PARAMETRIC AND SEMIPARAMETRIC ESTIMATION METHODS FOR BIVARIATE COPULA IN RAINFALL APPLICATION

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Specially dedicated to my beloved parents,
Mohd Lokoman bin Kasiran
and Normah binti Khalil,
all my siblings, my family and all my friends.
Thank you for your support, love, and encouragement.

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ABSTRACT

Hydrological phenomena such as drought, flood, and rainfall are one of the natural phenomena that often provide dependent multivariate variables. The correlation of the hydrologic dependent variables can be described by using copula. To determine a specified copula structure that fitted with the marginal variables, the copula dependence parameter needs to be estimated. This study focuses on the application of parametric and semiparametric approaches in estimating the copula dependence parameter. The performance of seven parameter estimation methods namely, maximum likelihood (ML) estimation, inference function of margins (IFM), maximization by parts (MBP), pseudo maximum likelihood (PML), the inversion of rank correlation coefficient approach based on Kendall's tau and Spearman's rho and maximum likelihood based on kernel density estimation (MLKDE) are compared in the simulation and empirical studies. The simulation and empirical studies are limited to the case of bivariate copulas. The result from the simulation study shows that the parametric approaches are inefficient when the marginal distributions are misspecified. Among the parametric approaches, MBP performs better than MLE and IFM. While, for semiparametric approaches, PML performs well and consistent for any correlation and sample size. The PML can be efficient and consistent with the parametric once the sample size is large. The empirical study is done by applying the estimation methods to identify the dependence of the daily rainfall at two rain gauge stations Station Kuala Krai and Station Ulu Sekor. The result from the empirical study is consistent with the result from the simulation study. Thus, it can be concluded that MBP is preferred when the copula and the marginal distributions are known. While, PML is preferred when the marginal distribution is unknown, where the situation is common in a real data application.

ABSTRAK

Fenomena hidrologi seperti kemarau, banjir dan hujan adalah salah satu fenomena semula jadi yang sering membentuk pemboleh ubah multivariat bersandar. Korelasi antara pembolehubah hidrologi tersebut boleh digambarkan dengan menggunakan *Copula*. Untuk memadankan struktur *Copula* tertentu yang sesuai dengan pembolehubah marginal, parameter bersandar *Copula* perlu dianggarkan dahulu. Kajian ini memberi tumpuan kepada penggunaan kaedah parametrik dan semiparametrik dalam menganggarkan parameter bersandar *Copula*. Prestasi tujuh kaedah penganggar iaitu anggaran kebolehjadian maksimum (MLE), fungsi taksiran marginal (IFM), kaedah pengoptimuman bahagian demi bahagian (MBP), pseudo kebolehjadian maksimum (PML), kaedah penyongsangan terhadap pekali kedudukan korelasi *Kendall tau* dan *Spearman rho* serta kebolehjadian maksimum berdasarkan anggaran ketumpatan kernel (MLKDE) telah dibandingkan dalam kajian simulasi dan kajian empirikal. Kajian simulasi dan kajian empirikal adalah terhad kepada kes *Copula* bivariat. Hasil daripada kajian simulasi menunjukkan bahawa pendekatan parametrik tidak efisien apabila taburan marginal disalah spesifikasikan. Antara kaedah parametrik, MBP mempunyai prestasi yang lebih baik daripada MLE dan IFM. Sementara itu, untuk kaedah semiparametrik, PML mempunyai prestasi yang lebih baik dan konsisten bagi setiap tahap korelasi dan saiz sampel. PML boleh menjadi efisien dan konsisten seperti parametrik apabila saiz sampel adalah besar. Kajian empirikal dilakukan dengan menggunakan kaedah anggaran untuk mengenal pasti kebersandaran hujan harian di dua stesen tolok hujan: Stesen Kuala Krai dan Stesen Ulu Sekor. Hasil kajian empirikal adalah konsisten dengan keputusan daripada hasil kajian simulasi. Secara kesimpulan, MBP lebih sesuai digunakan apabila *Copula* dan taburan marginal dapat dikenal pasti. Manakala PML lebih sesuai digunakan apabila taburan marginal tidak diketahui dan tidak dapat dikenalpasti. Situasi begini adalah biasa dalam aplikasi data sebenar.

