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The non-zero divisor graph of ring of integers modulo six and the Hamiltonian quaternion over integers modulo two

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Abstract. The study of graph theory was introduced and widely researched since many practical problems can be represented by graphs. A non-zero divisor graph is a graph in which its set of vertices is the non-zero elements of the ring and the vertices x and y are adjacent if and only if $xy \neq 0$. In this study, we introduced the non-zero divisor graphs of some finite commutative rings in specific the ring of integers modulo 6, $\mathbb{Z}_6$ and ring of Hamiltonian quaternion, $\mathbb{H}(\mathbb{Z}_2)$. First, the non-zero divisors of the commutative rings are found. Then, the non-zero divisor graphs are constructed. Finally, some properties of the graph, including the chromatic number, clique number, girth and the diameter are obtained.

1.0 Introduction

A ring is one of the fundamental algebraic structures consisting of a set with two binary operations known as multiplication and addition [1]. The structure of rings, their representations, special classes of rings and applications are studied in ring theory, generally. One of the most familiar examples of a ring is the ring of integers, $\mathbb{Z}$.The concepts of ring have been studied extensively by many researchers such as Cohn [1], Rotman [2], Harley and Hawkes [3] and Bourbaki [4]. A ring in which the multiplication is commutative is a commutative ring [5].

Meanwhile, most real-world situations can be described as diagrams consisting of a set of points together with lines joining certain pairs of these points. The mathematical abstraction of these situations has recently lead to the rise of researches in graphs. The concept of graph theory has many applications such as in industrial management, linear programming, game theory, statistical mechanics as well as fuzzy theory [6]. In fact, graph serves as a mathematical model for any system involving a binary relation. In this study, some concepts of rings are then applied to graph theory.

The concept of graphs related to commutative and non-commutative rings has been discovered by many researchers. The examples of graph related to rings are total graph, comaximal graph and zero-divisor graph. Throughout this research, ring theory is particularly applied to zero divisor graph. Beck [7] introduced idea of presenting zero-divisor graph of a commutative ring in 1988. Then, Anderson and Livingston [8] used the results obtained in the Beck's article and simplified Beck's definition of zero divisor graph. Anderson and Livingston defined the zero divisor graph as an undirected graph with the vertices of all zero divisors of the ring and two vertices are connected if the multiplication of these elements is equal to zero.

In 2003, the zero-divisor graph of a commutative Von Neumann regular ring is investigated by Anderson et al [9]. Von Neumann regular ring is a ring in which for each element of in the ring, there exists an element in the ring such that $x = x^2y$. In the following years, Redmond [10] has been extending the graph concept to non-commutative rings. He has been focusing on the directed zero divisor graphs and undirected zero divisor graphs of a non-commutative rings.

In this study, while considerable attention has been given to graph theory and ring theory, there is moderated research available in the literature on the properties of non-zero divisor graphs over some finite rings. Hence, the focus of this study is based on the determination and construction of the non-zero divisor graph of finite commutative rings and their graph properties.

This paper consists of four sections. The first section is the introduction, followed by preliminaries. Then, results and discussions are included in the next section. In the section, the non-zero divisors, non-zero divisor graphs and their properties are found. Finally, in the last section, all results obtained are summarized.

2.0 Preliminaries

In this section, some definitions and properties related to the main topic are presented.

Definition 1 [5] *Zero Divisor in a Ring*
If a and b are two nonzero elements of a ring R such that $ab = 0$, then a and b are called divisors of zero in R.

Next, some definitions related to graph theory are presented in this section. In the history of graph theory, the first article was published by Leonhard Euler on Konigsberg's Seven Bridges, where he investigated whether it is possible to follow a path that crosses each bridge exactly once and returns to the starting point [11]. This real world situation leads to the rise of graphs and its applications.

Definition 2 [12] *Graph*
A finite graph, denoted as Γ is an object consisting of two sets called the vertex set and edge set. The elements of $V(\Gamma)$, vertex set are called vertices while the elements of $E(\Gamma)$, edge set are called the edges.

Definition 3 [7] *Zero-Divisor Graph*
The zero divisor graph of R, denoted as $\Gamma(R)$, is a simple graph with vertices of all non-zero elements of a commutative ring, R such that two different elements x and y are adjacent if and only if $xy = 0$.

