THE NONABELIAN TENSOR SQUARES OF CERTAIN BIEBERBACH GROUPS WITH CYCLIC POINT GROUP OF ORDER TWO

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To my beloved husband and mother

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ABSTRACT

The torsion free crystallographic groups are called Bieberbach groups. These groups are extensions of a finite point group and a free abelian group of finite rank. The rank of the free abelian group is the dimension of Bieberbach group. In this research, Bieberbach groups with cyclic point group of order two and Bieberbach groups with the elementary abelian 2-group $C_2 \times C_2$ as point group are also considered. These groups are polycyclic since they are extensions of polycyclic groups. Using computational methods developed before for polycyclic groups, the nonabelian tensor squares for these Bieberbach groups are determined. The formulas for the nonabelian tensor squares of four Bieberbach groups with cyclic point group of order two and two Bieberbach groups with the elementary abelian 2-group $C_2 \times C_2$ as point group are given. For the abelian nonabelian tensor square, the formula obtained can be extended to calculate the nonabelian tensor squares of Bieberbach groups of arbitrary dimension. For the nonabelian cases, the nonabelian tensor squares of all Bieberbach groups with cyclic point group of order two and elementary abelian 2-group are nilpotent of class two and can be written as a direct product with the nonabelian exterior square as a factor. As a consequence, sufficient conditions for any group such that the nonabelian tensor square is abelian are obtained.

ABSTRAK

Kumpulan kristalografi yang bebas kilasan dikenali sebagai kumpulan Bieberbach. Kumpulan ini merupakan perluasan daripada kumpulan titik terhingga dan kumpulan abelan bebas dengan pangkat terhingga. Pangkat bagi kumpulan abelan bebas merupakan dimensi bagi kumpulan Bieberbach. Dalam penyelidikan ini, kumpulan Bieberbach dengan kumpulan titik kitaran berperingkat dua dan kumpulan Bieberbach dengan kumpulan abelan asas 2-kumpulan $C_2 \times C_2$ sebagai kumpulan titik dipertimbangkan. Kumpulan ini merupakan kumpulan polikitaran kerana ianya merupakan perluasan daripada kumpulan-kumpulan polikitaran. Dengan menggunakan kaedah pengiraan bagi kumpulan polikitaran yang telah diperkenalkan sebelum ini, kuasa dua tensor tak abelan bagi kumpulan-kumpulan ini telah ditentukan. Rumus bagi kuasa dua tensor tak abelan bagi empat kumpulan Bieberbach dengan kumpulan titik kitaran pada peringkat dua dan dua kumpulan Bieberbach dengan kumpulan abelan asas 2-kumpulan $C_2 \times C_2$ sebagai kumpulan titik diberikan. Bagi kes abelan, rumus bagi kuasa dua tensor tak abelan yang diperolehi boleh dilanjutkan untuk pengiraan bagi kuasa dua tensor tak abelan untuk kumpulan Bieberbach bagi sebarang dimensi. Bagi kes yang tak abelan, kuasa dua tensor tak abelan bagi semua kumpulan Bieberbach dengan kumpulan titik kitaran berperingkat dua dan kumpulan abelan asas 2-kumpulan adalah merupakan nilpoten dengan kelas dua dan boleh ditulis sebagai hasil darab langsung dengan kuasa dua perluaran tak abelan sebagai salah satu daripada faktornya. Akhir sekali, syarat-syarat cukup bagi sebarang kumpulan untuk mempunyai kuasa dua tensor tak abelan yang abelan turut diberikan.

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LIST OF SYMBOLS

$Ker(f)$	-	Kernel of a mapping f
$A\,char\,B$	-	A subgroup A is characteristic in B
$A \leq B$	-	A is a subgroup of B
$\langle a \rangle$	-	the cyclic subgroup generated by a
$A \triangleleft B$	-	A is a normal subgroup of B
G/H	-	the quotient group of G by H
$A \times B$	-	a direct product of A and B
${}^{y}x$	-	x conjugated by y, yxy^{-1}
$[a, b]$	-	the commutator of a and b, $aba^{-1}b^{-1}$
$H \cong G$	-	H is isomorphic to G
$G \wedge G$	-	the nonabelian exterior square of G
$H_2(G)$	-	the Schur Multiplier of G
C_n	-	a cyclic group of order n
$\langle X \mid R \rangle$	-	a group presented by generators X and relators R
$H \otimes K$	-	tensor product of H and K
$G' = [G, G]$	-	the derived subgroup of G
G^{ab}	-	the abelianization of G, i.e G/G'
$Z(G)$	-	center of G
$x \in G$	-	x is an element of G
$x \notin G$	-	x is not an element of G
${F_n}^{ab}$	-	a free abelian group of rank n

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CHAPTER 1

INTRODUCTION

1.1 Introduction

An important algebraic invariant of flat manifolds are their fundamental groups. These groups are known as Bieberbach groups and turn out to have many interesting algebraic properties. Bieberbach groups are torsion free crystallographic groups. These groups are extensions of a free abelian group L of finite rank by a finite group P. Then, there is a short exact sequence

$$1 \longrightarrow L \xrightarrow{\varphi} G \xrightarrow{\phi} P \tag{1.1}$$

such that $G/\varphi(L) \cong P$. Here, L is called the lattice group and P is a point group which is also known as a holonomy group. The dimension of G is the rank of L. G is called the Bieberbach group with point group P.

