}$, denoted by $Cay(D_{2n}, \{a^2, a^{n-2}\})$ is four cycle graphs of length $\frac{n}{2}$, denoted as $4C_{\frac{n}{2}}$.*

Proof. Suppose D_{2n} is the dihedral group of order $2n$, where $n \geq 5$ and n is even. Let $X^{(2)} = \{a^2, a^{n-2}\}$ be a subset of order two of D_{2n}. Assume that $n = 2k$, then, by using the definition of Cayley graph, it is proven that the followings are four cycles of length k for $Cay(D_{2n}, \{a^2, a^{n-2}\})$.

$$1 - a^2 - a^4 - \cdots - a^{2k-2} - 1 \qquad (5)$$
$$a - a^3 - a^5 - \cdots - a^{2k-1} - a \qquad (6)$$
$$b - a^2b - a^4b - \cdots - a^{2k-2}b - b \qquad (7)$$
$$ab - a^3b - a^5b - \cdots - a^{2k-1}b - ab \qquad (8)$$

First, it is proven that (5) is a cycle based on the followings:
(i) Since $(a^{2(i+1)})(a^{2i})^{-1} = a^{2i+2}a^{-2i} = a^2 \in X^{(2)}$, then a^{2i} is adjacent to $a^{2(i+1)}$.
(ii) Since $(a^{2k-2})(1)^{-1} = a^{2k-2} = a^{n-2} \in X^{(2)}$, then a^{2k-2} is adjacent to 1.
Next, it is proven that (6) is a cycle based on the followings:
(i) Since $(a^{2i+3})(a^{2i+1})^{-1} = a^{2i+3}a^{-2i-1} = a^{2i+3-2i-1} = a^2 \in X^{(2)}$, then a^{2i+1} is adjacent to a^{2i+3}.
(ii) Since $(a^{2k-1})(a)^{-1} = a^{2k-1-1} = a^{2k-2} = a^{n-2} \in X^{(2)}$, then a^{2k-1} is adjacent to a.
Next, (7) is proven a cycle based on the followings:
(i) Since $(a^{2(i+1)}b)(a^{2i}b)^{-1} = a^{2i+2}bb^{-1}a^{-2i} = a^2 \in X^{(2)}$, then $a^{2i}b$ is adjacent to $a^{2(i+1)}b$.
(ii) Since $(a^{2k-2}b)(b)^{-1} = a^{2k-2} = a^{n-2} \in X^{(2)}$, then $a^{2k-2}b$ is adjacent to b.
Finally, (8) is proven a cycle based on the followings:
(i) Since $(a^{2i+3}b)(a^{2i+1}b)^{-1} = a^{2i+3}bb^{-1}a^{-2i-1} = a^{2i+3-2i-1} = a^2 \in X^{(2)}$, then $a^{2i+1}b$ is adjacent to $a^{2i+3}b$.
(ii) Since $(a^{2k-1}b)(ab)^{-1} = a^{2k-1}bb^{-1}a^{-1} = a^{2k-1-1} = a^{2k-2} = a^{n-2} \in X^{(2)}$, then $a^{2k-1}b$ is adjacent to ab.
Therefore, all of them are cycles. There is no edge between all cycles since all Cayley graphs of D_{2n} with respect to the subset $X^{(2)}$ are 2-regular graphs. Thus, the Cayley graphs of D_{2n} with respect to the subset $X^{(2)} = \{a^2, a^{n-2}\}$, where $n \geq 5$ and n is even, is the four cycle graphs of length $\frac{n}{2}$, denoted as $4C_{\frac{n}{2}}$.

Theorem 13. *Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and n is even. Let $X^{(2)} = \{a^2, a^{n-2}\}$ be a subset of valency two of D_{2n}. Then, the Seidel energy of the Cayley graphs $Cay(D_{2n}, \{a^2, a^{n-2}\})$ for n even is $\sum_{i=1}^{n} |16 \cos(\frac{4i\pi}{n})|$.*