In this study, some graph properties will be further found after obtaining the non-zero divisor graph. The definition of graph properties are given in the following, starting with the clique number.

Definition 4 [13] *The Clique Number of* Γ
A complete subset of a graph, Γ_{sub} is called a clique. The maximum size of a clique in an undirected graph Γ is called the clique number of Γ and denoted by $\omega(\Gamma)$.

Definition 5 [13] *The Chromatic Number of* Γ
The smallest number of colors needed to color the vertices of Γ so that no two adjacent vertices share the same color is the chromatic number and usually denoted by $\chi(\Gamma)$.

Definition 6 [14] *The Diameter of* Γ
The diameter of Γ*, denoted as* $diam(\Gamma)$ *is the supremum of the degree from* x *to* y *where* x *and* y *are vertices of the graph.*

Definition 7 [14] *The Girth of* Γ
The girth of Γ*, denoted by* $girth(\Gamma)$*, is the length of a shortest cycle in* Γ*.*

3.0 Results And Discussions

In this section, the non-zero divisors and their graphs of some rings are discussed. The commutative rings that are being considered in this study are the ring of integers modulo six, denoted as $\mathbb{Z}_6$ and the ring of Hamiltonian quaternion over integers modulo two, denoted as $\mathbb{H}(\mathbb{Z}_2)$. The non-zero divisor graphs are then constructed for each ring.

3.1 The Non-Zero Divisor of Some Finite Commutative Rings

In this subsection, the non-zero divisors are calculated for commutative rings. The zero divisors will be determined first by using Definition 1. The propositions and proofs are discussed below.

Proposition 1. Let $\mathbb{Z}_6$ be the ring of integers modulo six. There are 21 pairs of elements where the multiplication of any two non-zero elements is not equal to zero.
Proof. To find the pairs of elements where the multiplication of these non-zero elements is not equal to zero, it is much more easier if the zero divisors are found first. By Definition 1, the set of zero divisors of $\mathbb{Z}_6$ are $\{2, 3, 4\}$ since $2 \cdot 3 = 3 \cdot 2$ and $3 \cdot 4 = 4 \cdot 3$. Since there are only five non-zero elements in $\mathbb{Z}_6$, the total pairs of elements are $5 \cdot 5 = 25$. Therefore, the pairs of elements where the multiplication of these pairs of those non-zero elements is not equal to zero is $25 - 4$ (pairs of elements from zero divisors) $= 21$.

Proposition 2. Let $\mathbb{H}(\mathbb{Z}_2)$ be the commutative ring of Hamiltonian quaternion in the form of $m + ni + sj + tk$ where $m, n, s, t \in \mathbb{Z}_2$. There are 207 pairs of elements where the multiplication of these non-zero elements is not equal to zero.
Proof. In group theory, the arbitrary elements of $\mathbb{H}(\mathbb{Z}_2)$ is in the form of $m + ni + sj + tk$ such that $m, n, s, t \in \mathbb{Z}_2$. Therefore, $\mathbb{H}(\mathbb{Z}_2)$ has 16 elements which are listed as follows:

$$\begin{aligned}\mathbb{H}(\mathbb{Z}_2) = \{&0, 1, j, k, i, i+j, i+k, j+k, 1+i, 1+j, 1+k, i+j+k, 1+i+j,\\ &1+i+k, 1+k+j, 1+i+j+k\}.\end{aligned}$$

By Definition 1, it is found that the set of zero divisors of $\mathbb{H}(\mathbb{Z}_2)$ is $\{1+i, 1+j, 1+k, i+j, i+k, j+k, 1+i+j+k\}$. There are nine pairs of elements in the ring that have product zero which are:

$$\begin{aligned}\{(p,q)|pq = qp = 0\} = \{&(1+i, j+k), (j+k, 1+i+j+k), (i+k, 1+i+j+k), (i+k, 1+j),\\ &(i+j, 1+k), (i+j, 1+i+j+k), (1+k, 1+i+j+k),\\ &(1+j, 1+i+j+k), (1+i, 1+i+j+k)\}.\end{aligned}$$

Since there are 15 non-zero elements in $\mathbb{H}(\mathbb{Z}_2)$, the total pairs of elements are $15 \cdot 15 = 225$. Therefore, the pairs of elements where the multiplication of these pairs of those non-zero elements is not equal to zero is $225 - 18$ (commuting pairs of elements from zero divisors) $= 207$.

