Flow Over Underwater Inclination Plane using Forced Korteweg-de Vries via Homotopy Analysis Method

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Abstract. A nonhomogeneous forced Korteweg-de Vries (fKdV) equation with some certain forcing term is investigated using a semi-analytic method called Homotopy Analysis Method (HAM). HAM is a summation of infinite series whereby its approximation solution converges immediately to the exact solution. Theoretical model, fKdV incorporating forcing term representing underwater bottom topography is used to analyze the wave interaction patterns over different inclination plane. The relationship between the three underwater sloping regions corresponding to wave interaction patterns are investigated. HAM solution found the higher inclination plane triggers higher amplitude wave interaction patterns.

INTRODUCTION

Forced KdV equation with various types of forcing term have been studied since four decades ago [1-5]. Nonhomogeneous KdV equation under various parameter setting is used to explained interaction of waves with certain topography [1]. Wu and Wu was the first one who discover numerically a phenomenon whereby a forcing disturbances generated by moving surface pressure or topography based on generalized Boussinesq model [6]. Experimentally, in 1980s, disturbance phenomenon for ship models moving in a towing tank with different transcritical speeds had been investigated related with fKdV equation [7-8]. Cole, [9] and Mei, [10] had investigated transcritical forcing of nonlinear long waves by surface pressure and seabed topographic perturbation. The basic mechanism underlying a phenomenon whereby a forcing disturbance moving steadily with a transcritical velocity in shallow water can successfully generate solitary waves upstream and downstream was investigated [4]. It is found that fKdV model admits external forcing disturbances when the surface pressure and bottom topography are entirely equivalent [11].

Shen in 1993 [12] derived the third order of KdV equation with forcing term and shows its solution properties. It is showed that there is little effect on the wave amplitude and period when the length is much greater than the water depth [13]. With some control parameter, Ong, Shen and Mohamad [14] had found several findings forced solitons generated by the forced KdV equation. In 2003, Efim Pelinovsky proposed an analytical model of tsunami generation using fKdV [15-16]. The solutions of fKdV equation can only be obtained by perturbation techniques or numerical [17]. Thus, it is essential to solve fKdV with a method that is reliable and accurate. Therefore, HAM is used to solve

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fKdV model in this research as unlike numerical, it does not involve discretization of variables and free from rounding off errors [18].

The homotopy analysis method was first introduced by Liao, [19] is a general analytic approach to get series solutions of various types of nonlinear equations, including algebraic equations, ordinary differential equations, partial differential equations, differential-integral equations, differential-difference equation, and coupled equations of them [20]. Perturbation and asymptotic techniques are strongly dependent upon physical parameters and often valid only for weakly nonlinear problems [21]. HAM consists of Lyapunov's small parameter method, the δ-expansion method and Adomian's decomposition method [22]. It also has a greater advantage in solving strong nonlinear problems without depending on a small parameter as in the perturbation approach. With these advantages and existence of high-performance computer and symbolic computation software, HAM had been successfully applied to various nonlinear differential equations in science, engineering and finance [23- 26]. Thus the validity, effectiveness, flexibility of the HAM [21] had been verified via all of these successful applications. Recently, forced KdV equation for a certain forcing term had been solved using HAM [27]. Characteristics of critical flow over various geometry using fKdV model investigated via HAM solution [28- 29].

The motivation of our work in this chapter is to investigate wave profile over a moving inclining underwater plane. Water waves move from flat bottom to inclining underwater plane is theoretically simulated. The forcing term in fKdV model is modified and solved by HAM, thereafter using symbolic software. The inclining underwater plane correspond with waves amplitude depicts a reasonable result.

Mathematical Formulation of Forced Korteweg-de Vries with underwater inclination plane

The fKdV model used in this work is [30]:

$$-\eta_t - \Delta\eta_x + \frac{3}{2}\eta\eta_x + \frac{1}{6}\eta_{xxx} + \frac{1}{2}f_x(x) = 0 \tag{1}$$

subject to initial condition [30-31],

$$\eta(x,0) = g(x) \tag{2}$$

where η is water wave elevation, f is forcing function and Δ is a critical parameter.

