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Projective surjectivity of quadratic stochastic operators on L^1 and its application

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ABSTRACT

A nonlinear Markov chain is a discrete time stochastic process whose transitions depend on both the current state and the current distribution of the process. The nonlinear Markov chain over an infinite state space can be identified by a continuous mapping (the so-called nonlinear Markov operator) defined on a set of all probability distributions (which is a simplex). In the present paper, we consider a continuous analogue of the mentioned mapping acting on L^1-spaces. Main aim of the current paper is to investigate projective surjectivity of quadratic stochastic operators (QSO) acting on the set of all probability measures. To prove the main result, we study the surjectivity of infinite dimensional nonlinear Markov operators and apply them to the projective surjectivity of the considered QSO. Furthermore, the obtained results are applied to the existence of the positive solution of some Hammerstein integral equations.

1. Introduction

Recently nonlinear Markov chains became an interesting subject in many areas of applied mathematics. These chains are discrete time stochastic processes whose transitions are governed by a stochastic hypermatrix $\mathcal{P} = (P_{i_1,\dots,i_m,k})_{i_1,\dots,i_m,k\in E}$, (where $E \subset \mathbb{N}$) which depends on both the current state and the current distribution of the process [1]. Such processes were introduced by McKean [2] and have been extensively studied in the context of the nonlinear Chapman- Kolmogorov equation [3] as well as the nonlinear Fokker-Planck equation [4,5]. On the other hand, we stress that such types of chains are generated by tensors (hypermatrices), therefore, this topic is closely related to the geometric and algebraic structures of tensors which have been systematically studied, and have wide applications in applied sciences. One of the intrinsic features of tensors is the concept of tensor eigenvalues and eigenvectors which turns out to be much more complex than that of the square matrix case (see for example, [6–9]).

Let us denote

$$S^E = \left\{ \mathbf{x} = (x_i)_{i\in E} \in \mathbb{R}^E \ : \ x_i \geq 0, \quad \sum_{i\in E} x_i = 1 \right\}. \tag{1.1}$$

By means of $\mathcal{P}$ one defines an operator $V : S^E \to S^E$ by

$$(V(\mathbf{x}))_k = \sum_{i_1,i_2,\dots,i_m\in E} P_{i_1 i_2\dots i_m,k} x_{i_1} x_{i_2} \dots x_{i_m}, \quad k \in E. \tag{1.2}$$

This operator is called an *m-ordered polynomial stochastic operator* (in short m-ordered PSO). We note that such a PSO has a direct connection with non-linear Markov operators which are intensively studied by many scientists (see [1] for recent review). Therefore, it is crucial to study several properties of such operators [6,8,10,11]. One of the important ones is its surjectivity. It turn out that when the set E is finite, in [12] we have established that the sturjectivity of V is equivalent to its orthogonal preserving property.

We notice that if the order of the operator is 2 (i.e. $m = 2$), then the nonlinear Markov operator V given by (1.2) is called a *discrete quadratic stochastic operator (DQSO)* which has many applications in population genetics [13]. Note that such kind of operators is traced back to Bernstein's work [14] where they arose from the problems of population genetics. We refer the reader to [3,15,16] as

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the exposition of the recent achievements and open problems in the theory of the quadratic stochastic operators (QSO) can be further researched. In [17–21] the surjectivity of DQSO and its relation with orthogonal preserving property of V have been investigated.

On the other hand, there has been much interest in recent years in self-organizing search methods in the field of quadratic stochastic operators. Recently, in [22] it was started to consider QSO over continuum state spaces, namely, on the set of σ-additive measures defined on $[0,1]$ (see also [23,24]). In particular, points of the unit interval $[0,1]$ serves to code (continuum valued) traits attributed to each individual from a considered population. The main aim of the present paper is further to investigate projective surjectivity of such kind of continuum analogous of QSO. First, in Section 3, we revise the results of [17], and in Section 4, we apply them to the projective surjectivity of QSO acting on the set of probability measures. We notice that very particular cases of QSO have been studied (see for example, [25–29,22,30]). In the last Section 5, a short application of the main result to the existence of positive solutions nonlinear Hammerstein integral equations is carried out. Certain Hammerstein integral equations associated with finite dimensional DQSO have been investigated in [31].

2. Discrete quadratic stochastic operators

In this section we give basic notations and some known results from the theory of discrete quadratic stochastic operators.

Let E be a subset of $\mathbb{N}$ such that $|E|\geq 2$ and S^E is a set given by (1.1). We notice that there is only two case for cardinality of E. So, in special cases we write S or S^{d-1} instead of S^E when E is either infinite or $|E|=d$, respectively. In what follows, by $\mathbf{e}_i$ we denote the standard basis in S^E, i.e. $\mathbf{e}_i=(\delta_{ik})_{k\in E}$ $(i\in E)$, where δ_{ij} is the Kronecker delta.

Let V be a mapping on S^E defined by

$$(V(\mathbf{x}))_k=\sum_{i,j\in E}P_{ij,k}x_ix_j, \qquad \forall k\in E, \tag{2.1}$$

here $P_{ij,k}$ are hereditary coefficients which satisfy

$$P_{ij,k}\geq 0, \quad P_{ij,k}=P_{ji,k}, \quad \sum_{k\in E}P_{ij,k}=1, \quad \forall i,j,k\in E \tag{2.2}$$

One can see that V maps S^E into itself and V is called *Discrete Quadratic Stochastic Operator (DQSO)* [32].

By *support* of $\mathbf{x}=(x_i)_{i\in E}\in S^E$ we mean a set $supp(\mathbf{x})=\{i\in E\ :\ x_i\neq 0\}$. Recall that two vectors $\mathbf{x},\mathbf{y}\in S^E$ are called *orthogonal* (denoted by $\mathbf{x}\perp\mathbf{y}$) if $supp(\mathbf{x})\cap supp(\mathbf{y})=\emptyset$. If $\mathbf{x},\mathbf{y}\in S^E$, then one can see that $\mathbf{x}\perp\mathbf{y}$ if and only if $\mathbf{x}\cdot\mathbf{y}=0$. Here, $\mathbf{x}\cdot\mathbf{y}=\sum_{i\in E}x_iy_i$.