3.2 The Non-Zero Divisor Graphs of Some Finite Commutative Rings

The non-zero divisor graphs of the commutative rings are constructed based on the non-zero divisors found in the previous subsection. The graph properties which are the chromatic and clique number, diameter and girth are obtained after constructing the non-zero divisor graphs. The propositions and proofs are discussed below.

In this research, a new notion of non-zero divisor graph is introduced in the following.

Definition 8 *The non-zero divisor graph of R, denoted by $\Gamma(R)$, is a simple graph with vertices of all non-zero elements of a ring, R such that two distinct elements x and y are adjacent if and only if $xy \neq 0$.*

Proposition 3. Let $\Gamma(\mathbb{Z}_6)$ be the non-zero divisor graph of $\mathbb{Z}_6$, then, $\Gamma(\mathbb{Z}_6)$ is an undirected graph with 5 vertices and 8 edges and the clique number with chromatic number of $\Gamma(\mathbb{Z}_6)$ are equal to four.

Proof. Based on Proposition 1, it can be seen that the total pair of elements (x, y) in which $x \cdot y \neq 0$ is equal to 21. To prevent the existence of loops in the non-zero divisors graphs, this is excluded $x \cdot x \neq 0$.

The non-zero divisor graph of $\mathbb{Z}_6$, $\Gamma(\mathbb{Z}_6)$ are constructed. Based on the graph, it can be seen that the clique number, $\omega(\Gamma(\mathbb{Z}_6))$ is four since the set of maximum clique in the ring is $\{1, 2, 4, 5\}$. There are four colors that can be applied on the vertices of $\Gamma(\mathbb{Z}_6)$ so that no two adjacent vertices share the same color. Therefore, the chromatic number, $\chi(\Gamma(\mathbb{Z}_6))$ is four. The diameter of $\Gamma(\mathbb{Z}_6)$ is two since the maximum distance from x to y for all $x, y \in \mathbb{Z}_6$ is two. The girth of $\Gamma(\mathbb{Z}_6)$ is three.

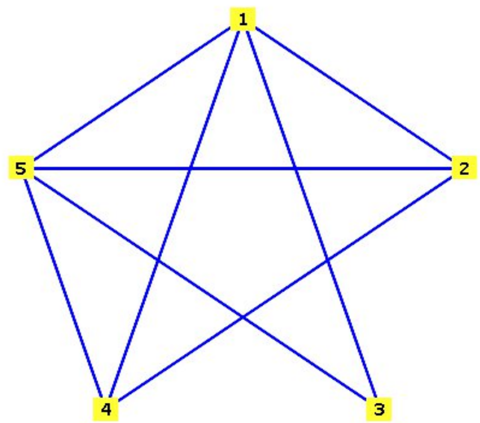

Figure 1. The non-zero divisor graph of $\mathbb{Z}_6$.

Proposition 4. Let $\Gamma(\mathbb{H}(\mathbb{Z}_2))$ be the non-zero divisor graph of $\mathbb{H}(\mathbb{Z}_2)$, then, $\Gamma(\mathbb{H}(\mathbb{Z}_2))$ is an undirected graph with 15 vertices and 96 edges. Furthermore, the clique number and chromatic number ofΓ $(\mathbb{H}(\mathbb{Z}_2))$ are 11.

Proof. Based on Proposition 2, it can be seen that the total pair of elements (x, y) in which $x \cdot y \neq 0$ is equal to 207. To prevent the existence of loops in the non-zero divisors graphs, this is excluded, $x \cdot x \neq 0$. Therefore,Γ $(\mathbb{H}(\mathbb{Z}_2))$ has the total edges of 96 edges.

The non-zero divisor graph of $\mathbb{H}(\mathbb{Z}_2)$, $\Gamma(\mathbb{H}(\mathbb{Z}_2))$ are constructed. Based on the graph, it can be seen that the clique number, $\omega(\Gamma(\mathbb{H}(\mathbb{Z}_2)))$ is 11 since the set of maximum clique in the ring is $\{1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 14\}$. There are 11 colors that can be applied on the vertices of $\Gamma(\mathbb{H}(\mathbb{Z}_2))$ so that no two adjacent vertices share the same color. Therefore, the chromatic number, $\chi(\Gamma(\mathbb{H}(\mathbb{Z}_2)))$ is 11. The diameter of $\Gamma(\mathbb{H}(\mathbb{Z}_2))$ is two since the maximum distance from x to y for all $x, y \in \mathbb{H}(\mathbb{Z}_2)$ is two. The girth of $\Gamma(\mathbb{H}(\mathbb{Z}_2))$ is three.