The nonabelian tensor square $G \otimes G$ of a group G is generated by the symbols $g \otimes h$, for all $g, h \in G$, subject to relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$$

for all $g, g', h, h' \in G$, where ${}^{g}g' = gg'g^{-1}$. The nonabelian tensor square is a specialisation of the more general nonabelian tensor product introduced by Brown and Loday [1]. This group construction has its roots in algebraic K-theory and topology extending ideas of Whitehead [2]. The nonabelian tensor square appears independently in Dennis's works [3] in K-theory and is based on the ideas of Miller [4].

By computing the nonabelian tensor square of group G, we mean finding a simple or standard form for expressing $G \otimes G$. The definition of the nonabelian tensor square gives no insight as to the group it describes or its structure. Starting in [1], methods in computational group theory have been invoked to investigate this problem. Many papers have appeared on the computation of the nonabelian tensor squares for various groups and classes of groups since the publication of Brown, Johnson and Robertson's seminal work. These include 2-generator nilpotent of class 2 ([5], [6] and [7]), metacyclic groups [8] and free nilpotent groups [10] .

1.2 Research Background

The computation of nonabelian tensor squares of the Bieberbach groups is not in the literature. In this research, the Bieberbach groups, G with cyclic point group of order two, C_2 are considered. These groups satisfy a short exact sequence as in (1.1) where L is a free abelian group of finite rank. Here, L and C_2 are polycylic groups. If L and P in (1.1) are polycyclic then G is polycyclic since the polycyclic groups are closed under forming extension. Hence, all the Bieberbach groups with point group C_2 which satisfy equation (1.1) are shown to be polycyclic groups. For any extension with an abelian image, the kernel must contain the derived subgroup G'. Since L is abelian then G' also is abelian. Therefore, the Bieberbach groups with point group C_2 are metabelian groups. Many of these groups (up to dimension 6) can be found in the Crystallographic Algorithms and Tables (CARAT) website [11]. However, in this research some of these groups are partitioned into several families. For example, the Bieberbach groups, G with cyclic point group C_2 are partitioned into eleven families which have similar structural characteristics.

The Bieberbach groups are crystallographic groups. Any new constructions of the crystallographic group might give some interest to the chemists. New properties of the crystallographic groups may appear when we calculate the nonabelian tensor squares of these groups. Hence, these reasons will motivate us for studying the nonabelian tensor squares of the Bieberbach groups with cyclic point group C_2 up to dimension 6.

Since the Bieberbach groups with point group C_2 are polycylic groups then some results of the nonabelian tensor squares for polycyclic groups as found in Blyth and Morse [12] and Morse [19] will be used to help us in studying the nonabelian tensor squares of these groups. One of the general results in [12] is given in the following theorem.

Theorem 1.1.
Let G be a polycyclic group. Then the nonabelian tensor square $G \otimes G$ is polycylic.

This general result provides an approach to computing the nonabelian tensor square of a polycyclic group G (finite and infinite). This result shows that $G \otimes G$ has a consistent polycyclic presentation. Such a presentation would allow us to exploit the structure of $G \otimes G$ of all other groups as in the following commutative diagram in [1], whose rows and columns all represent central extensions using the computational method.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & \Gamma(G^{ab}) & \xrightarrow{\psi} & J_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \nabla(G) & \longrightarrow & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \kappa'} & & \\
 & & 1 & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
\qquad (1.2)
$$

Here, G' is the derived subgroup of G, Γ is Whitehead's quadratic functor [2], $\nabla(G)$ is a central subgroup of $G \otimes G$, $H_2(G)$ is the the Schur Multiplier and $G \wedge G$ is a generalised exterior product of group G. These relations with well-known constructions suggest the interests in the computations of $G \otimes G$. These include 2-generator nilpotent groups of class 2 ([5], [6] and [7]), metacyclic groups [8] and free nilpotent groups [10]. Also in [20], Eick and Nickel described an effective algorithm for computing the Schur Multiplier, the nonabelian tensor square and the nonabelian exterior square of any polycyclic groups.

