RESEARCH ARTICLE

A Boundary Integral Equation Related to the Ahlfors Map

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Abstract The Ahlfors map is a mapping function that maps a multiply connected region onto a unit disk. This paper presents a new boundary integral equation related to the Ahlfors map of a bounded multiply connected region. The boundary integral equation is constructed from a boundary relationship satisfied by the Ahlfors map of a multiply connected region.

Keywords: Ahlfors map, multiply connected region, Kerzman-Stein kernel, Neumann kernel

1. Introduction

The Ahlfors map is a generalization of the Riemann map for a simply connected region. For a multiply connected region Ω of connectivity $M+1$, $M = 1, 2, \ldots,$ the Ahlfors map with the base point $a \in \Omega$ is a $M+1$-to-one map. It maps each boundary component of Ω one-to-one onto the unit disk and maps a to the origin [1, 2, 3]. The Ahlfors map might prove to be useful in the problem of fluid mechanics [1]. Bell [1] computed the Ahlfors map for a bounded multiply connected region without the knowledge of the zeros of the Ahlfors map. The method in [1] is based on the generalization of the Kerzman-Stein method in [4]. In addition, [2, 3] also computed the Ahlfors map for an annulus region. Some integral equations related to the Ahlfors map have been constructed in [5-8]. The integral equations in [5, 7, 8] are based on the boundary relationship satisfied by an analytic function involving the Kerzman-Stein kernel, the Neumann kernel, the Neumann-type kernel and the Kerzman-Stein type kernel. The methods in [5-8] however depend on the zeros of the Ahlfors map.

In this paper, we present a new boundary integral equation method related to the Ahlfors map of a bounded multiply connected region different from [5-8] using a boundary relationship satisfied by the Ahlfors map. The new boundary integral equation will be useful in computing the zeros of the Ahlfors map.

The organization of this paper is as follows: Section 2 presents some auxiliary materials related to the Ahlfors map. In Section 3 we give a new boundary integral equation related to the Ahlfors map for a bounded multiply connected region. In Section 4 we give a conclusion.

2. Auxiliary materials

Let Ω be a bounded multiply connected region with connectivity $M+1$, $M = 1, 2, \ldots$ consisting of $M+1$ smooth closed Jordan curves Γ_j, $j = 0, 1, \ldots, M$, i.e. $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \Gamma_M$. The curves $\Gamma_1, \ldots, \Gamma_M$ lie in the interior of the outer boundary Γ_0. The inner curves $\Gamma_1, \ldots, \Gamma_M$ have clockwise orientations and the outer curve Γ_0 has a counterclockwise orientation as shown in Figure 2.1.
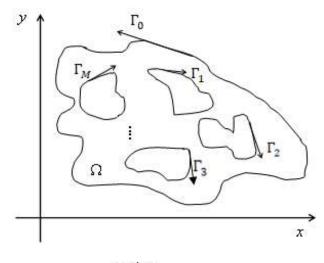
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z-plane

Figure 2.1 A bounded multiply connected region Ω

The curves Γ_j, $j = 0,1,\ldots,M$ are parameterized by $2\pi-$periodic twice continuously differentiable complex-valued functions $z_j(t)$ with non-vanishing first derivatives $z_j'(t) \neq 0$, $t \in J_j = [0, 2\pi]$. The total parameter domain J is defined as the disjoint union of $M+1$ intervals of J_j, $j = 0,1,\ldots,M$. The parameterization $z(t)$ of the whole boundary Γ on J is defined as $z(t) = z_j(t)$, $t \in J_j = [0, 2\pi]$. The unit tangent to the boundary Γ_j at $z_j(t)$ is given by

$$T\left(z_j(t)\right) = \frac{z_j'(t)}{\left|z_j'(t)\right|}. \tag{2.1}$$

The Szegö kernel is known to satisfy the Kerzman-Stein integral equation [4, 9]

$$S(z,a_0) + \int_{\Gamma} A(z,w)S(w,a_0)|dw| = g(z), \qquad a_0 \in \Omega, \tag{2.2}$$

where

$$A(z,w) = \begin{cases} \dfrac{1}{2\pi i}\left(\dfrac{T(w)}{z-w} - \dfrac{\overline{T(z)}}{\overline{z}-\overline{w}}\right), & z \neq w \in \Gamma, \\ 0, & z = w \in \Gamma, \end{cases}$$

$$g(z) = -\frac{1}{2\pi i}\frac{\overline{T(z)}}{\overline{z}-\overline{a_0}}.$$

The function $A(z,w)$ is known as the Kerzman-Stein kernel and it is continuous on the smooth boundary of Ω [4, 10]. It has the property that $A(z,w) = 0$ if z, w are on a circle. Numerical implementation of computing the Szegö kernel based on (2.2) is discussed in [10]. With $z = z(t)$ and $w = z(s)$, (2.2) becomes