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LIST OF SYMBOLS

$f_X(x)$	-	Probability Density Function for random variable *x*.
$f_Y(y)$	-	Probability Density Function for random variable *y*.
$F_x(x)$	-	Cumulative Distribution Function for random variable *x*.
$F_y(y)$	-	Cumulative Distribution Function for random variable *y*.
$F_x{}^{-1}(u)$	-	Inverse of the Cumulative Distribution Function for random variable *x*.
$F_y{}^{-1}(v)$	-	Inverse of the Cumulative Distribution Function for random variable *y*.
α	-	Corresponding parameter of $f_X(x)$
β	-	Corresponding parameter of $f_Y(y)$
$\hat{\alpha}$	-	Estimator for parameter α
$\hat{\beta}$	-	Estimator for parameter β

$C(u,v)$	-	Copula Function
$D_k(x)$	-	Debye Function
Φ_θ	-	The bivariate standard normal CDF
Φ_θ^{-1}	-	The inversed of univariate normal CDF
t_θ	-	The bivariate standard Student's *t* CDF
t_θ^{-1}	-	The inversed of the univariate Student's *t* CDF
df	-	Degree of freedom
n	-	Sample size
ρ	-	Spearman's rho
τ	-	Kendall's tau
θ	-	Copula Parameter
$\hat{\theta}$	-	Estimator for Copula Parameter
$\phi(t)$	-	Generator of Archimedean Copula
$\phi^{-1}(s)$	-	Inverse generator
$\ell_c(\alpha,\beta,\theta)$	-	Log-likelihood of the copula density function

$\ell_m(\alpha, \beta)$	-	Log-likelihood of marginal density functions
a	-	Shape parameter
b	-	Scale parameter
λ	-	Rate parameter
$K(y)$	-	Kernel function
h	-	Smoothing parameter for kernel density estimation.
$f''(y)$	-	Second derivative of $f(y)$
$o(y)$	-	Little o notation that shows the asymptotic behaviour of the given function.
$\hat{f}_{-i}\,(y_i)$	-	kernel estimator called as leave-one-out density estimator estimated from the data without the observation y_i

LIST OF ABBREVIATIONS

AIC	-	Akaike Information Criterion
PDF	-	Probability Density Function
CDF	-	Cumulative Distribution Function
GOF	-	Goodness-of-fit
IFM	-	Inference Function of Margins
MBP	-	Maximization by parts
ML	-	Maximum Likelihood
MLE	-	Maximum Likelihood Estimation
PML	-	Pseudo Maximum Likelihood
MLKDE	-	Maximum likelihood based on kernel density estimation.
ROT	-	Rule of thumb
LSCV	-	Least square cross validation

SJ	-	Sheater and Jones
MLKDE-rot	-	Maximum likelihood based on kernel density estimation by using bandwidth selector rule of thumb.
MLKDE-sj	-	Maximum likelihood based on kernel density estimation by using bandwidth selector Sheater and Jones plug-in.
MLKDE-lscv	-	Maximum likelihood based on kernel density estimation by using bandwidth selector least squares cross-validation.
MISE	-	Mean integrated squared error
AMISE	-	Approximate mean integrated square error
SE	-	Squared error
RMSE	-	Root mean squared error
iTAU	-	The inversion of the rank correlation coefficient Kendall's tau.
iRHO	-	The inversion of the rank correlation coefficient Spearman's rho.
NaNs	-	Undefined values
CV	-	Coefficient of variation

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CHAPTER 1

INTRODUCTION

1.1 Background of the study

Hydrological phenomena such as drought, flood and rainfall are one of the natural phenomena that often provide dependent multivariate observations. For example, from a drought phenomenon, we can observe the drought duration, intensity, and magnitude. According to Salvadori and De Michele (2007), those random variables play an important role between each other, where such an analysis of a joint distribution between the variables can identify the characteristics of drought. Therefore, it is necessary to find the joint distribution and estimate the dependence between the variables.