One of the objectives of this research is to investigate some of the central extensions in diagram (1.2). This includes whether the nonabelian tensor square of the Bieberbach groups with point group C_2 can be written as a direct product of $\nabla(G)$ and $G \wedge G$, and also some other structural properties when the nonabelian tensor squares of these groups are abelian or nilpotent of class 2.

1.3 Problem Statement

To calculate the nonabelian tensor squares of Bieberbach groups with elementary abelian 2-groups as the point group, particularly on the Bieberbach group with cyclic point group of order two.

1.4 Research Objectives

The objectives of this thesis are

(i) to compute the nonabelian tensor squares of the 28 Bieberbach groups with cyclic point groups of order two up to dimension 6

(ii) to determine the structural results when the nonabelian tensor squares of groups in (i) are abelian or nilpotent of class exactly 2

(iii) to determine the structural results when the nonabelian tensor squares of the groups in (i) are a direct product with one of the factors being the nonabelian exterior square and

(iv) to generalize computations based on some groups in (i) to arbitrary dimension.

1.5 Scope of Thesis

In this thesis, the groups considered are limited to four Bieberbach groups with point groups C_2 up to dimension 6.

1.6 Significance of Findings

The major contribution of this thesis will be new theoretical results on computing the nonabelian tensor squares for Bieberbach groups with cyclic point group of order two and some necessary conditions for any group such that the nonabelian tensor square is abelian. This thesis also contributes some generalisations of the nonabelian tensor squares of some of these groups of dimension n. Therefore, this thesis provides original results. Some of the results have been presented in national and international conferences and thus contribute to new findings in the field of group theory.

1.7 Thesis Outline

There are eight chapters in this thesis. Chapter 1 provides the introduction of the thesis. This chapter discusses research background of this thesis, the problem statement, research objectives, scope and the significance of the thesis.

In Chapter 2, studies of nonabelian tensor squares of some groups by several researchers are reviewed and compared. The methods of computing the nonabelian tensor square are discussed. The method of computing the nonabelian tensor squares of polycyclic groups that was introduced by Blyth and Morse [12] is presented and briefly discussed in one of the sections. The background and the application of GAP in this research are also presented in this chapter.

Chapter 3 presents some related definitions in group theory that will be used throughout the thesis. Some basic results and basic properties for Bieberbach groups with certain cyclic point groups are included. A list of commutator calculus are given and some new commutator identities are developed and proved. New structural results obtained for the nonabelian tensor squares of any group are given and proved.

Chapter 4 discusses the nonabelian tensor squares of two Bieberbach groups with point group C_2 in dimensions 2 and 3 respectively, given as $B_1(2)$ and $B_2(3)$. These two groups are the only groups in this group extension up to dimension 6 that have abelian nonabelian tensor squares. The generalizations of the nonabelian tensor squares of these two groups to n dimension which have abelian nonabelian tensor squares are also calculated. These calculations can be reduced by computing the nonabelian tensor squares of the groups $B_1(2)$, $B_2(3)$ and the ordinary tensor products of abelian groups. This will give a very nice general formula on computing the nonabelian tensor squares for Bieberbach groups with point group C_2 whose nonabelian tensor squares are abelian. It is also shown that only $B_2(3)$ has split nonabelian tensor square of group and can be written as the direct product of the exterior square of $B_2(3)$, $B_2(3) \wedge B_2(3)$ and a central subgroup of the nonabelian tensor square of $B_2(3)$, $\nabla(B_2(3))$.

Chapter 5 will complete the analysis of computing the nonabelian tensor squares of Bieberbach groups with point group C_2 by considering the nonabelian cases. Here, two families of groups at the lowest dimension, $B_3(3)$ and $B_4(5)$ are given as examples that each of them, has nonsplit and split nonabelian tensor squares respectively. For the nonabelian cases, the nonabelian tensor squares of these groups are nilpotent of class exactly 2. For the split case, the nonabelian tensor square of the group $B_4(5)$ can be written as a direct product with the nonabelian exterior square as a factor.

In Chapter 6, we will investigate the nonabelian tensor squares of the Bieberbach groups with elementary abelian 2-groups as point group. Here, two Bieberbach groups with point group $C_2 \times C_2$ in dimension 5, namely $B_5(5)$ and $B_6(5)$ are chosen as examples based on the structural properties of their nonabelian tensor squares. These examples are given in this thesis since it can lead us to the construction of sufficient conditions for groups to have abelian nonabelian tensor squares as in the following chapter.

The complete analysis of computing the nonabelian tensor squares of Bieberbach groups with cyclic point group of order two will be given in Chapter 7. Sufficient conditions on G such that the nonabelian tensor square $G \otimes G$ is abelian are given. In this chapter, only Bieberbach groups with abelian point groups will be shown as metabelian.

The thesis will be summarized in Chapter 8. Some suggestions for future research on computing the nonabelian tensor squares and other homological functors of the Bieberbach groups with other point groups will be given and pointed out.