$$S(z(t),a_0) + \int_J A(z(t),z(s))S(z(s),a_0)|z'(s)|ds = g(z(t)). \tag{2.3}$$

The derivative of the Szegö kernel has been derived in [7] as a solution of the integral equation

$$S'(z(t),a_0)z'(t) = g'(z(t))z'(t)$$

$$-\int_J \left[\frac{d}{dt} A(z(t),z(s))z'(t) \right] S(z(s),a_0) |z'(s)| ds, \qquad (2.4)$$

where

$$g'(z(t))z'(t) = -\frac{1}{2\pi i} \left[\frac{\overline{T'(z(t))z'(t)}}{z(t)-a_0} - \frac{\overline{T(z(t))z'(t)}}{(z(t)-a_0)^2} \right],$$

$$T'(z(t))z'(t) = \frac{z''(t)}{2|z'(t)|} - \frac{(z'(t))^2 \overline{z''(t)}}{2|z'(t)|^3},$$

and

$$\frac{d}{dt} A(z(t),z(s)) = \begin{cases} \dfrac{1}{2\pi i} \left[\dfrac{-T(z(s))z'(t)}{(z(t)-z(s))^2} - \dfrac{\overline{T'(z(t))z'(t)}}{(\overline{z(t)}-\overline{z(s)})} \right. \\[4mm] \left. -\dfrac{\overline{T'(z(t))z'(t)}}{(\overline{z(t)}-\overline{z(s)})} + \dfrac{\overline{T(z(t))z'(t)}}{(\overline{z(t)}-\overline{z(s)})^2} \right], \quad z(t) \neq z(s) \in \Gamma, \\[6mm] \dfrac{1}{4\pi|z'(t)|} \left[\dfrac{1}{3} \operatorname{Im}\left(\dfrac{z'''(t)}{z'(t)} \right) \right. \\[4mm] \left. -\operatorname{Re}\left(\dfrac{z''(t)}{z'(t)} \right) \operatorname{Im}\left(\dfrac{z''(t)}{z'(t)} \right) \right], \quad z(t) = z(s) \in \Gamma. \end{cases}$$

Let $f(z)$ be the Ahlfors function which maps Ω conformally onto the unit disk $E = \{ w : |w| < 1 \}$ which satisfies the conditions $f(a_j) = 0$, $j = 0, 1, \ldots, M$ and $f'(a_0) > 0$, where $a_j \in \Omega$ are the zeros of the Ahlfors map and $a_0 \in \Omega$ can be freely chosen. The boundary values for $f(z)$ is represented by

$$f(z_j(t)) = e^{i\theta_j(t)}, \qquad \Gamma_j : z = z_j(t), \quad 0 \leq t \leq 2\pi \in J_j, \qquad (2.5)$$

where $\theta_j(t)$, $j = 0, 1, \ldots, M$ are the boundary correspondence functions of the Ahlfors map on Γ_j. Differentiating both sides of (2.5), we have

$$f'(z_j(t))z_j'(t) = f(z_j(t))i\theta_j'(t). \qquad (2.6)$$

Taking modulus on both sides of (2.6) gives

$$\left| f'(z_j(t))z_j'(t) \right| = \left| f(z_j(t))i\theta_j'(t) \right|.$$

(2.7)

Dividing (2.6) by (2.7) and using (2.1), we get

$$f(z_j(t)) = \frac{T(z_j(t))}{i} \frac{\left| \theta_j'(t) \right| \left| f'(z_j(t)) \right|}{\theta_j'(t) \left| f'(z_j(t)) \right|}. \qquad (2.8)$$

The image of Γ_0 remains in a counterclockwise orientation, so $\theta_0'(t) > 0$ and the images of Γ_j, $j = 1, 2, \ldots, M$ are in reversed orientations, so $\theta_j'(t) > 0$.

Since $\dfrac{\left| \theta_j'(t) \right|}{\theta_j'(t)} = 1$, (2.8) becomes

$$f(z_j(t)) = \frac{T(z_j(t)) f'(z_j(t))}{i |f'(z_j(t))|}.$$

In general,

$$f(z) = \frac{T(z)f'(z)}{i|f'(z)|}, \qquad z \in \Gamma. \tag{2.9}$$

The interior values of the Ahlfors map $f(z)$ which is analytic on Ω can be obtained by using the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z} dw, \qquad z \in \Omega. \tag{2.10}$$

For points z closed to Γ, the method of Helsing and Ojala [11] can be used for the efficient numerical computation of (2.10).

