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Independence polynomial of the commuting and noncommuting graphs associated to the quasidihedral group

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Abstract. An independence polynomial is a type of graph polynomial from graph theory that store combinatorial information such as the graph properties or graph invariants. The independence polynomial of a graph contains coefficients that represent the number of independent sets of certain sizes and the degree of the polynomial denotes the independence number of the graph. A graph of group G is called commuting graph if the vertices are noncentral elements of G and two vertices are adjacent if and only if they commute in G. Meanwhile, a noncommuting graph of a group G has a vertex set that contains all noncentral elements of G and two vertices are adjacent if and only if they do not commute in G. Since the group properties can be presented as graph from graph theory, then the graph polynomial of such graph should also be identified. Therefore, in this research, the independence polynomials are determined for the commuting and noncommuting graphs that are associated to the quasidihedral group.

Keywords: Graph polynomial; independence polynomial; graph theory; group theory.

1. Introduction

A graph, $\Gamma = \left(V(\Gamma), E(\Gamma)\right)$ consists of $V(\Gamma)$, a nonempty set of vertices and $E(\Gamma)$, a set of edges which are unordered pairs of vertices from $V(\Gamma)$. Two vertices u and v in a graph Γ are called adjacent in Γ if u and v are connected by an edge e of Γ. Such an edge is called incident with the vertices u and v [1]. Throughout this research, the graphs considered are simple graphs in which each graph contains no loops and no two edges is incident with the same pair of vertices.

If (u, v) is an edge of graph Γ, then the vertex u is also called as the neighbor of the vertex v. Open neighborhood (or just neighborhood) of v is the set of all vertices in Γ that are adjacent to v, denoted as, $N(v) = \{u \in V(\Gamma) : (u, v) \in E, u \neq v\}$. Closed neighborhood of v in Γ is the set $N[v] = N(v) \cup \{v\}$ [2]. If the neighborhood of every $v \in V(\Gamma)$ is empty, then there is no edge in the graph. Such graph is named empty graph, E_n. If $n = 0$, then the graph is called a null graph [1].

Some common types of graphs from graph theory include the complete graph and the complete multipartite graph. A complete graph, K_n, is a graph with n vertices in which between each

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pair of distinct vertices contains only one edge. Meanwhile, a complete multipartite graph, $K_{n_1,n_2,\ldots,n_t}$, is a graph that has vertices partitioned into t subsets, each containing $n_1, n_2, \ldots, n_t$ elements, respectively, and vertices are adjacent if and only if they are from different subsets in the partition [1]. In addition to that, if $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are two simple graphs, then, the union of Γ_1 and Γ_2, denoted by $\Gamma_1 \cup \Gamma_2$, is the graph $\Gamma = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Furthermore, the complement of Γ, denoted by $\overline{\Gamma}$, contains vertex set, $V(\overline{\Gamma})$ equals to $V(\Gamma)$ and two vertices u and v are adjacent in $\overline{\Gamma}$ if and only if they are nonadjacent in Γ [2]. In fact, the complement of the disjoint union of complete graphs is the complete multipartite graph [3].

On top of that, graph polynomial is a polynomial that has been studied in graph theory such that it can store combinatorial information related to the structure of a graph. The independence polynomial is a type of graph polynomial that contains information related to the independent sets and independence number of a graph. Note that, an independent set of a graph Γ is the set of vertices in Γ such that no two distinct vertices are adjacent. Meanwhile, the independence number of a graph Γ, denoted by $\alpha(\Gamma)$, is the maximum number of vertices in an independent set of the graph [1]. Initially, the concept of the independent set polynomial by Hoede and Li in [4] turned out to be similar to the idea of the independence polynomial by Gutman and Harary in [5]. However, later, the name used in subsequent researches has always been the independence polynomial, in which Hoede and Li [4] have simplified its definition. An independence polynomial of a graph Γ is defined as the polynomial whose coefficient on x^k is given by the number of independent sets of size k in Γ. It is denoted as follows:

$$I(\Gamma; x) = \sum_{k=0}^{\alpha(\Gamma)} a_k x^k,$$

where a_k is the number of independent sets of size k in Γ and $\alpha(\Gamma)$ is the independence number of Γ [4].

Then, in [6] and [7], Levit and Mandrescu discussed on various results related to the independence polynomial and studied on the independence polynomial specifically for the well-covered graphs. Furthermore, Ferrin [8] presented the independence polynomials of some common types of graphs such as the book graph, the complete graph and the star graph, together with obtaining certain bound of roots for the independence polynomial. In general, previous works on the roots of certain types of polynomials have appealing geometrical interpretation and have led to major progress in other study areas like numerical algebraic geometries [9], polynomial stability [10] and theoretical computer science [11]. Makowsky *et al.* [12] deduced that the roots of graph polynomials received much concern and are significant when these polynomials model physical reality. Several preliminary works state that the location of roots of graph polynomials can give information about the structures or the families of the graphs such as in [13, 14].