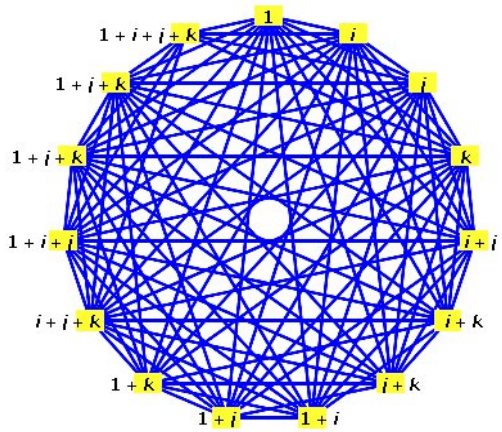

Figure 2. The non-zero divisor graph of $\mathbb{H}(\mathbb{Z}_2)$.

Theorem 1 Let Γ (R) be the non-zero divisor graph of a finite commutative ring with identity. Then, the diameter of the graph, $diam(\Gamma(R)) = 2$.

Proof. Let $a, b \in R$ and $a \cdot b = 0$ where a and b are the zero divisors of R. Since there is no edge connecting a and b, then the only path from a to b is by going through identity, $a - 1 - b$.

Remark: Note that this theorem is not applicable if R has no identity element. If the ring R has no identity, then the graph is not connected (in general). For example, if $I(\mathbb{Z}_{12}) = \{0, 2, 4, 6, 8, 10\}$ (an ideal of $\mathbb{Z}_{12}$), then is not connected because 6 is an isolated vertex.

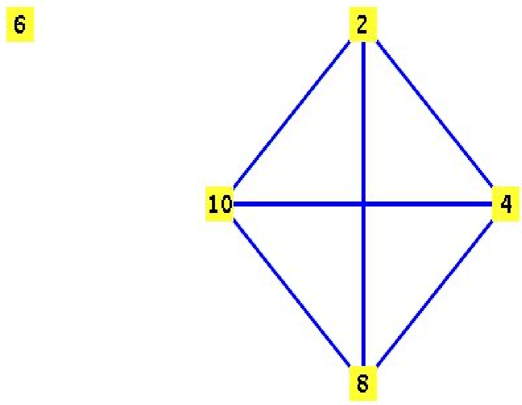

Figure 3. The non-zero divisor graph of $I(\mathbb{Z}_{12})$.

If $I(\mathbb{Z}_{14}) = \{0, 2, 4, 6, 8, 10, 12\}$ (an ideal of $\mathbb{Z}_{14}$), then it is a complete graph.

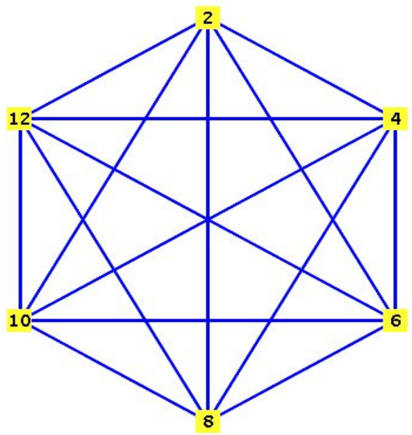

Figure 4. The non-zero divisor graph of $I(\mathbb{Z}_{14})$.

CONCLUSION

In this research, the non-zero divisors are determined using definition. The commutative rings that are being considered are the ring of integers modulo 6, $\mathbb{Z}_6$ and the Hamiltonian quaternion over ring of integers modulo two, $\mathbb{H}(\mathbb{Z}_2)$.

In the second part of this research, the non-zero divisor graphs of the same rings are constructed based on the non-zero divisors obtained. The results have shown that the non-zero divisor graphs of the commutative rings are undirected graphs. Lastly, the chromatic and clique number, girth and diameter of the graphs are found. The clique and chromatic number of the non-zero divisor graph, $\Gamma(R)$ are equal, $diam(\Gamma(R)) = 2$ and $girth(\Gamma(R)) = 3$.

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