The equation (1) is non-dimensionalized using the following parameter.

$$t^* = \frac{1}{6}t,\ \eta^* = \frac{3}{2}\eta,\ f^* = \frac{9}{2}f,\ \text{and}\ \Delta^* = 6\Delta\ . \tag{3}$$

Using (3), equation (1) rewritten with superscript is omitted,

$$\eta_t + \Delta\eta_x - 6\eta\eta_x - \eta_{xxx} - f_x(x) = 0 \tag{4}$$

Assume that the bottom obstacle moves with a constant velocity, V which is close to the linear velocity of propagation [15]. Then, the forcing term, f will be representing a new moving reference frame. It is easier to pass over the system of coordinates with the underwater moving obstacle. Let the forcing term, f corresponds to a moving frame of reference.

$$x^* = x - Vt\ \text{ and }\ t^* = t \tag{5}$$

From (5),

$$\frac{\partial}{\partial t} = -V\frac{\partial}{\partial x^*} + \frac{\partial}{\partial t^*}\ \text{ and }\ \frac{\partial}{\partial x} = \frac{\partial}{\partial x^*} \tag{6}$$

Using (6), the equation (4) is rewritten

$$\left(-V\frac{\partial\eta}{\partial x^*}+\frac{\partial\eta}{\partial t^*}\right)+\Delta\frac{\partial\eta}{\partial x^*}-6\eta\frac{\partial\eta}{\partial x^*}-\frac{\partial^3\eta}{\partial x^{*3}}=\frac{\partial f\left(x^*\right)}{\partial x^*} \tag{7}$$

Superscript is omitted and equation (7) is re-written,

$$\frac{\partial\eta}{\partial t}+(\Delta-V)\frac{\partial\eta}{\partial x}-6\eta\frac{\partial\eta}{\partial x}-\frac{\partial^3\eta}{\partial x^3}=\frac{\partial f\left(x\right)}{\partial x} \tag{8}$$

Let,

$$\mathrm{K}=\Delta-V \tag{9}$$

Hence, equation (8) can re-written,

$$\frac{\partial\eta}{\partial t}+\mathrm{K}\frac{\partial\eta}{\partial x}-6\eta\frac{\partial\eta}{\partial x}-\frac{\partial^3\eta}{\partial x^3}-\frac{\partial f(x)}{\partial x}=0. \tag{10}$$

The simplified forced KdV of equation (10) is investigated further with different types of forcing term in which represents different sloping bottom region.

HAM and Forced KdV

Considering specific initial conditions, the rule of solution expression and the rule of solution existence in HAM [21], solution of forced KdV equation (10) is obtained through this steps,
Considering, nonlinear partial differential equation,

$$N\left[U\left(x,t\right)\right]=0 \tag{11}$$

where N is a differential operator, $U\left(x,t\right)$ is unknown function , x and t is dependent variables.
Using zero-order deformation equation in HAM,

$$\left(1-q\right)\ell\left[U\left(x,t;q\right)-\eta_0\left(x,t\right)\right]=qc_0\mathcal{N}\left[U\left(x,t;q\right)\right] \tag{12}$$

where c_0 is convergence parameter, ℓ as the auxiliary linear operator satisfying

$$\ell\left[d\right]=0 \tag{13}$$

where d is constant.
Let, $q\in\left[0,1\right]$ is the embedding parameter,
in which it holds,

$$U\left(x,t;0\right)=\eta_0\left(x,t\right)\text{ and }U\left(x,t;1\right)=\eta\left(x,t\right). \tag{14}$$

Using Taylor Series,

$$U\left(x,t;q\right)=\eta_0\left(x,t\right)+\sum_{m=1}^{\infty}\eta_m\left(x,t\right)q^m, \tag{15}$$

where

$$\eta_m\left(x,t\right)=\frac{1}{m!}\left.\frac{\partial^m U\left(x,t;q\right)}{\partial q^m}\right|_{q=0}\quad;\; m\geq 1. \tag{16}$$

Since series convergent at $q=1$, equation (15) re-written

$$\eta(x,t)=\eta_0(x,t)+\sum_{m=1}^{\infty}\eta_m(x,t) \tag{17}$$

Let, initial guess function [30,32],

$$\eta_0(x,t)=\frac{-2e^x}{\left(1+e^x\right)^2} \tag{18}$$

And define the vectors,

$$\vec{\eta}_m(x,t)=\left\{\eta_0(x,t),\eta_1(x,t),\eta_2(x,t),........,\eta_m(x,t)\right\} \tag{19}$$

Differentiating the zero order deformation m-times with respect to the embedding parameter, $q=1$,

$$\ell\left[\eta_m(x,t)-\chi_m\eta_{m-1}(x,t)\right]=c_o R_m\left[\vec{\eta}_{m-1}(x,t)\right] \tag{20}$$

where

$$R_m\left[\vec{\eta}_{m-1}(x,t)\right]=\frac{1}{(m-1)!}\left\{\frac{\partial^{m-1}}{\partial q^{m-1}}\mathcal{N}\left[\sum_{n=0}\eta_n(x,t)q^n\right]\right\}\Bigg|_{q=0} \tag{21}$$

Hence, equation (10) will be employed in the HAM procedure as the follows

$$\mathcal{N}\left[\eta(x,t;q)\right]=\frac{\partial\eta(x,t;q)}{\partial t}+\mathrm{K}\frac{\partial\eta(x,t;q)}{\partial x}-6\eta(x,t;q)\frac{\partial\eta(x,t;q)}{\partial x}-\frac{\partial^3\eta(x,t;q)}{\partial x^3}-\frac{\partial f(x)}{\partial x} \tag{22}$$

