The Direct Product of p_i-Cayley Graph for Alt(4) and Sym(4)

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Abstract. A direct product graph is a graph that is formed from the direct product of two different graphs for two groups G and H, labelled as Γ_G and Γ_H. Suppose x_1 and y_1 be the elements in Γ_G and, x_2 and y_2 be the elements in Γ_H. Then, two vertices (x_1, x_2) and (y_1, y_2) are connected if x_1 and y_1 are connected in Γ_G, and x_2 and y_2 are connected in Γ_H . In this research, a new type of graph is introduced and constructed, namely the p_i-Cayley graph. This graph is constructed for the symmetric group of order 24 and alternating group of order 12. The graphs obtained are the regular graphs. Then, the direct product of the graphs obtained is also found.

INTRODUCTION

Cayley graph is a graph that is formed by considering the subsets of the group. In 2017, Tripi [1] explored the Cayley graph to study the pattern of bell ringing by using permutation groups. Khosravi and Mahmoudi [2] found the characterization of Cayley graph for rectangular groups. In graph theory, the direct product of graphs is one of the topics that has been studied by many researchers. Klavzar [3] has done a study on colouring of vertices for the direct product, Cartesian product and strong product of graphs. Then, Bresar *et al*. [4] in 2007 explored on the dominating number of direct products of graphs. In this paper, a new graph is introduced, called a p_i-Cayley graph. Then, the direct product of this graph is constructed. This paper consists of four sections. The introduction section is followed by the preliminaries section in which some basic definitions in group theory and graph theory are stated. The third section discusses the results of this research. The last section gives a conclusion.

PRELIMINARIES

Related definitions such as the definitions of direct product of graphs, symmetric group and alternating group that are used in this research are stated as follows.

Definition 1 *[4] (Direct product of graphs)*
For graphs Γ_G and Γ_H, the direct product $\Gamma_G \times \Gamma_H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (x, y) and (v, w) are adjacent if and only if $xv \in E(G)$ and $yw \in E(H)$.

Definition 2 *[5] (Symmetric group)*

Let A be the finite set $\{1, 2, \ldots, n\}$. The group of all permutations of A is the symmetric group on n letters, and is denoted by Sym(n). Note that Sym(n) has $n!$ elements, where $n! = n(n-1)(n-2)\ldots(3)(2)(1)$.

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Definition 3 *[6] (Alternating group)*

The set of all even permutations in Sym(n) forms a subgroup of Sym(n) for each $n \geq 2$. This subgroup is called the alternating group of degree n, denoted by Alt(n). The order of Alt(n) is $\frac{1}{2}n!$.

RESULTS AND DISCUSSION

The focus of this research is on the alternating group of order 12, Alt(4) and the symmetric group of order 24, Sym(4).

The elements of Alt(4) are as listed below:

$$Alt(4) = \{(1), (13)(24), (12)(34), (14)(23), (123), (132), (234), (142), (243), (143), (134), (124)\},$$

while the elements of Sym(4) are listed as follows:

$$\begin{aligned} Sym(4) = \{&(1), (12), (23), (24), (13), (14), (34), (13)(24), (12)(34), (14)(23), (123), (132), (234),\\ &(142), (243), (143), (134), (124), (1234), (1342), (1423), (1243), (1324), (1432)\}. \end{aligned}$$

The main steps in this research are the construction of p_i-Cayley graph and direct product of graphs obtained for Alt(4) and Sym(4).

A new graph is defined, which is called a p_i-Cayley graph. The definition of p_i-Cayley graph is given in the following.