Based on the traditional approach, the joint distribution has been described using bivariate or multivariate distribution functions such as bivariate gamma, bivariate normal or multivariate normal distribution. However, there are limitations to this approach which made it difficult to execute mathematically. The marginal distributions must belong to the same family of the joint distribution function and the marginal parameters may affect the dependence between the variables. In addition, Salvadori and De Michele (2007) stated that canonical Pearson's coefficient of linear correlation, ρ is usually used as the dependence parameter between variables in

hydrology process. However, the canonical Pearson's coefficient may shows $\rho = 0$ in some cases which means there is no dependency between the variables although the variables are obviously not independent. This is because it only shows a linear dependence. Thus, to overcome this problem, a flexible method called copula method is introduced.

Copula method was introduced by Sklar (1959). A copula function is a joint distribution function of a combination of two or more uniform marginal distributions. This method can overcome the limitations of the traditional approach because it allowed us to specify any distribution function to the marginal distributions and then choose any copula to construct the dependence structure of the variables. In the work of Zhang and Singh (2007), they have proved that the copula method is able to derive bivariate joint distributions of rainfall variables that have different marginal distributions and without assuming the variables to be normal or independent. Many different copula families that able to cover a wide scope of dependence structures have been proposed and developed, for example, Archimedean, Gaussian, and Student's *t* copula families. Further information on copula families is discussed in Chapter 2 of this study.

In hydrologic application, the most copula families that have been used for analysis are Archimedean copula families. According to Nelsen (2006) and Zhang and Singh (2007), Archimedean copulas that usually have closed form are very popular and desirable in constructing the dependence structure of the hydrologic variables. It is because of the ease in constructing the functions and they can be applied when the variables correlation is either positive or negative.

Archimedean copulas are divided into two groups, symmetric and asymmetric copulas as mentioned by Chen et al (2013). The symmetric Archimedean family is directed by one dependence parameter, θ. They stated that the limitation of the symmetric Archimedean copula is that it can only measure one dependence structure between two variables, where all possible pairs of variables that can be paired up will

have the same dependence structure. Thus, symmetric Archimedean copulas are only suitable for structuring the dependence of two variables, but inadequate for more than two variables. To overcome the limitation of the symmetric Archimedean copula, an asymmetric copula is constructed. The asymmetric copula is a nested form of the symmetric copula. Consequently, the asymmetric copula will able to describe different dependence structures between two or more variables. Other than Archimedean copulas, elliptical copulas such as Gaussian and Student's *t* copula also have been widely used for an analysis of multivariate hydrologic variables. Elliptical copula family is implicit copulas where they do not have a closed form.

To determine a specified copula structure that fitted with the marginal variables, the parameters of the copula function need to be estimated first. There are many parameter estimation methods have been proposed and developed for estimating the dependence parameter of the copula. These methods are classified into three categories, parametric approaches, semiparametric approaches, and nonparametric approaches. For hydrological analysis, parametric and semi-parametric approaches are the most common estimation methods that have been used to estimate the copula parameter. However, the nonparametric method is very rarely used because no specific parametric forms are assumed for either the copula or the marginal distributions and the copula is estimated based on empirical distributions by simple observation on the data sets. Therefore, determining an empirical copula relies on the amount of the available observation data and the formation of an empirical copula depended on a large amount of the data which is one of the limitations in hydrologic application.

In parametric approaches, the marginal distributions are assumed to follow a parametric distribution. The parameters of interest are marginal parameters and copula dependence parameter. Parametric methods are popular because they estimate the estimator precisely. However, they have a weakness against a misspecified marginal parametric distribution. For that reason, the semiparametric approach is implemented to overcome the problem by assuming the marginal distributions to be nonparametric, which allow marginal empirical cumulative distribution functions be plugged into the

marginal functions. Thus, the copula dependence parameter is the only parameter of interest in the semiparametric approach.