Definition 2.1. A DQSO V given by (2.1) is called *orthogonality preserving DQSO (OP DQSO)* if for any $\mathbf{x},\mathbf{y}\in S$ with $\mathbf{x}\perp\mathbf{y}$ one has $V(\mathbf{x})\perp V(\mathbf{y})$.

Recall [32,33] that a DQSO $V:S^E\to S^E$ is called *Volterra* if

$$P_{ij,k}=0 \text{ if } k\notin\{i,j\}, \quad \forall i,j,k\in E. \tag{2.3}$$

Remark 2.2. In [33] it was given an alternative definition Volterra operator in terms of extremal elements of S^E.

One can check [32,33] that a DQSO V is Volterra if and only if one has

$$(V(\mathbf{x}))_k=x_k\left(1+\sum_{i\in E}a_{ki}x_i\right), \qquad \forall k\in E,$$

where $a_{ki}=2P_{ik,k}-1$ for all $i,k\in E$. One can see that $a_{ki}=-a_{ik}$. This representation leads us to the following definition.

Definition 2.3. A DQSO $V:S^E\to S^E$ is called π-Volterra if there is a permutation π of E such that V has the following form

$$(V(\mathbf{x}))_k=x_{\pi(k)}\left(1+\sum_{i\in E}a_{\pi(k)i}x_i\right), \qquad \forall k\in E,$$

where $a_{\pi(k)i}=2P_{i\pi(k),k}-1$, $a_{\pi(k)i}=-a_{i\pi(k)}$ for any $i,k\in E$.

In [34,24] it has been proved the following result.

Theorem 2.4. *Let $|E|=d$ and V be a DQSO on S^{d-1}. Then the following statements are equivalent:*

(i) *V is orthogonality preserving;*
(ii) *V is surjective;*
(iii) *V is π-Volterra QSO.*

We emphasize that there is a big difference between finite and infinite dimensional settings. It is known [17] that in the infinite dimensional setting, some implication of Theorem 2.4 fails.

Theorem 2.5. [25] *Let V be infinite dimensional DQSO such that $V(\mathbf{e}_i)=\mathbf{e}_{\pi(i)}$ for some permutation π of $\mathbb{N}$. Then the following statements are equivalent:*

(i) *V is surjective;*
(ii) *V is orthogonality preserving;*
(iii) *V is $\pi-$Volterra QSO.*

Theorem 2.6. [25] *Let V be a surjective and OP infinite dimensional DQSO. Then V is a π-Volterra for some permutation $\pi:\mathbb{N}\to\mathbb{N}$.*

3. Surjectivity of DQSO

In this section, we are going to provide a sufficient condition for the surjectivity of infinite dimensional DQSOs.

Let E be a subset of $\mathbb{N}$. We denote

$$\mathbf{B}_1^+=\left\{\mathbf{x}=(x_i)_{i\in E}\in\mathbb{R}^E\ :\ x_i\geq 0, \quad \forall i\in E \text{ and } \sum_{j\in E}x_j\leq 1\right\}.$$

We can extend each DQSO V from S^E to $\mathbf{B}_1^+$ by the same formula (2.1). Then the following crucial result is true.

Lemma 3.1. *Let V be a DQSO on $\mathbf{B}_1^+$. Then one has*

$$V(S^E)\subset S^E, \qquad V(\mathbf{B}_1^+\setminus S^E)\subset\mathbf{B}_1^+\setminus S^E.$$

Proof. For a given $r\geq 0$ we denote $S_r^E:=rS^E$. Then it is obvious that

$$\mathbf{B}_1^+=\bigcup_{r\in[0,1]}S_r^E.$$

Take any $r\in[0,1]$ and an arbitrary $\mathbf{x}\in S_r^E$. One can check that $(V(\mathbf{x}))_k\geq 0$ for all $k\geq 1$. Furthermore, using (2.2) we get

$$\sum_{k\in E}(V(\mathbf{x}))_k=\sum_{k\in E}\sum_{i,j\in E}P_{ij,k}x_ix_j=\sum_{i,j\in E}x_ix_j=r^2.$$

From this we find $V(\mathbf{x})\in S_{r^2}^E$. Hence,

$$V(S^E)\subset S^E \text{ and } V(\mathbf{B}_1^+\setminus S^E)\subset\mathbf{B}_1^+\setminus S^E,$$

which completes the proof. $\square$

Theorem 3.2. *Let V be a DQSO on $\mathbf{B}_1^+$. Then the following statements are equivalent:*

(i) *V is surjective on $\mathbf{B}_1^+$;*
(ii) *V is surjective on S^E;*
(iii) *V is surjective on $\mathbf{B}_1^+\setminus S^E$.*

Proof. Thanks to Lemma 3.1 the implication $(i) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (iii)$. Assume that V is surjective on S^E. Let $r \in (0,1)$, then we define an operator $T_r : S_r^E \to S^E$ as follows $T_r(\mathbf{x}) = r^{-1}\mathbf{x}$ for all $\mathbf{x} \in S_r^E$. We notice that T_r is a bijection. Then keeping in mind $V(rS^E) = r^2 V(S^E)$ and $r^2 S^E = S_{r^2}^E$, one gets

$$V(S_r^E) = V(rT_r(S_r^E)) = V(rS^E) = S_{r^2}^E.$$

From the arbitrariness of r and $V(\mathbf{0}) = \mathbf{0}$, we find

$$V\Big(\bigcup_{r\in[0,1)} S_r^E\Big) = \bigcup_{r\in[0,1)} S_r^E,$$

which means $V(\mathbf{B}_1^+ \setminus S^E) = \mathbf{B}_1^+ \setminus S^E$.

$(iii) \Rightarrow (i)$. One can see that $V(S_r^E) \subset S_{r^2}^E$ for any $r \in [0,1)$. Due to the surjectivity of V on $\bigcup_{r\in[0,1)} S_r^E$, we conclude

$$V(S_r^E) = S_{r^2}^E, \qquad \forall r \in [0,1).$$

Then, for any $r > 0$, one has

$$V(S^E) = V(r^{-1}S_r^E) = r^{-2}V(S_r^E) = r^{-2}S_{r^2}^E = S^E.$$

The last one together with $V(\mathbf{B}_1^+ \setminus S^E) = \mathbf{B}_1^+ \setminus S^E$ implies that $V(\mathbf{B}_1^+) = \mathbf{B}_1^+$. This completes the proof. □

Remark 3.3. Thanks to Theorem 3.2, to establish the surjectivity of DQSO V on $\mathbf{B}_1^+$ it is enough to consider it only on S^E.