Definition 4 *Let G be a group and $S_{(p_i)}$ be non-empty subset of G. Let the order of G be the product of power of primes, $|G| = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$, $S_{(p_i)} = \{x \in G : |x| = p_i^n, n \in \mathbb{N}, n = 1, 2, \ldots, \alpha_i\}$, and $S^{-1} = S$. The p_i-Cayley graph, denoted as $Cay_{p_i}(G, S_{(p_i)})$, is a graph where G is the set of vertices of the group and two different vertices, g and h, are connected if $gh^{-1} \in S_{(p_i)}$ for all $g, h \in G$.*

The direct product of p_i-Cayley graphs for Alt(4)

In this section, the direct product of p_i-Cayley graphs for Alt(4) is constructed which is presented in the following lemmas, followed by a theorem.

Lemma 1 *The 2-Cayley graph of the alternating group of order 12, Alt(4) is the regular graph where each vertex has degree three.*

Proof
The order of Alt(4) is 12, which can be written as $2^2 \times 3$. Let $S_{(2)}$ be a subset of Alt(4) where all the elements are order of 2^n where $n = 1, 2$. Then, based on Definition 4, $S_{(2)} = \{(13)(24), (12)(34), (14)(23)\}$. The 2-Cayley graph of Alt(4) denoted as $Cay_2(Alt(4), S_{(2)})$ is constructed by taking all elements of Alt(4) as the vertices and form the edges that satisfy Definition 4 i.e. g and h are connected if $gh^{-1} \in S_{(2)}$ for all $g, h \in Alt(4)$. The connections between vertices to form edges can be seen in Table 1. The • symbol in the table represents the connection between two distinct vertices.

TABLE 1: Relation between the vertices

Vertices	(1)	(123)	(132)	(234)	(13)(24)	(142)	(12)(34)	(243)	(143)	(134)	(124)	(14)(23)
(1)					•		•					•
(123)						•		•		•		
(132)				•					•		•	
(234)			•						•		•	
(13)(24)	•						•					•
(142)		•						•		•		
(12)(34)	•				•							•
(243)		•				•				•		
(143)			•	•							•	
(134)		•				•		•				
(124)			•	•					•			
(14)(23)	•				•		•					

The following examples are to show the connectivity between two vertices.
Let $g = (142)$ and $h = (134)$ be the vertices for $Cay_2(Alt(4), S_{(2)})$. The inverse for h, h^{-1} is (143). Then,

$$\begin{aligned} gh^{-1} &= (142) \cdot (143) \\ &= (12)(34) \in S_{(2)} \end{aligned} \tag{1}$$

Since $gh^{-1} \in S_{(2)}$ for $g = (142)$ and $h = (134)$, then g and h are connected in $Cay_2(Alt(4), S_{(2)})$.

Let $g = (13)(24)$ and $h = (123)$ be the vertices for $Cay_2(Alt(4), S_{(2)})$. The inverse for h, h^{-1} is (132). Then,

$$\begin{aligned} gh^{-1} &= (13)(24) \cdot (132) \\ &= (234) \notin S_{(2)} \end{aligned} \tag{2}$$

Since $gh^{-1} \notin S_{(2)}$ for $g = (13)(24)$ and $h = (123)$, then g and h are not connected in $Cay_2(Alt(4), S_{(2)})$.
Therefore, based on the vertices and edges, the regular graph of 2-Cayley graph is constructed as shown in Figure 1.

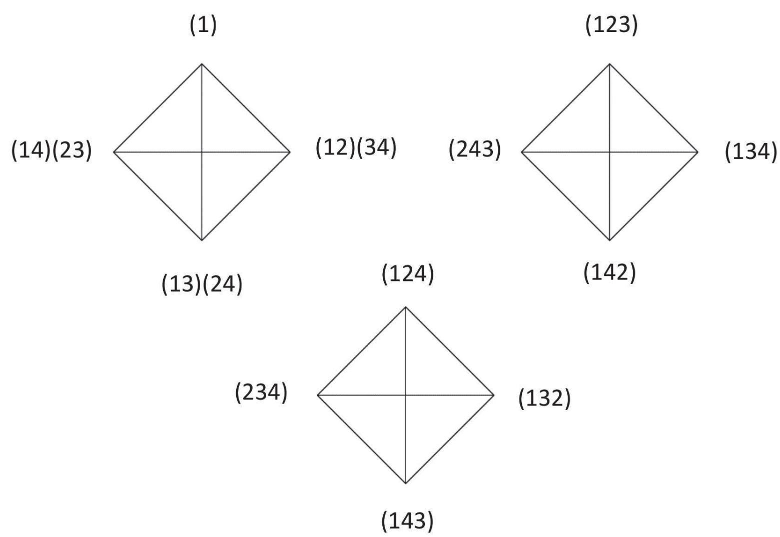

FIGURE 1. The 2-Cayley graph of Alt(4)