Proof. Suppose D_{2n} is the dihedral group of order $2n$, where $n \geq 3$ and n is even. Let $X^{(2)} = \{a^2, a^{n-2}\}$ be a subset of valency two of D_{2n}. By Theorem 12, the Cayley graph of D_{2n} with respect to the subset $\{a^2, a^{n-2}\}$, denoted as $Cay(D_{2n}, \{a^2, a^{n-2}\})$ is $4C_{\frac{n}{2}}$. Therefore, the eigenvalues of $Cay(D_{2n}, \{a^2, a^{n-2}\})$ are $\{2\cos(\frac{4\pi j}{n}); j = 0, 1, \ldots, n - 1\}$ with

multiplicity 4. Similarly, as in the proof of Theorem 9, the Seidel energy of the Cayley graphs $Cay(D_{2n}, \{a^2, a^{n-2}\})$ where $n \geq 3$ and n is even is as given in the following:

$$
\begin{aligned}
SE(Cay(D_{2n}, \{a^2, a^{n-2}\})) &= \sum_{i=1}^{n} | -2\lambda_i| \\
&= \sum_{i=1}^{n} | -2(8)\cos(\frac{4i\pi}{n})| \\
&= \sum_{i=1}^{n} |16\cos(\frac{4i\pi}{n})|.
\end{aligned}
$$

The following example is provided to illustrate the previous theorem.

Example 3. *Let D_{12} be the dihedral group of order 12, where $D_{12} = \left\langle a, b | a^6 = b^2 = 1, bab = a^{-1} \right\rangle$ and $X^{(2)} = \{a^2, a^4\}$ be the subset of D_{12} of valency two. Then, the Cayley graph of D_{12} with respect to $X^{(2)}$, denoted by $Cay(D_{12}, \{a^2, a^4\})$ is the union of four cycle graphs of three vertices, $4C_3$. Therefore, the Seidel energy of the graph, $\sum_{i=1}^{6} |16\cos(\frac{4i\pi}{6})| = \sum_{i=1}^{6} |16\cos(\frac{2i\pi}{3})|.$*

Next theorem provides the Cayley graph for the fourth case which is for the subset $\{a^{\frac{n}{2}}, a^i b\}$ while n is even.

Theorem 14. *Let D_{2n} be the dihedral group of order $2n$, where $n \geq 4$ and n is even. Let $X^{(2)} = \{a^{\frac{n}{2}}, a^i b\}$ be a subset of valency two of D_{2n}. Then, the Seidel energy of the Cayley graphs $Cay(D_{2n}, \{a^{\frac{n}{2}}, a^i b\})$ is $\sum_{i=0}^{3} |2n\cos(\frac{\pi i}{2})|.$*

Proof. Suppose D_{2n} is the dihedral group of order $2n$, where $n \geq 3$ and n is even. Let $X^{(2)} = \{a^{\frac{n}{2}}, a^i b\}$ be a subset of valency two of D_{2n}. The Cayley graph of D_{2n} with respect to the subset $\{a^{\frac{n}{2}}, a^i b\}$, denoted as $Cay(D_{2n}, \{a^{\frac{n}{2}}, a^i b\})$ is $\frac{n}{2}C_4$. It is easy to see that the eigenvalues of $Cay(D_{2n}, \{a^{\frac{n}{2}}, a^i b\})$ are $\lambda_i = \{2\cos(\frac{\pi j}{2}); j = 0, 1, 2, 3\}$ with multiplicity $\frac{n}{2}$. By using the same step as in the previous proof, the Seidel energy of the Cayley graphs $Cay(D_{2n}, \{a^{\frac{n}{2}}, a^i b\})$ is as given in the following:

$$
\begin{aligned}
SE(Cay(D_{2n}, \{a^{\frac{n}{2}}, a^i b\})) &= \sum_{i=1}^{n} |\theta_i| \\
&= \sum_{i=0}^{3} | -2n\cos(\frac{\pi i}{2})| \\
&= \sum_{i=0}^{3} |2n\cos(\frac{\pi i}{2})|.
\end{aligned}
$$

An example is provided to illustrate the previous theorem.

Example 4. *Let D_8 be the dihedral group of order 8 where $D_8 = < a, b | a^4 = b^2 = 1, bab = a^{-1} >$ and $X = \{a^2, b\}$ be a subset of D_8. The Cayley graph of D_8 with respect to the subset X, $Cay(D_8, \{a^2, b\})$ are the cycle graph $2C_4$. Therefore, the Seidel energy of the graph is $\sum_{i=0}^{3} |8\cos(\frac{\pi i}{2})|.$*

3.3. The Seidel Energy of the Cayley Graphs Associated to the Generating Subset of Dihedral Group

The followings are the lemmas and theorems for the Seidel eigenvalues and Seidel energy of the Cayley graph with respect to the generating subset associated to dihedral group.