Let us recall the Cauchy Product which has the following form:

$$\left(\sum_{i=1}^{\infty} x_i\right)^m = \sum_{i_1,\dots,i_m\in\mathbb{N}} x_{i_1}\cdots x_{i_m}, \qquad \forall m \in \mathbb{N}, \tag{3.1}$$

where $\sum_{i=1}^{\infty} x_i < \infty$.

Theorem 3.4. *Let V be a surjective DQSO on S. Then there exists a sequence $\{j_k\}_{k\geq 1} \subset \mathbb{N}$ such that $P_{j_k j_k,k} = 1$ for all $k \in \mathbb{N}$.*

Proof. Let us denote

$$I_k = \left\{ j \in \mathbb{N} : \ P_{jj,k} = 1 \right\}$$

First of all, we show that surjectivity of V implies $I_k \neq \emptyset$ for any $k \in \mathbb{N}$. Thanks to the surjectivity of V, for every $k \in \mathbb{N}$ there is an $\mathbf{x}^{(k)} \in S$ such that

$$V(\mathbf{x}^{(k)}) = \mathbf{e}_k. \tag{3.2}$$

Now, we consider two cases.

Case 1: Let $\left|supp(\mathbf{x}^{(k)})\right| = 1$. Then one can find a number $j_k \in \mathbb{N}$ such that $supp(\mathbf{x}^{(k)}) = \{j_k\}$ and

$$(V(\mathbf{x}^{(k)}))_k = P_{j_k j_k,k} x_{j_k}^2 = 1,$$

which yields that $P_{j_k j_k,k} = 1$. Hence, we get $j_k \in I_k$.

Case 2: Let $|supp(\mathbf{x}^{(k)})| > 1$ and $A := supp(\mathbf{x}^{(k)})$. By (3.2) one finds

$$(V(\mathbf{x}^{(k)}))_k = \sum_{i,j\in A} P_{ij,k} x_i x_j = 1. \tag{3.3}$$

Now suppose that there are $\bar{i}, \bar{j} \in A$ such that $P_{\bar{i}\bar{j},k} < 1$. Then

$$\begin{aligned}(V(\mathbf{x}^{(k)}))_k &= \sum_{i,j\in A} P_{ij,k} x_i x_j \\ &\leq \sum_{\{i,j\}\subset A\setminus\{\bar{i},\bar{j}\}} x_i x_j + P_{\bar{i}\bar{j},k} x_{\bar{i}} x_{\bar{j}} \\ &< \sum_{i,j\in A} x_i x_j.\end{aligned}$$

The last inequality together with (3.1) implies

$$(V(\mathbf{x}^{(k)}))_k < 1,$$

which contradicts to (3.2). So, we conclude that

$$P_{ij,k} = 1, \qquad \forall i,j \in A.$$

In particular, we have $P_{ii,k} = 1$ for any $i \in A$ which mans that $A \subset I_k$. Hence, we immediately get $I_k \neq \emptyset$ for any $k \geq 1$.

Consequently, we infer that $I_k \neq \emptyset$ for every $k \geq 1$. Now, we can define a sequence $\{j_k\}_{k\geq 1}$ by $j_k = \inf I_k$. Due to the construction of I_k, one gets $P_{j_k j_k,k} = 1$ for all $k \in \mathbb{N}$. This completes the proof. □

Next result gives a sufficient condition for the surjectivity of DQSO.

Theorem 3.5. *Let V be a DQSO on S. Assume that there exists a sequence $\{j_n\}_{n\geq 1} \subset \mathbb{N}$ such that*

$$P_{j_n j_m,k} = 0, \qquad \forall k \notin \{n,m\}. \tag{3.4}$$

Then V is surjective.

Proof. Let $I := \{j_n\}_{n\geq 1}$ be a subset of $\mathbb{N}$ for which (3.4) is satisfied. Let us define a new cubic matrix $\tilde{\mathcal{P}} = (\tilde{P}_{ij,k})_{i,j,k\geq 1}$ by

$$\tilde{P}_{ij,k} = \begin{cases} P_{\alpha(i)\alpha(j),k}, & k \in \{i,j\}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha(k) = j_k$ for all $k \geq 1$. We consider a DQSO $\tilde{V}$ is given by

$$(\tilde{V}(\mathbf{x}))_k = \sum_{i,j\geq 1} \tilde{P}_{ij,k} x_i x_j, \qquad \forall \mathbf{x} \in S.$$

Due to the construction of $\tilde{\mathcal{P}}$ one concludes that $\tilde{V}$ is a Volterra DQSO. Then, thanks to Theorem 2.5, the operator $\tilde{V}$ is a surjection on S.

Let us denote $S_I = \{\mathbf{x} \in S : supp(\mathbf{x}) \subset I\}$. Then it is obvious that operator $T : S \to S_I$ given by $T(\mathbf{x})_k = x_{\alpha(k)}$ is a bijection. Furthermore, we have $\tilde{V} = V \circ T$. Keeping in mind that $\tilde{V}$ is surjective and T is bijection, we infer that $V = \tilde{V} \circ T^{-1}$ is surjective, which completes the proof. □

We stress that unfortunately, we are not able to prove that (3.4) is necessary for the surjectivity of V. However, all construced examples show it is true. So, we may formulate the following conjecture.

Conjecture 3.6. *Let V be a surjective DQSO on S. Then there exists a sequence $\{j_n\}_{n\geq 1} \subset \mathbb{N}$ such that (3.4) holds.*

4. Quadratic stochastic operators on L^1 and associated DQSO

In this section, we consider QSO on L^1 and construct associated DQSO. Let $(X, \mathcal{F}, \lambda)$ be a measurable space with a σ-finite measure λ. By $L^1(X, \mathcal{F}, \lambda)$ we define an usual L^1 space. We notice that L^1 can be identified with the set of all measures (signed ones) absolutely continuous w.r.t. λ. Namely, for every non negative $f \in L^1(X, \mathcal{F}, \lambda)$ we can define a measure μ_f by

$$\mu_f(A) = \int_A f\,d\lambda, \qquad \forall A \in \Omega.$$

Therefore, in what follows, we may identify measures with functions and visa versa.