Lemma 2 *The 3-Cayley graph of the alternating group of order 12, Alt(4) is the regular graph where each vertex has degree eight.*

Proof

The order of Alt(4) is 12, which can be written as $2^2 \times 3$. Let $S_{(3)}$ be a subset of Alt(4) where all the elements are order of 3^n where $n = 1$. Then, based on Definition 4, $S_{(3)} = \{(123), (132), (234), (142), (243), (143), (134), (124)\}$. The 3-Cayley graph of Alt(4) denoted as $Cay_3(Alt(4), S_{(3)})$ is constructed by taking all elements of Alt(4) as the vertices and the relation between the vertices for 3-Cayley graph can be seen in Table 2. The • symbol in the table represents the connection between two distinct vertices.

TABLE 2: Relation between the vertices

Vertices	(1)	(123)	(132)	(234)	(13)(24)	(142)	(12)(34)	(243)	(143)	(134)	(124)	(14)(23)
(1)		•	•	•		•		•	•	•	•	
(123)	•		•	•	•		•		•		•	•
(132)	•	•			•	•	•	•		•		•
(234)	•	•			•	•	•	•		•		•
(13)(24)		•	•	•		•		•	•	•	•	
(142)	•		•	•	•		•		•		•	•
(12)(34)		•	•	•		•		•	•	•	•	
(243)	•		•	•	•		•		•		•	•
(143)	•	•			•	•	•	•		•		•
(134)	•		•	•	•		•		•		•	•
(124)	•	•			•	•	•	•		•		•
(14)(23)		•	•	•		•		•	•	•	•	

The following example is to show the connectivity between two vertices for $Cay_3(Alt(4), S_{(3)})$.
Let $g = (123)$ and $h = (12)(34)$ be the vertices for $Cay_3(Alt(4), S_{(3)})$. The inverse for h, h^{-1} is (12)(34). Then,

$$\begin{aligned} gh^{-1} &= (123) \cdot (12)(34) \\ &= (134) \in S_{(3)} \end{aligned} \tag{3}$$

Since $gh^{-1} \in S_{(3)}$ for $g = (123)$ and $h = (12)(34)$, then g and h are connected in $Cay_3(Alt(4), S_{(3)})$.

Let $g = (123)$ and $h = (243)$ be the vertices for $Cay_3(Alt(4), S_{(3)})$. The inverse for h, h^{-1} is (234). Then,

$$\begin{aligned} gh^{-1} &= (123) \cdot (234) \\ &= (12)(34) \notin S_{(3)} \end{aligned} \tag{4}$$

Since $gh^{-1} \notin S_{(3)}$ for $g = (123)$ and $h = (243)$, then g and h are not connected in $Cay_3(Alt(4), S_{(3)})$.
Therefore, based on the vertices and edges, the regular graphs of 3-Cayley graph is constructed as shown in Figure 2.