And equation (20) and (21) is rewritten

$$\ell\left[\eta_m(x,t)-\chi_m\eta_{m-1}(x,t)\right]=c_o\left[\frac{\partial\eta_{m-1}}{\partial t}+\mathrm{K}\frac{\partial\eta_{m-1-i}}{\partial x}-6\left(\sum_{i=0}^{m-1}\eta_i\frac{\partial\eta_{m-1-i}}{\partial x}\right)-\frac{\partial^3\eta_{m-1}}{\partial x^3}-\frac{1}{2}\frac{\partial f_{m-1}}{\partial x}\right] \tag{23}$$

with

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases} \tag{24}$$

And initial condition,

$$\eta_m(x,t;0)=0\ ,\ m\ge 1 \tag{25}$$

The N-th order approximation of $\eta(x,t)$ in HAM is given by

$$\eta(x,t)\approx\eta_0(x,t)+\sum_{m=1}^{N}\eta_m(x,t) \tag{26}$$

where, it is must be one of the solutions, if the HAM converges [19-22].

HAM methodology incorporating forcing KdV models from equations (11) till (26) are used to obtain analytic approximate solution of fKdV model.

Bottom Topography of inclination plane and forcing function

Three different input of forcing terms representing underwater bottom topography used in solving equation (10). The theoretical plane is completely flat at $x < 0$ and the plane is rising in a scale over $x > 0$. Therefore, in comply with this scenario, the underwater sloping region is represented using piecewise function as below.

$$f(x) = \begin{cases} -h & ,x < 0 \\ -h + \dfrac{1}{b}x & ,x \geq 0 \end{cases} \tag{27}$$

where h is the depth of water, and $1/b$ is the scale of sloping plane.

Three different steepness of plane is investigated using forcing term in fKdV equation. The bottom topography had a flat bottom at $x < 0$ and the topography will have sloping plane after x > 0. The value of b is determined at 2, 21 and 40 so that the inclination planes becomes at 1:2, 1:21 and 1:40.

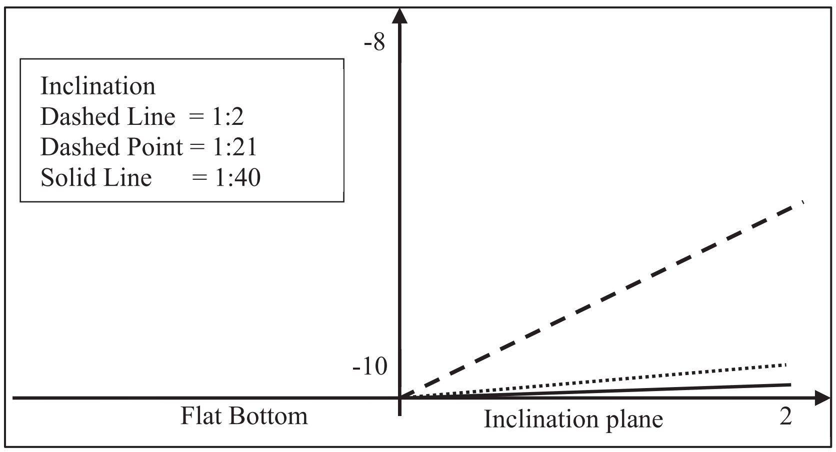

FIGURE 1. Bottom Topography of three different inclination planes

Figure 1 depicts the shapes of sloping plane which represents as underwater bottom topography. The sloping plane of 1:2 is approximately 26.56 degrees whereas the 1:21 and 1:40 is 2.73 degrees and 1.43 degrees. The angle of plane 1:21 and 1:40 is very small compared to plane 1:2.

HAM SOLUTION OF THE FKDV MODEL FLOW

HAM solution of fKdV model for the sloping plane found at fifth order approximation by using symbolic software. Using equation (26), HAM solution rewritten,

$$\eta(x,t) \approx \eta_0(x,t) + \sum_{m=1}^{5} \eta_m(x,t) \tag{28}$$

$$\eta(x,t) = \eta_0(x,t) + \eta_1(x,t) + \eta_2(x,t) + \eta_3(x,t) + \eta_4(x,t) + \eta_5(x,t) \tag{29}$$

HAM solution obtained is,

$$\begin{aligned}\eta(x,t) = \frac{e^x}{(1+e^x)^2} - \frac{c_0 t}{(1+e^x)^5}(f(x) + 5e^x f(x) + 10e^{2x} + 10e^{3x} f(x) + 5e^{4x} b + 6e^{2x} \\ -6e^{3x} - \mathrm{K}e^{x} - \mathrm{K}e^{2x} + \mathrm{K}e^{3x} + \mathrm{K}e^{4x} + 3e^{x} + ...\end{aligned} \tag{30}$$

Specific solution obtained, for each sloping plane with substitution of value of b as chosen earlier in Figure 1. The term, K in the solution is the criticality parameter in which we choose a value near to zero. Figure 2 shows the convergence of c_0-curve in the HAM solution of the three sloping underwater topography.