It is known that the Ahlfors map can be represented in terms of the Szegö kernel $S(z,a_0)$ and the Garabedian kernel $L(z,a_0)$. It is given by [4]

$$f(z) = \frac{S(z,a_0)}{L(z,a_0)}, \qquad z \in \Omega \cup \Gamma, \qquad a_0 \in \Omega. \tag{2.11}$$

Since $L(z,a_0) = i\overline{S(z,a_0)T(z)}$ [1], thus (2.11) can be written as

$$f(z(t)) = \frac{S(z(t),a_0)T(z(t))}{i\overline{S(z(t),a_0)}}, \qquad z(t) \in \Gamma. \tag{2.12}$$

Thus the boundary values for the Ahlfors map can be determined completely from the boundary values of the Szegö kernel.

Differentiating both sides of (2.12), we get

$$f'(z(t))z'(t) = \frac{1}{i}\Bigg[\frac{S(z(t),a_0)T'(z(t))z'(t)}{\overline{S(z(t),a_0)}}$$
$$+ \frac{S'(z(t),a_0)z'(t)T(z(t))}{\overline{S(z(t),a_0)}}$$
$$- \frac{\overline{S'(z(t),a_0)z'(t)}S(z(t),a_0)T(z(t))}{\left(\overline{S(z(t),a_0)}\right)^2}\Bigg]. \tag{2.13}$$

In [12, 13] an alternative formula for the derivative of the boundary correspondence function of the Ahlfors map has been derived by using (2.6), (2.12) and (2.13), i.e.

$$\theta'(t) = 2\,\text{Im}\Bigg[\frac{S'(z(t),a_0)z'(t)}{S(z(t),a_0)} \Bigg] + \text{Im}\Bigg[\frac{z''(t)}{z'(t)} \Bigg]. \tag{2.14}$$

Suppose that $b(z)$, $Q(z)$ and $H(z)$ are complex-valued functions defined on Γ such that $b(z) \neq 0$, $Q(z) \neq 0$, $H(z) \neq 0$ and $\dfrac{\overline{H(z)}}{T(z)Q(z)}$ satisfies the Hölder condition on Γ, where $T(z)$ is the unit tangent vector on Γ. The function $P(z)$ is said to satisfy the interior relationship if $P(z)$ is analytic in Ω, $P(z) \neq 0$ on Γ, and Hölder continuous on $\Omega \cup \Gamma$, and satisfies the non-homogeneous boundary relationship [14, p. 33]

$$P(z) = \frac{b(z)T(z)Q(z)\overline{P(z)}}{G(z)} + \overline{H(z)}, \qquad z \in \Gamma, \tag{2.15}$$

where $G(z)$ is analytic in Ω, Hölder continuous on Γ, and $G(z) \neq 0$ on Γ. The boundary relationship

(2.15) also has the following equivalent form [14, p. 33]

$$G(z) = \frac{\overline{b(z)T(z)Q(z)P(z)^2}}{|P(z)|^2} + \frac{G(z)H(z)}{P(z)}.$$ (2.16)

The following theorem gives an integral equation for an analytic function $P(z)$ satisfying the interior non-homogeneous boundary relationship (2.15) or (2.16).

Theorem 2.1. [14, p. 34] Let $u(z)$ and $v(z)$ be any complex-valued functions that are defined on Γ. If the function $P(z)$ satisfies the boundary relationship (2.15) or (2.16), then

$$\frac{1}{2}\left[v(z) + \frac{u(z)}{T(z)Q(z)}\right]P(z) + \text{PV}\int_\Gamma K(z,w)P(w)|dw|$$

$$+ b(z)u(z)\left[\sum_{a_j \in \Gamma} \text{Res}_{w=a_j} \frac{P(w)}{(w-z)G(w)}\right]^- = -u(z)\overline{L_{\overline{R}}(z)}, \quad z \in \Gamma,$$ (2.17)

where

$$K(z,w) = \frac{1}{2\pi i}\left[\frac{b(z)u(z)}{b(w)(\overline{w}-\overline{z})\overline{Q(w)}} - \frac{v(z)T(w)}{w-z}\right],$$

and

$$L_{\overline{R}}(z) = -\frac{1}{2}\frac{H(z)}{Q(z)T(z)}$$

$$+ \text{PV}\frac{1}{2\pi i}\int_\Gamma \frac{\overline{b(z)}H(w)}{b(w)(w-z)Q(w)T(w)}dw.$$

The symbol "−" in the superscript denotes the complex conjugate and the sum in (2.17) is over all those zeros $a_0, a_1, \ldots, a_M$ of $G(z)$ that lie inside Ω. There is no residue term if the function $G(z)$ has no zeros in Ω. Theorem 2.1 generalizes the result in [5] which is limited to a doubly connected region.