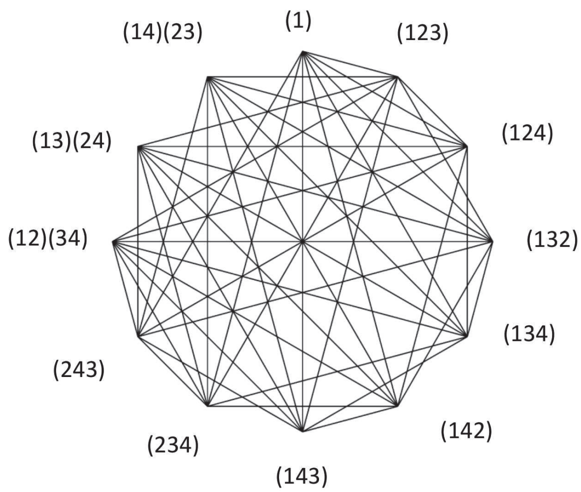

FIGURE 2. The 3-Cayley graph of Alt(4)

Theorem 1 *The direct product of 2-Cayley graph and 3-Cayley graph for the alternating group of order 12, Alt(4) is a regular graph with 144 vertices where each vertex has degree 24.*

Proof
From Lemma 1, the 2-Cayley and 3-Cayley graphs of Alt(4) are found to be regular graphs. Since the order of both groups is 12, then the number of elements of direct product for these two graphs is 144 elements, where these elements are the vertices of the graph. There are 144 vertices for direct product of 2-Cayley graph and 3-Cayley graph for Alt(4), $Cay_2(Alt(4), S_{(2)}) \times Cay_3(Alt(4), S_{(3)})$.

Let (a, b) is any vertices in $Cay_2(Alt(4), S_{(2)}) \times Cay_3(Alt(4), S_{(3)})$ where a is the vertices in $Cay_2(Alt(4), S_{(2)})$ and b is the vertices in $Cay_3(Alt(4), S_{(3)})$. Let $((1), (1))$ as one of the vertices for $Cay_2(Alt(4), S_{(2)}) \times Cay_3(Alt(4), S_{(3)})$. Based on Definition 1, $((1), (1))$ is connected with (a, b) if (1) and a are connected in $Cay_2(Alt(4), S_{(2)})$, and (1) and b are connected in $Cay_3(Alt(4), S_{(3)})$.

From Table 1, $(1) \in Cay_2(Alt(4), S_{(2)})$ is connected with (12)(34), (13)(24) and (14)(23). From Table 2, $(1) \in Cay_3(Alt(4), S_{(3)})$ is connected with (123), (132), (234), (142), (243), (143), (134) and (124).

Hence, $((1), (1))$ is connected with $((12)(34), (123))$, $((12)(34), (132))$, $((12)(34), (234))$, $((12)(34), (142))$, $((12)(34), (243))$, $((12)(34), (143))$, $((12)(34), (134))$, $((12)(34), (124))$, $((13)(24), (123))$, $((13)(24), (132))$, $((13)(24), (234))$, $((13)(24), (142))$, $((13)(24), (243))$, $((13)(24), (143))$, $((13)(24), (134))$, $((13)(24), (124))$, $((14)(23), (123))$, $((14)(23), (132))$, $((14)(23), (234))$, $((14)(23), (142))$, $((14)(23), (243))$, $((14)(23), (143))$, $((14)(23), (134))$ and $((14)(23), (124))$. The degree for $((1), (1))$ is 24.

The same procedure are repeated for all other vertices in $Cay_2(Alt(4), S_{(2)}) \times Cay_3(Alt(4), S_{(3)})$ and it can be concluded that all vertices have degree 24. Therefore, $Cay_2(Alt(4), S_{(2)}) \times Cay_3(Alt(4), S_{(3)})$ is a regular graph with 144 vertices where each vertex has degree 24.

In the next section, the direct product of p_i-Cayley graphs for Sym(4) are presented by using same the procedures as in Lemma 1, Lemma 2 and Theorem 1.

The direct product of p_i-Cayley graphs for Sym(4)

In this section, the direct product of p_i-Cayley graphs for Sym(4) is constructed as written in the following lemmas and theorem.