In the hydrological analysis, the most common parameter estimation methods that have been used are Maximum likelihood (ML) estimation and Inference Function of Margins (IFM) for parametric approaches. Whereas, pseudo maximum likelihood (PML) and rank correlation coefficient of Kendall's tau and Spearman's rho methods have been used for semiparametric approaches. Among these five methods, Kendall's tau method is the most popular method for estimating bivariate copula probably because it has a closed form of one-to-one relationship between rank correlation, tau (τ) and the copula parameter, θ which has made the estimation process become easier. Vandenberghe et al. (2010) and Chen et al. (2015) also preferred to use Kendall's tau method than ML estimation or PML because it is easier to estimate the copula parameter based on Kendall's tau rank correlation coefficient rather than finding the fitted marginal distributions and maximizing a log-likelihood function that leads to a complicated algorithm.

Other than five estimation methods mentioned above, there are two other copula estimation methods that have been developed, which are maximization by parts (MBP) under the parametric approach and maximum likelihood based on kernel density estimation (MLKDE) under semiparametric approach. Song et al. (2005) proposed MBP to overcome some loss made by IFM. Meanwhile, MLKDE has the same structure as PML, where the difference in MLKDE is the marginal distributions are estimated by kernel density estimation. There are large research of hydrological studies that use ML estimation, IFM, PML, the inversion of rank correlation coefficient approach based on Kendall's tau and Spearman's rho to estimate the copula dependence parameter. However, studies that implement MBP and MLKDE are rare to find in hydrologic application literature.

1.2 Problem statement

Recently, there has been an increase of interest in joining distribution functions of multivariate hydrologic observations using the copula method. Copula method is able to assess the relation between the variables without concerning a specific marginal distribution. The copula is estimated using parametric and semiparametric approaches. The most common methods that have been used in estimating the copula parameter are Maximum likelihood (ML) estimation and Inference Function of Margins (IFM) for parametric approaches and pseudo maximum likelihood (PML), and the rank correlation coefficient approach based on of Kendall's tau and Spearman's rho for semiparametric approaches.

Although ML estimation, IFM, PML, and the inversion of the rank correlation coefficient approach based on Kendall's tau and Spearman's rho have been widely used for hydrologic analysis, there are limited comparative studies that focus on copula estimation methods in a hydrologic application. In addition, there are two other copula estimation methods that have been developed but rarely used in hydrologic analyses, the methods are maximization by parts (MBP) under the parametric approach and maximum likelihood based on kernel density estimation (MLKDE) under semiparametric approach. These seven estimation methods have different steps and techniques to estimate the parameter. Therefore, a comparison study is important to evaluate the performance of the estimation methods. This study is conducted to compare the precision and the performance of seven parameter estimation methods for copula in a hydrologic application.

1.3 Research questions

The problem statement raises several research questions. The questions are listed as follow:

i. How to estimate the copula dependence parameter, θ using parametric and semiparametric estimation methods?
ii. What is the performance of parametric and semiparametric estimation in terms precision based on their value of root mean square error (RMSE)?
iii. Which parameter estimation methods that are suitable and efficient for estimating the dependence parameter of hydrologic variables?

1.4 Objectives of the study

The objectives of this study are listed as follows:

i. To estimate the copula dependence parameter, θ using parametric and semiparametric estimation methods.
ii. To evaluate and to compare the performance of parametric and semiparametric estimation methods for copula in terms of efficiency and precision.
iii. To identify the estimation methods that are suitable, efficient, and precise in estimating the dependence parameter of hydrologic variables.