By $S(X)$ we denote the set of all probability measures on X which are absolutely continuous w.r.t. λ.

Recall that a collection of measurable sets $\mathcal{B} = \{B_k\}_{k\geq 1}$ is called a *partition of* X (w.r.t. λ) if it satisfies

(1) $X = \bigcup_{k\geq 1} B_k$;
(2) $B_i \cap B_j = \emptyset$ for all $i \neq j$;
(3) $0 < \lambda(B_k) < \infty$ for every $k \geq 1$.

We denote by $\mathcal{P}(X)$ the set of all partitions of X. Since λ is σ-finite, we infer that $\mathcal{P}(X) \neq \emptyset$.

Let us take $\mathcal{B} = \{B_k\}_{k\geq 1} \in \mathcal{P}(X)$. For any $\mathbf{x} \in \ell^1$ we define a measure $\mu_{\mathbf{x}}^{\mathcal{B}}$ on $\mathcal{F}$ as follows:

$$\mu_{\mathbf{x}}^{\mathcal{B}}(A) = \sum_{k=1}^{\infty} \frac{x_k}{\lambda(B_k)} \lambda(A \cap B_k), \qquad \forall A \in \mathcal{F}. \tag{4.1}$$

We notice that $\mu_{\mathbf{x}}^{\mathcal{B}}$ is not probability measure when $\|\mathbf{x}\|_{\ell^1} \neq 1$. A natural question arises: for what kind of $\mathbf{x} \in \ell^1$ is it true $\mu_{\mathbf{x}}^{\mathcal{B}} \in S(X)$?

Recall that $S = \{\mathbf{x} \in \ell^1 : x_i \geq 0, \ \forall i \geq 1 \text{ and } \| \mathbf{x} \|_{\ell^1} = 1\}$. Then the following result holds.

Lemma 4.1. *Let $\mu_{\mathbf{x}}^{\mathcal{B}}$ be a measure given by (4.1). Then $\mu_{\mathbf{x}}^{\mathcal{B}} \in S(X)$ iff $\mathbf{x} \in S$.*

Proof. Let us assume that $\mu_{\mathbf{x}}^{\mathcal{B}} \in S(X)$. We have $\mu_{\mathbf{x}}^{\mathcal{B}}(B_k) = x_k$ for every $k \in \mathbb{N}$. It yields that $0 \leq x_k \leq 1$ for every $k \in \mathbb{N}$. On the other hand, we obtain

$$1 = \mu_{\mathbf{x}}^{\mathcal{B}}(X) = \mu_{\mathbf{x}}^{\mathcal{B}}\Big(\bigcup_{k\geq 1} B_k\Big) = \sum_{k\geq 1} x_k,$$

which implies that $\mathbf{x} \in S$.

Now we suppose that $\mathbf{x} \in S$. Then $\mu_{\mathbf{x}}^{\mathcal{B}}(X) = \sum_{k\geq 1} x_k = 1$. This means that $\mu_{\mathbf{x}}^{\mathcal{B}}$ is a probability measure on X. Moreover, it is obvious that the measure given by (4.1) is absolutely continuous w.r.t. λ. Hence, we infer that $\mu_{\mathbf{x}} \in S(X)$. $\square$

Remark 4.2. For a given partition $\mathcal{B}$ of X, thanks to Lemma 4.1 there exists a one-to-one correspondence between S and $M(X, \mathcal{B}) := \{\mu_{\mathbf{x}}^{\mathcal{B}} \in S(X) : \ \mathbf{x} \in \ell^1\}$. In other words, every $\mu \in M(X, \mathcal{B})$ is uniquely defined by the values $\mu(B_k)$, $k \geq 1$.

Proposition 4.3. *Let $\mathcal{B} \in \mathcal{P}(X)$. Then $M(X, \mathcal{B})$ is a convex and closed set w.r.t. strong convergence. Moreover, $M(X, \mathcal{B})$ is not compact w.r.t. weak convergence.*

Proof. One can see that $T : S \to M(X, \mathcal{B})$ given by $T\mathbf{x} = \mu_{\mathbf{x}}^{\mathcal{B}}$ is a bijection. Then any sequence on $M(X, \mathcal{B})$ is defined by a sequence $\{\mathbf{x}^{(n)}\}_{n\geq 1} \subset S$. It is obvious that if $\mathbf{x}^{(n)} \stackrel{\|\cdot\|_{\ell_1}}{\longrightarrow} \mathbf{x}$ then $\lim_{n\to\infty} \mu_{\mathbf{x}^{(n)}}^{\mathcal{B}}(A) = \mu_{\mathbf{x}}^{\mathcal{B}}(A)$ for all $A \in \mathcal{F}$.

Let us pick a sequence $\{\mu_{\mathbf{x}^{(n)}}^{\mathcal{B}}\}_{n\geq 1} \subset M(X, \mathcal{B})$. Assume that $\mu(A) = \lim_{n\to\infty} \mu_{\mathbf{x}^{(n)}}^{\mathcal{B}}(A)$ for every $A \in \mathcal{F}$. Then we have $\mu \in S(X)$. On the other hand, we obtain

$$\mu(B_k) = \lim_{n\to\infty} x_k^{(n)}, \qquad \forall k \geq 1.$$

The last one together with $\mu(X) = 1$ implies that $\mathbf{x}^{(n)}$ converges on S w.r.t. ℓ^1-norm. Hence, we conclude that T is a homeomorphism. Then due to closedness and convexity of S we infer that $M(X, \mathcal{B})$ has the same topological properties. We notice that S is not compact w.r.t. ℓ^1-norm. Consequently, $M(X, \mathcal{B})$ is not a compact w.r.t. the weak convergence. $\square$

Lemma 4.4. *Let $\tilde{S}(X) = \big\{\mu_f \in S(X) : f$ is a simple function on $L^1\big\}$. Then*

$$\tilde{S}(X) = \bigcup_{\mathcal{B}\in\mathcal{P}(X)} M(X, \mathcal{B}). \tag{4.2}$$