Other than the graphs in graph theory itself, there are also graphs that are related to group theory. The algebraic properties of groups can be analyzed and described by using the graph properties from graph theory. Among graphs that are associated to groups include the commuting graph and the noncommuting graph of a group. The commuting graph of a group G, denoted as Γ_G^{comm}, is defined as the graph whose vertices are the noncentral elements of G, that is $|V(\Gamma_G^{comm})| = |G| - |Z(G)|$ in which two distinct vertices are adjacent if and only if they commute in G [15]. Meanwhile, the noncommuting graph of a group G, denoted as Γ_G^{nc}, is defined as the graph whose vertex set contains the noncentral elements of G, that is $|V(\Gamma_G^{nc})| = |G| - |Z(G)|$ in which two distinct vertices are adjacent if and only if they do not commute in G [16].

Apart from determining certain graph invariants, namely the independence number, the chromatic number and the clique number for those graphs related to group theory, there are also researches that have established other properties for the graphs associated to groups such as the genus [17], the topological indices [18] and some types of energies [19] .

The independence polynomial can also be determined for those graphs that are associated to group. However, a group that has many elements can have larger number of vertices in the graph that will lead to difficulties in computing its independence polynomial, one by one. Therefore, this research is interested to simplify the process by determining the general form of the independence polynomials for the commuting and noncommuting graphs associated to the quasidihedral group. The quasidihedral group, also called semidihedral group, is a nonabelian group of order 2^n in which $n \geq 4, n \in \mathbb{N}$. This group, denoted as QD_{2^n}, can be expressed in a group presentation, $\left\langle a, b : a^{2^{(n-1)}} = b^2 = 1, bab^{-1} = a^{2^{(n-2)}-1} \right\rangle$. The center of QD_{2^n} is denoted as $Z\left(QD_{2^n}\right) = \left\{1, a^{2^{(n-2)}}\right\}$ [20].

2. Preliminaries

The following are some preliminaries in graph theory that are needed to assist in obtaining the results of this research. To begin with, the following theorem states the independence polynomial for the union of two graphs.

Theorem 1. [8] *Let Γ_1 and Γ_2 be two vertex-disjoint graphs. The independence polynomial for the union of two graphs is stated as follows:*

$$I\left(\Gamma_1 \cup \Gamma_2; x\right) = I\left(\Gamma_1; x\right) \cdot I\left(\Gamma_2; x\right).$$

Next, some properties related to the independence polynomial of graph are given in the following propositions.

Proposition 1. [8] *The independence polynomial of a complete graph on n vertices is*

$$I(K_n; x) = 1 + nx.$$

Proposition 2. [8] *The independence polynomial of m complete graphs each of n_i vertices is*

$$I\left(\bigcup_{i=1}^{m} K_{n_i}; x\right) = \prod_{i=1}^{m}\left(1 + n_i x\right).$$

Additionally, Hoede and Li [4] have stated that the independence polynomial of a graph is equal to the clique polynomial of the complement of the graph. Since the clique polynomial of the union of two vertex-disjoint graphs, Γ_1 and Γ_2 is given as $C(\Gamma_1 \cup \Gamma_2; x) = C\left(\Gamma_1; x\right) + C\left(\Gamma_2; x\right) - 1$, then the following proposition is obtained.

Proposition 3. *Let $\Gamma = K_{n_1, n_2, \ldots, n_t}$ be a complete multipartite graph with $n_1 + n_2 + \ldots + n_t$ vertices. Then the independence polynomial of Γ,*

$$I\left(K_{n_1, n_2, \ldots, n_t}; x\right) = \left(1 + x\right)^{n_1} + \left(1 + x\right)^{n_2} + \ldots + \left(1 + x\right)^{n_t} - \left(t - 1\right).$$