1.5 Scopes of the study

This study focuses on the application of parametric and semiparametric approaches in estimating the copula dependence parameter. The performance of seven parameter estimation methods namely, maximum likelihood (ML) estimation, inference function of margins (IFM), maximization by parts (MBP), pseudo maximum likelihood (PML), the inversion of rank correlation coefficient approach based on Kendall's tau and Spearman's rho and maximum likelihood based on kernel density estimation (MLKDE) are compared in the simulation and empirical studies. The simulation and empirical studies are limited to the case of bivariate copulas.

In the simulation study, simulation data are generated from Clayton copula as the true copula with four different values of true copula parameter dependence that are corresponding to Kendall's tau, $\tau = 0.20, 0.50, 0.60$, and 0.80. The sample sizes of the generated data are set to $n = 50, 100, 1000$, and 5000. 500 repetitions of data generation and estimation process are done for each combination of different data sample size, n and copula dependence level, θ.

While, for the empirical study, rainfall data are used as the empirical data. The data are selected from two Kelantan rain gauge stations which are located in the north-east of Peninsular Malaysia. The selected rain gauge stations are Station Kuala Krai, 5522047 (Station A) and Station Ulu Sekor, 5520001 (Station B). Three types of marginal distributions are considered in fitting the hydrologic variables: Gamma, Weibull and Exponential distributions. The marginal information is used in the estimation process done by the parametric approach. For the joint distribution function, six copulas that are usually used in the hydrologic application are selected. The copulas are Gumbel, Clayton, Frank, Ali-Mikhail-Haq, Gaussian, and Student's t copulas.

1.6 Significance of the study

The analysis in this study involves the implementation of copula method in combining the hydrologic variables and estimating the copula dependence parameter, θ. This study allows the characteristics of marginal distributions, parameter estimation methods, and copula families to be recognized. Furthermore, the main significance of this study is it will become as another example of application and comparative study of parametric and semiparametric estimation approaches in estimating the dependence parameter of copulas model since there are only a few previous studies that compare the parameter estimation methods for copulas in a hydrologic application. Other parameter estimation methods such as MBP and MLKDE which are rarely used in hydrologic analyses are also discovered in this study.

In the simulation study, the research process leads in developing the methodology for simulating data of marginal distributions based on the given true marginal and copula distributions and the true value of the dependence parameter. The simulation process allows the generation of n sample sizes of data and desirable repetitions of estimation process for each combination of different data sample size, n and copula dependence level, θ.

In addition, the performance of the methods based on the measured root of mean square error (RMSE) comparison can give statistical evidence in choosing which the parameter estimation methods that are more accurate and efficient to estimate the copula dependence parameter. This is important because the copula dependence parameter will affect the precision in estimating the copula function that is fitted to the data. This study also provides the result of computational performance for the seven estimation methods.

1.7 Research outline

This dissertation report consists of five chapters. Chapter 1 starts with the introduction of copula and parameter estimation methods in the background of the study. Then, it is followed by the statement of the problem and the questions that arise in the problem statement. After that, the purpose or the objectives of the study and the scope that are used in the study are highlighted. Finally, the possible significance or contributions that the study can provide are also presented in this chapter.

Chapter 2 consists of the general review about copula function and the expression of some copula families that have been widely used in hydrologic analyses. Some parameter estimation methods that can be used to estimate the copula dependence parameter are reviewed and the algorithm or mathematical formulation of some estimation methods are also presented. In addition, previous comparative studies that focus on copula parameter estimation methods and hydrologic analysis studies using copulas are discussed.

Chapter 3 describes the research methodology. It consists a brief explanation about the simulation and empirical studies for comparing the performance and efficiency of the parametric and semiparametric estimation methods. The procedures that are used for the both studies are also explained. The steps that involved in the parametric and semiparametric estimation methods are also described in this chapter.

Chapter 4 presents all the results and findings of the simulation and empirical studies. In this chapter, the performance of the seven copula estimation methods is compared based on the measured root mean squared error (RMSE) and time spend of each method. Finally, in the last chapter, Chapter 5, concludes the research project based on the results and findings from the simulation and empirical studies. Some recommendations that need to be done for further research are also suggested.

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