Lemma 4. *Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X = \{b, ab, \ldots, a^{n-1}b\}$ be the generating subset of D_{2n}. Therefore, the Seidel eigenvalues of $Cay(D_{2n}, \{b, ab, \ldots, a^{n-1}b\})$ are $2n - 1$ with multiplicity 1 and -1 with multiplicity $2n - 1$.*

Proof. Consider the dihedral group D_{2n} of order $2n$ where $n \geq 3$ and $X = \{b, ab, \ldots, a^{n-1}b\}$ is the generating subset of D_{2n}. From Proposition 8, the Cayley graphs of D_{2n} with respect to the generating subset X, $Cay(D_{2n}, \{b, ab, \ldots, a^{n-1}b\})$ are the complete bipartite graph $K_{n,n}$. Recalled from Proposition 6, since the Seidel spectrum of a complete bipartite graph $K_{m,n}$ is $\{(m + n - 1)^1, (-1)^{m+n-1}\}$, then since in this case, $m = n$ for $K_{n,n}$, thus the Seidel spectrum $Spec(K_{n,n}) = \{(n + n - 1)^1, (-1)^{n+n-1}\} = \{(2n - 1)^1, (-1)^{2n-1}\}$. Therefore, the Seidel eigenvalues are $\theta_i = 2n - 1$ with multiplicity 1 and $\theta_i = -1$ with multiplicity $2n - 1$.

Theorem 15. *Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X = \{b, ab, \ldots, a^{n-1}b\}$ be the generating subset of D_{2n}. The Seidel energy of the Cayley graphs of D_{2n} with respect to the generating subset X, $SE(Cay(D_{2n}, \{b, ab, \ldots, a^{n-1}b\})) = 4n - 2$.*

Proof. Consider the dihedral group D_{2n} of order $2n$ where $n \geq 3$ and $X = \{b, ab, \ldots, a^{n-1}b\}$ is the generating subset of D_{2n}. From Proposition 8 and Lemma 4, the Cayley graphs of D_{2n} with respect to the generating subset X, $Cay(D_{2n}, \{b, ab, \ldots, a^{n-1}b\})$ are the complete bipartite graph $K_{n,n}$ with Seidel eigenvalues $\theta_i = 2n - 1$ with multiplicity 1 and $\theta_i = -1$ with multiplicity $2n - 1$. Therefore, the Seidel energy of the Cayley graphs of D_{2n} with respect to the generating subset X, $SE(Cay(D_{2n}, \{b, ab, \ldots, a^{n-1}b\})) = (1)|(2n-1)| + (2n-1)|(-1)| = 2n - 1 + 2n - 1 = 4n - 2$.

The following example is provided to illustrate the previous theorem.

Example 5. *Let D_6 be the dihedral group of order 6, where $D_6 = \langle a, b | a^3 = b^2 = 1, bab = a^{-1} \rangle$ and $X = \{b, ab, a^2b\}$ be the generating subset of D_6. From Proposition 8, $Cay(D_{2n}, \{b, ab, \ldots, a^{n-1}b\}) = K_{n,n}$ gives the $Cay(D_6, \{b, ab, a^2b\})$ as complete bipartite graph $K_{3,3}$. By Lemma 4, the Seidel eigenvalues of $Cay(D_6, \{b, ab, a^2b\})$ are $\theta_i = 5$ with multiplicity 1 and $\theta_i = -1$ with multiplicity 5. This gives the Seidel energy of the Cayley graphs, $SE(Cay(D_6, \{b, ab, a^2b\})) = 4(3) - 2 = 10$.*

Conclusion

As conclusion, the Seidel energy of the Cayley graphs associated to the dihedral groups D_{2n}, with respect to subsets of order one and two as well as the generating set are computed are summarized in the following table.

Table 1. The Seidel energy of the Cayley graphs with respect to subsets of order one and two of dihedral groups, $D_{2n}, (n \geq 3)$

Group	Order	Cases	Seidel Energy
$D_{2n}, (n \geq 3)$	1	$\{a \mid \text{for all } a \in D_{2n}\}$	$2n$
	2	$\{a, a^{n-1}\}$	$\sum_{i=1}^{n} \lvert 8\cos(\frac{2\pi i}{n}) \rvert$
		$\{a^2, a^{n-2}\}$ for n odd	$\sum_{i=1}^{n} \lvert 8\cos(\frac{2\pi i}{n}) \rvert$
		$\{a^2, a^{n-2}\}$ for n even	$\sum_{i=1}^{n} \lvert 16\cos(\frac{4\pi i}{n}) \rvert$
		$\{a^{\frac{n}{2}}, a^i b\}$	$\sum_{i=0}^{3} \lvert 2n\cos(\frac{\pi i}{2}) \rvert$
	Generating set	$\{b, ab, \ldots, a^{n-1}b\}$	$4n - 2$

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