Proof. Suppose that the complement of a complete multipartite graph is the union of some complete graphs and the independence polynomial of a graph is equal to the clique polynomial of the complement of the graph. Accordingly, the independence polynomial of the complete multipartite graph is determined as follows:

$$\begin{aligned}
I\Big(K_{n_1,n_2,\ldots,n_t};x\Big) &= C\Big(\overline{K_{n_1,n_2,\ldots,n_t}};x\Big)\\
&= C\Big(K_{n_1}\cup K_{n_2}\cup\ldots\cup K_{n_t};x\Big)\\
&= C\Big(K_{n_1};x\Big)+C\Big(K_{n_2}\cup K_{n_3}\cup\ldots\cup K_{n_t};x\Big)-1\\
&= C\Big(K_{n_1};x\Big)+C\Big(K_{n_2};x\Big)+C\Big(K_{n_3}\cup\ldots\cup K_{n_t};x\Big)-1-1\\
&= C\Big(K_{n_1};x\Big)+C\Big(K_{n_2};x\Big)+C\Big(K_{n_3};x\Big)+\\
&\quad\; C\Big(K_{n_4}\cup\ldots\cup K_{n_t};x\Big)-1-1-1\\
&= C\Big(K_{n_1};x\Big)+C\Big(K_{n_2};x\Big)+C\Big(K_{n_3};x\Big)+C\Big(K_{n_4};x\Big)+\\
&\quad\; C\Big(K_{n_4}\cup\ldots\cup K_{n_t};x\Big)-1-1-1-1\\
&= C\Big(K_{n_1};x\Big)+C\Big(K_{n_2};x\Big)+\ldots+C\Big(K_{n_t};x\Big)-\\
&\quad\; \underbrace{1-1-\ldots-1}_{t-1\text{ times}}\\
&= \big(1+x\big)^{n-1}+\big(1+x\big)^{n_2}+\ldots+\big(1+x\big)^{n_t}-\big(t-1\big).
\end{aligned}$$

Apart from that, from previous researches, based on the algebraic properties of groups, graphs associated to finite groups can be constructed and are expressed in general forms. For instance, the commuting graph and the noncommuting graph associated to QD_{2^n} consist of certain types of graphs from graph theory, namely the union of some complete graphs and the multipartite graph, respectively. Both graphs are determined by Dutta and Nath [21] and can be stated as in the following theorems.

Theorem 2. *Suppose that QD_{2^n} is the quasidihedral group of order 2^n, where $n \geq 4$, $n \in \mathbb{N}$. Then the commuting graph of QD_{2^n},*

$$\Gamma^{comm}_{QD_{2^n}} = \left(\bigcup_{i=1}^{2^{(n-2)}} K_2\right) \cup K_{2^{(n-1)}-2}.$$

Theorem 3. *Suppose that QD_{2^n} is the quasidihedral group of order 2^n, where $n \geq 4$, $n \in \mathbb{N}$. Then the noncommuting graph of QD_{2^n} ,*

$$\Gamma^{nc}_{QD_{2^n}} = K_{\underbrace{2,2,\ldots,2}_{2^{(n-2)}\ times},2^{(n-1)}-2}.$$

Other than obtaining the independence polynomials associated to the quasidihedral group, this research is also interested in establishing the roots for the independence polynomials obtained. In order to do that, since a polynomial is always a continuous function, then the following theorems will be applied.

Theorem 4. [22] *Suppose that a function f is continuous at each point of a closed interval $[a, b]$ such that $f(a) \cdot f(b) < 0$. Then there is at least one $c \in (a, b)$ such that $f(c) = 0$.*

Theorem 5. [23] *Suppose that f is a function and f' is its derivative. If $f'(x) > 0$ on an interval, then f is increasing on that interval. Meanwhile, if $f'(x) < 0$ on an interval, then f is decreasing on that interval.*

Theorem 6. [23] *Suppose that c is a critical point of a continuous function f and let f' be the derivative of f. The point c is in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.*

3. Main Results

This section is divided into two subsections. The first subsection is on the independence polynomial of the commuting graph associated to the quasidihedral group and the second subsection is on the independence polynomial of the noncommuting graph associated to the quasidihedral group.

3.1. The Independence Polynomial for the Commuting Graph of Quasidihedral Group

The following theorem presents the general form of the independence polynomial for the commuting graph of quasidihedral group.

Theorem 7. *Suppose that QD_{2^n} is the quasidihedral group of order 2^n, $n \geq 4$, $n \in \mathbb{N}$ and $\Gamma^{comm}_{QD_{2^n}}$ is its commuting graph. Then, the independence polynomial of $\Gamma^{comm}_{QD_{2^n}}$,*

$$I(\Gamma^{comm}_{QD_{2^n}}; x) = \left(1 + \left(2^{(n-1)} - 2\right) x\right)\left(1 + 2x\right)^{2^{(n-2)}}.$$