Proof. It is clear that $\bigcup_{\mathcal{B}\in\mathcal{P}(X)} M(X, \mathcal{B}) \subset \tilde{S}(X)$. Indeed, for any $\mu_{\mathbf{x}}^{\mathcal{B}}$ we define a simple function

$$f_{\mu_{\mathbf{x}}^{\mathcal{B}}}(u) = \frac{x_k}{\lambda(B_k)}, \qquad \forall u \in B_k, \ \forall k \geq 1,$$

which satisfies

$$\mu_{\mathbf{x}}^{\mathcal{B}}(A) = \int_A f_{\mu_{\mathbf{x}}^{\mathcal{B}}} d\lambda, \qquad \forall A \in \mathcal{F}.$$

Now, we take an arbitrary $\mu_f \in \tilde{S}(X)$. Then for any $i \geq 1$ we have a measurable set $A_i = \{u \in X : f(u) = y_i\}$. One may assume that $\lambda(A_i) > 0$ for every $i \geq 1$. We notice that if $\lambda(A_i) < \infty$ for each $i \in \mathbb{N}$ then $\mathcal{A} = \{A_i\}_{i\geq 1}$ is a partition of X and $\mu_f = \mu_{\mathbf{x}}^{\mathcal{A}}$, where $\mathbf{x} = (\mu_f(A_1), \mu_f(A_2), \dots) \in S$.

If $\lambda(A_j) = \infty$ for some $j \geq 1$ then one has $y_j = 0$ (otherwise f is not integrable). Hence, $\mu_f(A_j) = 0$. So, without loss of generality we may assume that $y_1 = 0$, $\lambda(A_1) = \infty$ and $y_i > 0$, $\lambda(A_i) < \infty$ for any $i > 1$. Pick any partition $\{B_k\}_{k\geq 1}$ of X and define a new partition $\tilde{\mathcal{B}}$ of X as follows:

$$\tilde{B}_k = \begin{cases} A_1 \cap B_{\frac{k+1}{2}}, & \text{if } k \text{ is odd}, \\ A_{\frac{k+2}{2}}, & \text{if } k \text{ is even}. \end{cases}$$

Then $\mu_f = \mu_{\tilde{\mathbf{y}}}^{\tilde{\mathcal{B}}}$, where coordinates of $\tilde{\mathbf{y}} \in S$ are given by

$$\tilde{y}_k = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ \mu_f(A_{\frac{k+2}{2}}), & \text{if } k \text{ is even}. \end{cases}$$

The arbitrariness of μ_f yields that $\tilde{S}(X) \subset \bigcup_{\mathcal{B}\in\mathcal{P}(X)} M(X, \mathcal{B})$. The last one together with $\bigcup_{\mathcal{B}\in\mathcal{P}(X)} M(X, \mathcal{B}) \subset \tilde{S}(X)$ implies (4.2). $\square$

Due to the density argument, from Lemma 4.4 we immediately infer the following result.

Proposition 4.5. *One has $S(X) = \overline{\bigcup_{\mathcal{B}\in\mathcal{P}(X)} M(X, \mathcal{B})}$, here the closure in sense of weak convergence.*

4.1. Projective surjectivity of QSO

Let $(X, \mathcal{F}, \lambda)$, as before, be a measurable space with a σ-finite measure λ. Now, we consider a measurable function $P : X \times X \times \mathcal{F} \to [0; 1]$ which satisfies the following conditions:

$$P(u, v, A) = P(v, u, A), \qquad \forall u, v \in X, \quad \forall A \in \mathcal{F}, \tag{4.3}$$

$$P(u, v, \cdot) \in S(X), \qquad \forall u, v \in X. \tag{4.4}$$

This function is called *transition kernel*, and defines a *Quadratic Stochastic Operator* (in short QSO) by

$$(\mathcal{V}\mu)(A) = \int_X \int_X P(u, v, A) d\mu(u) d\mu(v), \qquad \forall \mu \in S(X), \ \forall A \in \mathcal{F}. \tag{4.5}$$

One can check that $\mathcal{V} : S(X) \to S(X)$. Moreover, we always mean that equivalent transition kernels define the same QSO on $S(X)$.

Remark 4.6. Let $(X, \mathcal{F}, \lambda)$ be as before. If the transition kernel is defined by

$$P(x, y, A) = \int_A q(x, y, z) d\lambda(z)$$

(where $q : X \times X \times X \to \mathbb{R}_+$ is a $\mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F}$ -measurable, nonnegative function with $q(x, y, z) = q(y, x, z)$ for any $x, y, z \in X$, and $\int_X q(x, y, z) d\lambda(z) = 1$ for every $(x, y) \in X \times X$), then the corresponding QSO is called *kernel QSO* [27,28]. One can see QSO given by (4.4) is more general than kernel QSO.

Definition 4.7. A QSO $\mathcal{V}$ given by (4.5) is called *projective surjection* if it is surjective on $M(X, \mathcal{B})$ for some $\mathcal{B} \in \mathcal{P}(X)$.

Now we are going to find QSO's which are projective surjection.

For a given QSO $\mathcal{V}$ we associate DQSO (this DQSO depends on a partition $\{B_k\}_{k\geq 1}$) $V_{\mathcal{B}} : S \to S$ by

$$(V_{\mathcal{B}}(\mathbf{x}))_k = \sum_{i,j\geq 1} P_{ij,k}^{\mathcal{B}} x_i x_j, \qquad \forall k \geq 1, \tag{4.6}$$

where

$$P^{\mathcal{B}}_{ij,k} = \frac{1}{\lambda(B_i)\lambda(B_j)}\int_{B_i}\int_{B_j} P(u,v,B_k)d\lambda(u)d\lambda(v), \qquad \forall k \geq 1. \tag{4.7}$$

Lemma 4.8. *Let $\mathcal{B} = \{B_k\}_{k\geq 1} \in \mathcal{P}(X)$. Then for every $\mathbf{x} \in S$ it holds*

$$(\mathcal{V}\mu^{\mathcal{B}}_{\mathbf{x}})(B_k) = (V_{\mathcal{B}}(\mathbf{x}))_k, \qquad \forall k \geq 1.$$