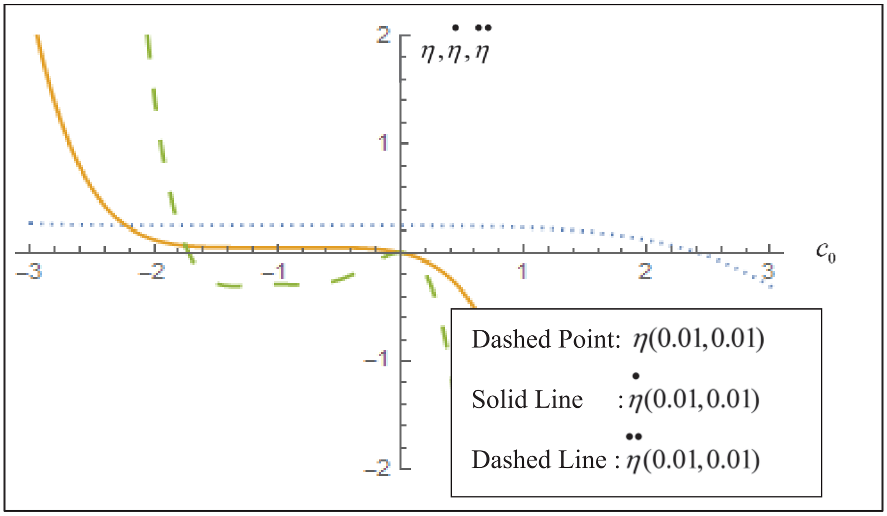

FIGURE 2. The c_0-curves according to 5[th]-order approximation for fKdV with bottom topography of equation (27).

Based on the horizontal line segment, the c_0 value determined at -1. The significance of c_0-curve and the methodology to choose a suitable c_0-values is essential in obtaining a good approximation [21].

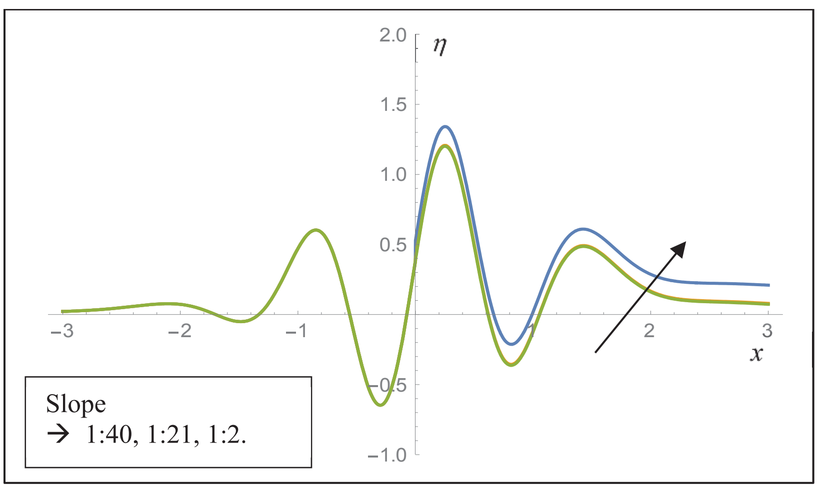

FIGURE 3. Wave Profile over three inclination plane

Figure 3 depicts the wave profile over three sloping planes. No differences found in the wave profile found between the three cases of sloping plane at the upstream as the region is entirely flat over $x < 0$. But different waves showed up at the downstream as the waves travel over different sloping region over $x \geq 0$. The transition of waves from flatten region to sloping region exhibited a high amplitude waves over the centric region. Although fKdV contains weak nonlinearity and weak dispersion, the wave characteristic in Figure 3 reveals peaked waves corresponds with the critical parameter. In addition to that, higher sloping regions triggered higher amplitude waves.

Conclusion

Forced Korteweg-de Vries equation with forcing term representing three inclination planes was theoretically investigated using HAM. Analytic approximation solution found using the concept of homotopy in HAM. The solution reveals water wave profile interacted over inclination planes. The wave profile is similar for flat bottom and the elevation of waves differs when it across different steepness planes. Physically, the steepness plane is 1:2 in which it obtained highest elevated water wave profile compared to inclination plane of 1:21 and 1:40. In summary, it is shown that higher steeper planes induced higher amplitude water waves.

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