Proof. Let QD_{2^n} be the quasidihedral group of order 2^n, $n \geq 4$, $n \in \mathbb{N}$ and $\Gamma^{comm}_{QD_{2^n}}$ be its commuting graph. From Theorem 2, $\Gamma^{comm}_{QD_{2^n}} = \left(K_{2^{(n-1)}-2}\right) \cup \left(\bigcup_{i=1}^{2^{(n-2)}} K_2\right)$. By using Theorem 1, Proposition 1 and Proposition 2, the independence polynomial of $\Gamma^{comm}_{QD_{2^n}}$ is obtained as follows:

$$\begin{aligned} I(\Gamma^{comm}_{QD_{2^n}}; x) &= I \quad \left(\left(K_{2^{(n-1)}-2}\right) \cup \left(\bigcup_{i=1}^{2^{(n-2)}} K_2\right); x\right) \\ &= I\left(K_{2^{(n-1)}-2}; x\right) \cdot I\left(\left(\bigcup_{i=1}^{2^{(n-2)}} K_2\right); x\right) \\ &= \left[1 + (2^{(n-1)} - 2)x\right]\left(1 + 2x\right)^{2^{(n-2)}}. \end{aligned}$$

Next, the roots for the independence polynomial of the commuting graph for quasidihedral group are determined.

Theorem 8. *Suppose that QD_{2^n} is the quasidihedral group of order 2^n, $n \geq 4$, $n \in \mathbb{N}$ and $\Gamma^{comm}_{QD_{2^n}}$ is its commuting graph. Then, the independence polynomial of $\Gamma^{comm}_{QD_{2^n}}$, denoted by $I(\Gamma^{comm}_{QD_{2^n}}; x)$, has two non-integer roots, namely $-\frac{1}{2^{(n-1)}-2}$ and $-\frac{1}{2}$.*

Proof. Let QD_{2^n} be the quasidihedral group of order 2^n, $n \geq 4$, $n \in \mathbb{N}$ and $\Gamma^{comm}_{QD_{2^n}}$ be its commuting graph. By Theorem 7, the independence polynomial of $\Gamma^{comm}_{QD_{2^n}}$ is denoted by $I(\Gamma^{comm}_{QD_{2^n}}; x) = \left(1 + \left(2^{(n-1)} - 2\right) x\right)\left(1 + 2x\right)^{2^{(n-2)}}$. Let $I(\Gamma^{comm}_{QD_{2^n}}; x) = 0$, then either $1 + \left(2^{(n-1)} - 2\right) x = 0$ or $\left(1 + 2x\right)^{2^{(n-2)}} = 0$. Accordingly, from those two conditions, the roots obtained are $-\frac{1}{2^{(n-1)}-2}$ and $-\frac{1}{2}$.

3.2. The Independence Polynomial for the Noncommuting Graph of Quasidihedral Group

The following theorem presents the general form of the independence polynomial for the noncommuting graph of quasidihedral group.

Theorem 9. *Suppose that QD_{2^n} is the quasidihedral group of order 2^n where $n \geq 4$, $n \in \mathbb{N}$. Then the independence polynomial of the noncommuting graph of QD_{2^n},*

$$I(\Gamma^{nc}_{QD_{2^n}}; x) = \left(2^{(n-2)}\right)\left(2x + x^2\right) + \left(1 + x\right)^{2^{(n-1)}-2}.$$

Proof. Let QD_{2^n} be the quasidihedral group of order 2^n, $n \geq 4$, $n \in \mathbb{N}$ and $\Gamma^{nc}_{QD_{2^n}}$ be its noncommuting graph. From Theorem 3, $\Gamma^{nc}_{QD_{2^n}} = K_{\underbrace{2,2,\ldots,2}_{2^{(n-2)}\text{ times}},(2^{(n-1)}-2)}$.

By Proposition 3, $I\left(K_{n_1,n_2,\ldots,n_t}; x\right) = (1+x)^{n_1} + (1+x)^{n_2} + \ldots + (1+x)^{n_t} - (t-1)$. Hence, the independence polynomial of $\Gamma^{nc}_{QD_{2^n}}$ is determined as follows:

$$\begin{aligned}
I(\Gamma^{nc}_{QD_{2^n}}; x) &= I\left(K_{\underbrace{2,2,\ldots,2}_{2^{(n-2)}\text{ times}},(2^{(n-1)}-2)}; x\right) \\
&= \underbrace{(1+x)^2 + (1+x)^2 + \ldots + (1+x)^2}_{2^{(n-2)}\text{ times}} + (1+x)^{2^{(n-1)}-2} - \\
&\quad \left[\left(2^{(n-2)} + 1\right) - 1\right] \\
&= \left(2^{(n-2)}\right)(1+x)^2 + (1+x)^{2^{(n-1)}-2} - 2^{(n-2)} \\
&= \left(2^{(n-2)}\right)\left(1 + 2x + x^2 - 1\right) + (1+x)^{2^{(n-1)}-2} \\
&= \left(2^{(n-2)}\right)\left(2x + x^2\right) + (1+x)^{2^{(n-1)}-2}.
\end{aligned}$$

Next, the roots for the independence polynomial of the noncommuting graph for quasidihedral group are determined.