Proof. Let $\mathcal{B} = \{B_k\}_{k\geq 1}$ be a partition of X and $\mathbf{x} \in S$. Then for any $k \geq 1$ we have

$$\begin{aligned}(\mathcal{V}\mu^{\mathcal{B}}_{\mathbf{x}})(B_k) &= \int_X\int_X P(u,v,B_k)d\mu^{\mathcal{B}}_{\mathbf{x}}(u)d\mu^{\mathcal{B}}_{\mathbf{x}}(v)\\ &= \sum_{i,j\geq 1}\frac{x_ix_j}{\lambda(B_i)\lambda(B_j)}\int_{B_i}\int_{B_j} P(u,v,B_k)d\lambda(u)d\lambda(v)\\ &= \sum_{i,j\geq 1} P^{\mathcal{B}}_{ij,k}x_ix_j\\ &= (V_{\mathcal{B}}(\mathbf{x}))_k.\end{aligned}$$

□

Proposition 4.9. *Let $\mathcal{V}$ be a QSO given by (4.5) and $\mathcal{B} \in \mathcal{P}(X)$. If $P(u,v,\cdot) \in M(X,\mathcal{B})$ for every $(u,v) \in X^2$ then $\mathcal{V}(M(X,\mathcal{B})) \subset M(X,\mathcal{B})$.*

Proof. Let $\mathcal{B} = \{B_k\}_{k\geq 1}$ be partition of X. Assume that $P(u,v,\cdot) \in M(X,\mathcal{B})$ for every $(u,v) \in X^2$. Then for arbitrary $(u,v) \in X^2$ we obtain

$$P(u,v,A_k) = \frac{\lambda(A_k)}{\lambda(B_k)}P(u,v,B_k), \qquad \forall A_k \subset B_k,\ \forall k \geq 1. \tag{4.8}$$

For any $\mathbf{x} \in S$ we define $\mathbf{y} \in S$ as follows $y_k = (V_{\mathcal{B}}(\mathbf{x}))_k$, $k \geq 1$. The due to Lemma 4.8 we get

$$(\mathcal{V}\mu^{\mathcal{B}}_{\mathbf{x}})(B_k) = y_k, \qquad \forall k \geq 1.$$

Let us establish that $\mathcal{V}\mu^{\mathcal{B}}_{\mathbf{x}} = \mu^{\mathcal{B}}_{\mathbf{y}}$. Take an arbitrary measurable $A \in \mathcal{F}$ and denote $A_k = A \cap B_k$ for every $k \geq 1$. Keeping in mind (4.8) one gets

$$\begin{aligned}(\mathcal{V}\mu^{\mathcal{B}}_{\mathbf{x}})(A) &= \int_X\int_X P(u,v,A)d\mu^{\mathcal{B}}_{\mathbf{x}}(u)d\mu^{\mathcal{B}}_{\mathbf{x}}(v)\\ &= \sum_{i,j\geq 1}\frac{x_ix_j}{\lambda(B_i)\lambda(B_j)}\sum_{k\geq 1}\int_{B_i}\int_{B_j} P(u,v,A_k)d\lambda(u)d\lambda(v)\\ &= \sum_{i,j\geq 1}\frac{x_ix_j}{\lambda(B_i)\lambda(B_j)}\sum_{k\geq 1}\frac{\lambda(A_k)}{\lambda(B_k)}\int_{B_i}\int_{B_j} P(u,v,B_k)d\lambda(u)d\lambda(v)\\ &= \sum_{k\geq 1}\frac{\lambda(A_k)}{\lambda(B_k)}y_k\\ &= \mu^{\mathcal{B}}_{\mathbf{y}}(A),\end{aligned}$$

which yields $\mathcal{V}\mu^{\mathcal{B}}_{\mathbf{x}} = \mu^{\mathcal{B}}_{\mathbf{y}}$. The arbitrariness of $\mathbf{x} \in S$ implies $\mathcal{V}(M(X,\mathcal{B})) \subset M(X,\mathcal{B})$. The proof is complete. □

Remark 4.10. We point out that a QSO is given by (4.5) with $P(u,v,\cdot) \in M(X,\mathcal{B})$ for every $(u,v) \in X^2$, has been considered in [28], and its domain is $M(X,\mathcal{B})$.

Now we are going to find sufficiency conditions for the projective surjectivity of QSO given by (4.5).

Theorem 4.11. *Let $(X,\mathcal{F},\lambda)$ be a measurable space with a σ-finite measure λ and $\mathcal{B} = \{B_k\}_{k\geq 1} \in \mathcal{P}(X)$. Assume that the transition kernel P of QSO $\mathcal{V}$ satisfies the followings conditions:*

(i) *$P(u,v,\cdot) \in M(X,\mathcal{B})$, for all $(u,v) \in X^2$;*
(ii) *there exists a sequence $\{j_n\}_{n\geq 1} \subset \mathbb{N}$ such that*

$$P(u,v,B_k) = \frac{\lambda(B_k\cap B_n)}{2\lambda(B_n)} + \frac{\lambda(B_k\cap B_m)}{2\lambda(B_m)},\ \forall (u,v) \in B_{j_n}\times B_{j_m},\ \forall k \in \mathbb{N}. \tag{4.9}$$

Then $\mathcal{V}$ is projective surjection.

Proof. Assume that all conditions of the theorem hold. From the condition (i), according to Proposition 4.9 we have $\mathcal{V} : M(X,\mathcal{B}) \to M(X,\mathcal{B})$.

Now, let us show that $\mathcal{V}(M(X,\mathcal{B})) = M(X,\mathcal{B})$. For any triple $(n,m,k) \in \mathbb{N}^3$ from (4.9) after simple calculations, we get

$$P^{\mathcal{B}}_{j_nj_m,k} = P^{\mathcal{B}}_{j_mj_n,k} = \begin{cases}1, & \text{if } n = m = k;\\ \frac{1}{2}, & \text{if } n = k \neq m;\\ 0, & \text{if } k \notin \{n,m\}.\end{cases}$$

Hence, by Theorem 3.5, the corresponding DQSO $V_{\mathcal{B}}$ is a surjection. So, for any $\mathbf{y} \in S$ one can find $\mathbf{x} \in S$ such that $V_{\mathcal{B}}(\mathbf{x}) = \mathbf{y}$. Consequently, Lemma 4.8 implies that

$$(\mathcal{V}\mu^{\mathcal{B}}_{\mathbf{x}})(B_k) = \mu^{\mathcal{B}}_{\mathbf{y}}(B_k), \qquad \forall k \in \mathbb{N}.$$

From the last one, keeping in mind $\mathcal{V}\mu^{\mathcal{B}}_{\mathbf{x}} \in M(X,\mathcal{B})$ thanks to Remark 4.2 we infer

$$\mathcal{V}\mu^{\mathcal{B}}_{\mathbf{x}} = \mu^{\mathcal{B}}_{\mathbf{y}}.$$

Finally, the arbitrariness of $\mathbf{y} \in S$ yields $\mathcal{V}(M(X,\mathcal{B})) = M(X,\mathcal{B})$. This completes the proof. □

Remark 4.12. We notice that the conclusion of the last theorem will be true if (4.9) holds almost everywhere in $B_{j_n}\times B_{j_m}$.