We next derive a new boundary integral equation related to the Ahlfors map of a bounded multiply connected region using Theorem 2.1.

3. Integral equation related to the Ahlfors map

The following theorem gives another integral equation satisfied by $\theta'(t)$ where $\theta(t)$ is the boundary correspondence function of the Ahlfors map of a bounded multiply connected region onto a unit disk.

Theorem 3.1. *For all $z \in \Gamma$, the function $\theta'(t)$ of the Ahlfors map for the multiply connected region satisfies*

$$i\theta'(s) + \frac{1}{\pi}\text{PV}\int_J \frac{z'(s)}{z(s)-z(t)}\theta'(t)dt$$

$$= 2z'(s)\sum_{j=0}^M \frac{1}{z(s)-a_j},$$ (3.1)

where a_j, $j = 0,1,\ldots,M$ are the zeros of the Ahlfors map.

Proof. The Ahlfors map for a multiply connected region can be written as

$$f(z) = \prod_{j=0}^M (z-a_j)g(z), \quad z \in \Omega,$$ (3.2)

where $g(z)$ is analytic in Ω and $g(z) \neq 0$ for all $z \in \Omega$. Applying log on both sides of (3.2) gives

$$\log f(z) = \sum_{j=0}^{M} \log(z - a_j) + \log g(z), \quad z \in \Gamma. \tag{3.3}$$

Differentiate on both sides of (3.3), we get

$$\frac{f'(z)}{f(z)} = \sum_{j=0}^{M} \frac{1}{z - a_j} + \frac{g'(z)}{g(z)}, \tag{3.4}$$

so that

$$D(z) = \frac{f'(z)}{f(z)} - \sum_{j=0}^{M} \frac{1}{z - a_j} = \frac{g'(z)}{g(z)} \tag{3.5}$$

is analytic in Ω and hence (3.5) can be written as

$$\frac{f'(z)}{f(z)} = D(z) + \sum_{j=0}^{M} \frac{1}{z - a_j}, \tag{3.6}$$

Squaring both sides of (2.9) gives

$$f(z)^2 = -\frac{T(z)^2 f'(z)^2}{|f'(z)|^2}. \tag{3.7}$$

Since $|f'(z)|^2 = f'(z)\overline{f'(z)}$ and $f(z) = \dfrac{1}{\overline{f(z)}}$, (3.7) becomes

$$\frac{\overline{f'(z)}}{\overline{f(z)}} = -\frac{T(z)^2 f'(z)}{f(z)}. \tag{3.8}$$

Taking conjugate to both sides of (3.8), we get

$$\frac{f'(z)}{f(z)} = -\overline{T(z)}^2 \left(\overline{\frac{f'(z)}{f(z)}} \right). \tag{3.9}$$

Substituting (3.6) into (3.9), we obtain

$$D(z) + \sum_{j=0}^{M} \frac{1}{z - a_j} = -\overline{T(z)}^2 \left[\overline{D(z)} + \sum_{j=0}^{M} \frac{1}{\overline{z} - \overline{a}_j} \right],$$
$$\tag{3.10}$$
$$D(z) = -\overline{T(z)}^2 \overline{D(z)} - \sum_{j=0}^{M} \frac{\overline{T(z)}^2}{\overline{z} - \overline{a}_j} - \sum_{j=0}^{M} \frac{1}{z - a_j}.$$

Letting $u(z) = 0$ and $v(z) = 2$ in Theorem 2.1 we have

$$P(z) + \mathrm{PV}\int_{\Gamma} K(z, w) P(w) |dw| = 0, \tag{3.11}$$

where

$$K(z, w) = -\frac{1}{2\pi i} \left[\frac{2T(w)}{w - z} \right].$$

Comparing (2.15) and (3.10) gives the following assignments

$$P(z) = D(z), \quad Q(z) = 1, \quad G(z) = 1, \quad b(z) = -\overline{T(z)},$$

and

$$H(z) = -\sum_{j=0}^{M} \frac{T(z)^2}{z - a_j} - \sum_{j=0}^{M} \frac{1}{\overline{z} - \overline{a}_j}.$$

Substituting the above assignments into (3.11) and using (3.5), we obtain

$$\frac{f'(z)}{f(z)} - \sum_{j=0}^{M} \frac{1}{z-a_j} - 2\text{PV} \int_{\Gamma} \frac{1}{2\pi i} \left[\frac{T(w)}{w-z} \right] \left[\frac{f'(w)}{f(w)} - \sum_{j=0}^{M} \frac{1}{w-a_j} \right] |dw| = 0. \tag{3.12}$$