Theorem 10. *Suppose that QD_{2^n} is the quasidihedral group of order 2^n, $n \geq 4$, $n \in \mathbb{N}$ and $\Gamma^{nc}_{QD_{2^n}}$ is its noncommuting graph. Then, the independence polynomial of $\Gamma^{nc}_{QD_{2^n}}$, denoted by $I(\Gamma^{nc}_{QD_{2^n}}; x)$, has two non-integer roots, call them $x_1$and x_2, such that $0 > x_1 > -1 > x_2 > -2$.*

Proof. Let QD_{2^n} be the quasidihedral group of order 2^n, $n \geq 4$, $n \in \mathbb{N}$ and $\Gamma^{nc}_{QD_{2^n}}$ be its noncommuting graph. From Theorem 9, the independence polynomial of $\Gamma^{nc}_{QD_{2^n}}$ is denoted as $I(\Gamma^{nc}_{QD_{2^n}}; x) = \left(2^{(n-2)}\right)\left(2x + x^2\right) + (1+x)^{2^{(n-1)}-2}$. Let the polynomial be $f(x)$ and its derivative, $f'(x) = \left(2^{(n-2)}\right)(2 + 2x) + \left(2^{(n-1)} - 2\right)(1+x)^{2^{(n-1)}-3}$. Then,

$$\begin{aligned}
f(0) &= \left(2^{(n-2)}\right)\left(2(0) + 0^2\right) + (1+0)^{2^{(n-1)}-2} = 1\,, \\
f(-1) &= \left(2^{(n-2)}\right)\left(2(-1) + (-1)^2\right) + (1-1)^{2^{(n-1)}-2} = -(2^{(n-2)})\,, \\
f(-2) &= \left(2^{(n-2)}\right)\left(2(-2) + (-2)^2\right) + (1-2)^{2^{(n-1)}-2} = (-1)^{2^{(n-1)}-2} = 1\,.
\end{aligned}$$

Since $f(-1) \cdot f(0) < 0$ and $f(-2) \cdot f(-1) < 0$, then by Theorem 4, there exist roots $x_1 \in (-1, 0)$ and $x_2 \in (-2, -1)$, such that $f(x_1) = 0$ and $f(x_2) = 0$, respectively. Note that, $f'(-1) = \left(2^{(n-2)}\right)(2 + 2(-1)) + \left(2^{(n-1)} - 2\right)(1-1)^{2^{(n-1)}-3} = 0$ and thus, by Theorem 6, f has a critical point at -1. Furthermore, for every $x \in (0, \infty)$, $f'(x) > 0$ since all the terms in $f'(x)$ become positive. Therefore, by Theorem 5, f is always increasing on the interval $(0, \infty)$. Meanwhile, for every $x \in (-\infty, -2)$, since $\left(2^{(n-2)}\right)(2+2x) < 0$ and $\left(2^{(n-1)} - 2\right)(1+x)^{2^{(n-1)}-3} < 0$, then $f'(x) < 0$. Thus, f is always decreasing on the interval $(-\infty, -2)$. Hence, there are only two non-integer roots for $I(\Gamma^{nc}_{QD_{2^n}}; x)$.

4. Conclusion

In this paper, the general form of the independence polynomials for the commuting graph and noncommuting graph are established for the quasidihedral group. The degree of each independence polynomial is the independence number for the graphs, respectively, and always depending on the values of n. For the independence polynomial of the commuting graph, the number of independent sets of size one is found to be the number of vertices of the graph, namely $\left|V(\Gamma^{comm}_{QD_{2^n}})\right| = 2^n - 2$. Similarly, for the independence polynomial of the noncommuting graph, the number of independent sets of size one is also representing the number of vertices of the graph, namely $\left|V(\Gamma^{nc}_{QD_{2^n}})\right| = 2^n - 2$. This is because each vertex of the graph always belong to the independent sets of size one and hence, there are as many independent sets of size one as the number of vertices. Furthermore, the roots established for the independence polynomials obtained in this research are all not integers. The independence polynomial for the commuting graph of quasidihedral group has roots that can be determined in exact form. Meanwhile, the roots established for the independence polynomial for the noncommuting graph of quasidihedral group is obtained in the form of bound since the exact roots cannot be determined.

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