For any measurable set $A \subset X$ we define

$$\mathcal{E}_A = \left\{(x,y) \in (X\setminus A)^2 :\ P(x,y,A) \neq 0\right\}.$$

Theorem 4.13. *Let $(X,\mathcal{F},\lambda)$ be a measurable space with a σ-finite measure λ and $\mathcal{B} = \{B_k\}_{k\geq 1} \in \mathcal{P}(X)$. Then there is only one QSO $\mathcal{V}$ whose transition kernel satisfies the followings:*

(i) *$P(u,v,\cdot) \in M(X,\mathcal{B})$, $\quad\forall (u,v) \in X^2$;*
(ii) *$\lambda(\mathcal{E}_{B_k}) = 0$ for every $k \in \mathbb{N}$.*

Moreover, $\mathcal{V}$ is projective surjection.

Proof. From (i) we have $P(u,v,\cdot) = \mu^{\mathcal{B}}_{\mathbf{x}(u,v)}$ for any $u,v \in X$. Without loss of generality we may replace the condition (ii) to

$$\mu^{\mathcal{B}}_{\mathbf{x}(u,v)}(B_k) = 0, \qquad \forall (u,v) \in (X\setminus B_k)^2,\ \forall k \in \mathbb{N}.$$

Then, for any $(u,v) \in B_n\times B_m$

$$\mu^{\mathcal{B}}_{\mathbf{x}(u,v)}(B_n) + \mu^{\mathcal{B}}_{\mathbf{x}(u,v)}(B_m) = 1.$$

Keeping in mind $\mathbf{x}(u,v) = \mathbf{x}(v,u)$, from the last one, we have $\mathbf{x}(u,v) = \frac{1}{2}\mathbf{e}_n + \frac{1}{2}\mathbf{e}_m$ for every $(u,v) \in B_n\times B_m$. Hence,

$$P(u,v,A) = \frac{\lambda(A\cap B_n)}{2\lambda(B_n)} + \frac{\lambda(A\cap B_m)}{2\lambda(B_m)}, \qquad \forall (u,v) \in B_n\times B_m,\ \forall A \in \mathcal{F}. \tag{4.10}$$

We notice that (4.10) implies (4.9) for the sequence $\{n\}_{n\geq 1}$. Then Theorem 4.11 implies that $\mathcal{V}$ is projective surjection. The proof is complete. □

Corollary 4.14. *Let $(X,\mathcal{F},\lambda)$ be a measurable space with a σ-finite measure λ and $\mathcal{B} = \{B_k\}_{k\geq 1} \in \mathcal{P}(X)$. There is only one QSO $\mathcal{V}$ whose transition kernel satisfies the followings:*

(i) *$P(u,v,\cdot) \in M(X,\mathcal{B})$, $\quad\forall u,v \in X$;*
(ii) *$\lambda(\mathcal{E}_A) = 0$ for every $A \in \mathcal{F}$.*

Moreover, $\mathcal{V}$ is projective surjection.

5. Application

In this section we give a direct application of the projective surjectivity of QSO to the existence of positive solutions of certain nonlinear integral equations.

Let $(X, \mathcal{F}, \lambda)$ be a measurable space with a σ-finite measure λ. Let us consider the following nonlinear Hammerstein integral equation:

$$\int_X \int_X K(u, v, t)x(u)x(v)d\lambda(u)d\lambda(v) = \varphi(t), \tag{5.1}$$

where K is some positive kernel and $\varphi \in L^1_+$ is a given function.

We note that this type of equation appeared in several problems of astrophysics, mechanics, and biology. Here in the equation, $K : X^3 \to \mathbb{R}$ and $\varphi : X \to \mathbb{R}$ are given functions, and $x : \Omega \to \mathbb{R}$ is an unknown one. Generally speaking, in order to solve the nonlinear Hammerstein integral Eq. (5.1) over some functions space, one should impose on some constraints on $K(\cdot, \cdot, \cdot)$. There are several works where the existence of solutions the above given equation have been carried out by means of contraction methods (see [35–38]). In this section, we are going to another approach for the existence of positive solutions of (5.1). In what follows, we consider the Eq. (5.1) over L^1-spaces.

Multiplying (5.1) by a function g from L^∞ and integrating it, we obtain

$$\int_X \int_X \int_X g(t)K(u, v, t)x(u)x(v)d\lambda(u)d\lambda(v)d\lambda(t) = \int_X g(t)\varphi(t)d\lambda(t). \tag{5.2}$$

We stress that the arbitrariness of g implies that (5.2) and (5.1) are equivalent.

Now, assume that there is a transition kernel P such that

$$\int_X \int_X \int_X g(t)K(u, v, t)x(u)x(v)d\lambda(u)d\lambda(v)d\lambda(t) = \int_X \int_X \int_X g(t)P(u, v, dt)x(u)x(v)d\lambda(u)d\lambda(v)$$

for all $x \in L^1$ and $g \in L^\infty$.

Then (5.2) is reduced to

$$\int_X \int_X \int_X g(t)P(u, v, dt)x(u)x(v)d\lambda(u)d\lambda(v) = \int_X g(t)\varphi(t)d\lambda(t).$$

Now, taking $g = \chi_A$, $A \in \mathcal{F}$, we arrive at

$$(\mathcal{V}\mu_x)(A) = \mu_\varphi(A),$$

where, as before, $\mu_x(A) = \int_A x(u)d\lambda(u)$. Assume that

$$\int_X \varphi d\lambda = 1.$$

Hence, the integral Eq. (5.1) is reduced to

$$\mathcal{V}\mu = \mu_\varphi, \tag{5.3}$$

where $\mu \in S(X)$.

Hence, the following result is true.