Lemma 3 *The 2-Cayley graph of the symmetric group of order 24, Sym(4) is a regular graph where each vertex has degree 15.*

Proof
The order of Sym(4) is 24, which can be written as $2^3 \times 3$. Let $S_{(2)}$ be a subset of Sym(4) where all the elements are order of 2^n where $n = 1, 2, 3$. There are 15 elements in $S_{(2)}$ which are (12), (23), (24), (13), (14), (34), (13)(24), (12)(34), (14)(23), (1234), (1342), (1423), (1243), (1324), (1432).

The 2-Cayley graph of Sym(4) denoted as $Cay_2(Sym(4), S_{(2)})$ is constructed by taking all elements of Sym(4) as the vertices and form the edges that satisfy Definition 4 i.e. g and h are connected if $gh^{-1} \in S_{(2)}$ for all $g, h \in$.

Let (12) $\in$ as one of the vertices in $Cay_2(Sym(4), S_{(2)})$. The product of (12) with the inverses of (1), (143), (243), (1324), (13)(24), (134), (14)(23), (1423), (234), (123), (124), (132), (142), (12)(34) and (34) is the elements in $S_{(2)}$. For example, $(12) \cdot (143)^{-1} = (12) \cdot (134) = (1342)$. Hence, the degree of (12) is 15.

The same calculations are repeated for other vertices. It can be concluded that all vertices have degree 15. Therefore, the 2-Cayley graph of the symmetric group of order 24, Sym(4) is a regular graph where each vertex has degree 15.

Lemma 4 *The 3-Cayley graph of the symmetric group of order 24, Sym(4) is a regular graph where each vertex has degree eight.*

Proof
The proof is similar to the proof of Lemma 2 since $S_{(3)}$ in Alt(4) is similar to $S_{(3)}$ in Sym(4).

Theorem 2 *The direct product of 2-Cayley graph and 3-Cayley graph for the symmetric group of order 24, Sym(4) is a regular graph with 576 vertices where each vertex has degree 120.*

Proof
From Lemma 1, the 2-Cayley and 3-Cayley graphs of Sym(4) are found to be regular graphs. Since the order of both groups is 24, then the number of elements of direct product for these two graphs is 576 elements, where these elements are the vertices of the graph. There are 576 vertices for direct product of 2-Cayley graph and 3-Cayley graph for Sym(4), $Cay_2(Sym(4), S_{(2)}) \times Cay_3(Sym(4), S_{(3)})$.

Let (a, b) is any vertices in $Cay_2(Sym(4), S_{(2)}) \times Cay_3(Sym(4), S_{(3)})$ where a is any vertex in $Cay_2(Sym(4), S_{(2)})$ and b is any vertex in $Cay_3(Sym(4), S_{(3)})$. The vertex a is connected with 15 vertices in $Cay_2(Sym(4), S_{(2)})$ and the vertex b is connected with eight vertices in $Cay_3(Sym(4), S_{(3)})$. Hence, vertex (a, b) is connected with 120 vertices in $Cay_2(Sym(4), S_{(2)}) \times Cay_3(Sym(4), S_{(3)})$. Therefore, $Cay_2(Sym(4), S_{(2)}) \times Cay_3(Sym(4), S_{(3)})$ is a regular graph with 576 vertices where each vertex has degree 120.

Conclusion

In this paper, a new type of graph is introduced, named as p_i-Cayley graph. The 2-Cayley and 3-Cayley graphs for Alt(4) and Sym(4) are constructed based on the prime factorization of the groups' order. The 2-Cayley graph and 3-Cayley graph of Alt(4) turned out to be regular graphs with degree three and degree eight, respectively. Moreover, the 2-Cayley and 3-Cayley graphs for Sym(4) turned out to be regular graphs with degree 15 and degree eight, respectively. The direct product between two different graphs formed another graph, named as a direct product graph. In this paper, the direct product graph of the 2-Cayley and 3-Cayley graphs for Alt(4) is found to be a regular graph with 144 vertices where each vertex has degree 24, while the direct product graph of the 2-Cayley and 3-Cayley graphs for Sym(4) is a regular graph with 576 vertices where each vertex has degree 120.

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