Rearranging (3.12) and since $T(w)|dw| = dw$, gives

$$\frac{f'(z)}{f(z)} - 2\text{PV} \int_{\Gamma} \frac{1}{2\pi i} \frac{1}{w-z} \frac{f'(w)}{f(w)} dw$$

$$+ 2\text{PV} \sum_{j=0}^{M} \int_{\Gamma} \frac{1}{2\pi i} \frac{1}{w-z} \frac{1}{w-a_j} dw = \sum_{j=0}^{M} \frac{1}{z-a_j}. \tag{3.13}$$

Using the fact that [15, p. 91]

$$\text{PV} \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{w-z} \frac{1}{w-a_j} dw = -\frac{1}{2(z-a_j)},$$

and after some simplifications (3.13) becomes

$$\frac{1}{2} \frac{f'(z)}{f(z)} - \text{PV} \int_{\Gamma} \frac{1}{2\pi i} \frac{1}{w-z} \frac{f'(w)}{f(w)} dw = \sum_{j=0}^{M} \frac{1}{z-a_j}. \tag{3.14}$$

Letting $z = z(s)$ and $w = z(t)$, and multiplying (3.14) by $z'(s)$, gives

$$\frac{1}{2} \frac{f'(z(s))z'(s)}{f(z(s))}$$

$$+ \frac{1}{2\pi i} \text{PV} \int_{J} \frac{z'(s)}{z(s)-z(t)} \frac{f'(z(t))}{f(z(t))} z'(t) dt = z'(s) \sum_{j=0}^{M} \frac{1}{z(s)-a_j}. \tag{3.15}$$

Applying (2.6) to (3.15) yields the integral equation (3.1).
This completes the proof of Theorem 3.1.
Observe that, by taking the imaginary part on both sides of (3.1), it reduces to the integral equation derived in [7], i.e.

$$\theta'(s) + \int_{J} N^*(s,t) \theta'(t) dt = 2\,\text{Im} \left[\sum_{j=0}^{M} \frac{z'(s)}{z(s)-a_j} \right], \tag{3.16}$$

where [16, 17]

$$N^*(s,t) = \begin{cases} \dfrac{1}{\pi} \text{Im} \left[\dfrac{z'(s)}{z(s)-z(t)} \right], & t \neq s, \\[4mm] \dfrac{1}{2\pi} \text{Im} \left[\dfrac{z''(s)}{z'(s)} \right], & t = s, \end{cases}$$

and $\theta'(t)$ is defined in (2.14). The kernel $N^*(s,t)$ is the adjoint of the classical Neumann kernel $N(s,t) = N^*(t,s)$. It is well known that $\lambda = -1$ is an eigenvalue of the Neumann kernel $N(s,t)$ with multiplicity M [17]. The eigenfunctions of $N(s,t)$ corresponding to the eigenvalue $\lambda = -1$ are $\left\{ \chi^{[1]}, \chi^{[2]}, \ldots, \chi^{[M]} \right\}$, where [18]

$$\chi^{[k]}(\xi) = \begin{cases} 1, & \xi \in \Gamma_k, \ k = 1, 2, \ldots, M, \\ 0, & \text{otherwise.} \end{cases}$$

By the Fredholm alternative theorem [15], the integral equation (3.16) is solvable if and only if the given function $g(s) = 2\,\text{Im} \left[\sum_{j=0}^{M} \frac{z'(s)}{z(s)-a_j} \right]$ satisfies the M orthogonality conditions

$$\left(g, \chi^{[k]}\right) = \int_J g(t) \chi^{[k]}(t) dt = 0, \qquad k = 1, 2, \ldots, M.$$

Perhaps these M orthogonality conditions could be used to find the M unknown zeros $a_1, \ldots, a_M$ in the right-hand side of the integral equation (3.16). If Ω is a simply connected region, then (3.16) reduces to the Warschawski's integral equation for Riemann map as given in [15, p. 394-395] with $R(0) = 0$, i.e.

$$\theta'(s) + \int_J N^*(s,t) \theta'(t) dt = 2 \operatorname{Im}\left[\frac{z'(s)}{z(s)}\right]. \tag{3.17}$$

If is a doubly connected region, equation (3.1) reduces to the result given in [19] by a different approach.

4. Conclusion

In this paper, we have presented a new boundary integral equation (3.1) related to the Ahlfors map of a bounded multiply connected region which contains the special cases (3.16) and (3.17).

Acknowledgements

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