Theorem 5.1. *Let $(X, \mathcal{F}, \lambda)$ be a measurable space with a σ-finite measure λ and $\mathcal{B} = \{B_k\}_{k\geq 1} \in \mathcal{P}(X)$. Assume that QSO $\mathcal{V}$ is a projective surjection on $M(X, \mathcal{B})$. Then for any $\mu_\varphi \in M(X, \mathcal{B})$ the Eq. (5.3) has a solution in $M(X, \mathcal{B})$.*

Credit Author Statement

The all authors of this paper are equaly contributed for the realization of the results.

Declaration of Competing Interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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References

[1] Kolokoltsov VN. Nonlinear markov processes and kinetic equations. New York: Cambridge Univ Press; 2010.
[2] McKean HP. A class of markov processess associated with nonlinear parabolic equations. Proc Natl Acad Sci USA 1966;56:1907–11.
[3] Ganikhodzhaev R, Mukhamedov F, Rozikov U. Quadratic stochastic operators and processes: results and open problems. Infin Dimens Anal Quant Prob Relat Top 2011;14:279–335.
[4] Frank TD. Nonlinear fokker-Planck equations fundamentals and applications. Springer-Verlag, Berlin; 2005.
[5] Raftery A. A model of high-order markov chains. J Royal Statist Soc 1985;47:528–39.
[6] Cui LB, Li W, Ng MK. Birkhoff–von neumann theorem for multistochastic tensors. SIAM J Matrix Anal Appl 2014;35:956–73.
[7] Li W, Ng MK. On the limiting probability distribution of a transition probability tensor. Linear And Multilinear Algebra 2014;62:362–85.
[8] Li C-K, Zhang S. Stationary probability vectors of higher-order markov chains. Linear Algebra Appl 2015;473:114–25.
[9] Qi L, Luo Z. Tensor analysis: spectral theory and special tensors. SIAM; 2017.
[10] Saburov M. Ergodicity of nonlinear markov operators on the finite dimensional space. Nonlin Anal Theor Methods Appld 2016;143:105–19.
[11] Qi L. Eigenvalues of a real supersymmetric tensor. J Symbolic Comput 2005;40:1302–24.
[12] Mukhamedov F, Embong AF. On non-linear markov operators: surjectivity vs orthogonal preserving property. Lin Multilin Alg 2018;66:2183–90.
[13] Jamilov UU. Quadratic stochastic operators corresponding to graphs. Lobachevskii J Math 2013;34:148–51.
[14] Bernstein SN. The solution of a mathematical problem concerning the theory of heredity. Ann Math Statistics 1924;13:53–61.
[15] Lyubich YI. Mathematical structures in population genetics. Berlin, Springer-Verlag; 1992.
[16] Mukhamedov F, Ganikhodjaev N. Quantum quadratic operators and processes. Berlin: Springer; 2015.
[17] Mukhamedov F, Khakimov ON, Embong AF. On surjective second order non–linear markov operators and associated nonlinear integral equations. Positivity 2018;22:1445–59.
[18] Mukhamedov F, Khakimov O, Embong AF. Solvability of nonlinear integral equations and surjectivity of non-linear markov operators. Math Methods in Appl Sci 2020;43(15):9102–18.
[19] Mukhamedov F, Khakimov O, Embong AF. On omega limiting sets of infinite dimensional volterra operators. Nonlinearity 2020;33:5875–904.
[20] Mukhamedov F, Embong AF. Infinite dimensional orthogonality preserving nonlinear markov operators. Lin Multilin Alg 2021;69(3):526–50.
[21] Mukhamedov F, Embong AF, Rosli A. Orthogonal preserving and surjective cubic stochastic operators. Ann Funct Anal 2017;8:490–501.
[22] Ganikhodjaev N, Saburov M, Muhitdinov R. On lebesgue nonlinear transformations. Bull Korean Math Soc 2017;54:607–18.
[23] Mukhamedov F. On l_1-weak ergodicity of nonhomogeneous discrete markov processes and its applications. Rev Mat Complut 2013;26:799–813.
[24] Mukhamedov F, Taha MH. On volterra and orthoganality preserving quadratic stochastic operators. Miskloc Math Notes 2016;17:457–70.
[25] Badocha M, Bartoszek W. Quadratic stochastic operators on banach lattices. Positivity 2018;22:477–92.
[26] Bartoszek K, Domsta J, Pulka M. Weak stability of centred quadratic stochastic operators. Bulletin Malays Math Sci Soc 2019;42:1813–30.
[27] Bartoszek K, Pulka M. Asymptotic properties of quadratic stochastic operators acting on the l^1 space. Nonlinear Anal Theory Methods Appl 2015;114:26–39.
[28] Bartoszek K, Pulka M. Prevalence problem in the set of quadratic stochastic operators acting on l^1. Bulletin Malays Math Sci Soc 2018;41:159–73.
[29] Ganikhodjaev NN, Akin H, Mukhamedov F. On the ergodic principle for markov and quadratic stochastic processes and its relations. Linear Algebra Appl 2006;416:730–41.
[30] Mukhamedov F, Saburov M. On homotopy of volterrian quadratic stochastic operator. Appl Math & Inform Sci 2010;4:47–62.
[31] Ganikhodzhaev R, Mukhamedov F, Saburov M. Elliptic quadratic operator equations. Acta Appl Math 2019;159(1):29–74.

[32] Mukhamedov FM. On infinite dimensional volterra operators. Russian Math Surveys 2000;55:1161–2.
[33] Mukhamedov F, Akin H, Temir S. On infinite dimensional quadratic volterra operators. J Math Anal Appl 2005;310:533–56.
[34] Akin H, Mukhamedov F. Orthogonality preserving infinite dimensional quadratic stochastic operators. AIP Conf Proc 2015;1676:020008.
[35] Atkinson KE. A survey of numerical methods for solving nonlinear integral equations. J Integral Equations 1992;4:15–46.
[36] Banas J, Martinon A. Monotonic solutions of a quadratic integral equation of volterra type. Comput Math Appl 2004;47:271–9.
[37] Krasnosel'skii MA. Topological methods in the theory of nonlinear integral equations. Pergamon; 1964.
[38] Some B. Some recent numerical methods for solving nonlinear hammerstein integral equations. Math and Computer Modelling 1993;18(9):55–62.