The Incidence Matrix of the Non-Normal Subgroup Graph for Some Dihedral Groups

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Abstract. There are various types of matrices associated with graph in the field of graph theory. A graph can be represented in the form of its adjacency matrix or its incidence matrix. These matrices have their own way of representing a complex graph which have many vertices, edges and directions. In this paper, the incidence matrices of the non-normal subgroup graphs are determined for some dihedral groups.

INTRODUCTION

A various types of matrices have been applied in graph theory such as adjacency matrix, distance matrix and incidence matrix. A complex graph can be represented by using all of these matrices. The adjacency matrix had been studied in the conjugate graph of some metacylic 2-groups by Alimon et al. [1] and in the energy of commuting graph, non-commuting graph, conjugate graph and conjugacy class graph of the same group [2]. The distance matrix had been applied in the study of the distance spectrum and distance energy of complement of subgroup graphs of dihedral group [3]. The researches for the incidence matrix of some graphs had been done in the C_n^+ graph by Thiagarajan et al. [4] and in the semigraph [5]. Meanwhile, the incidence matrix had been applied in the graph-theoretic method for finding the Moore-Penrose inverse matrix [6]. Bapat and Azimi [6] obtained a graph theoretic description for a distance regular graph.

In this paper, the incidence matrix of the non-normal subgroup graph for some dihedral groups is determined. The dihedral group can be expressed in a group representation as follows:

$$D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle, n \geq 3, n \in \mathrm{N}.$$

This paper is divided into three sections.The first section provides some backgrounds on the application of matrices in graph theory. Some definitions that are related to the non-normal subgroup graph and the incidence matrix are stated in Section 2. In the next section, the results are presented on the incidence matrices for some dihedral groups, followed by the conclusion.

PRELIMINARIES

In this section, some definitions used throughout in this research are given. The following definition is used to determine the non-normal subgroup graph of some finite groups.

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Definition 1 *[7] Let G be a group and H be a non-normal subgroup of G. The subgroup graph $\Gamma_H^{NN}(G)$ is a directed graph with vertex set G such that x is the initial vertex and y is the terminal vertex of an edge if and only if $x \neq y$ and $xy \in H$.*

Next is the definition of the incidence matrix of a directed graph.

Definition 2 *[8] Let Γ be a graph with $V(\Gamma) = \{1, \ldots, n\}$ and $E(\Gamma) = \{e_1, \ldots, e_m\}$. Suppose each edge of Γ is assigned an orientation, which is arbitrary but fixed. The incidence matrix of Γ denoted by $Q(\Gamma)$ is the $n \times m$ matrix defined as follows: the rows and columns of $Q(\Gamma)$ are indexed by $V(\Gamma)$ and $E(\Gamma)$, respectively, the (i, j)-entry of $Q(\Gamma)$ is 0 if vertex i and edge e_j are not incident, and otherwise it is 1 or -1 according whether e_j originates or terminates at i, respectively.*

RESULTS AND DISCUSSIONS

In this section, the incidence matrices of the non-normal subgroup graphs are determined for some dihedral groups. The results of the matrices are shown in the form of propositions. Based on Definition 1, the non-normal subgroup graphs are determined first to get the incidence matrix. The incidence matrices are stated in the following propositions.

Proposition 1 *Let G be the dihedral group of order six, D_6 and H be a non-normal subgroup of G. Then the incidence matrix of the non-normal subgroup graph for H, $Q(\Gamma_H^{NN}(G))$ is stated as follows:*

$$\begin{bmatrix} -1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Proof: Assume G is a dihedral group of order six, D_6 with the elements are $\{1, a, a^2, b, ab, a^2b\}$. The non-normal subgroups of D_6 are $H_1 = \{1, b\}$, $H_2 = \{1, ab\}$ and $H_3 = \{1, a^2b\}$. By Definition 1, the set of vertices for the non-normal subgroup graph is the set of elements of D_6 with x as an initial vertex and y as a terminal vertex of an edge. These x and y are two distinct points with $xy \in H$. For example, let $x = a$ and $y = a^2$. Since $xy = a \cdot a^2 \in H$, there is a direction from $x = a$ to $y = a^2$. Hence, the non-normal subgroup graph of $H_1 = \{1, b\}$, $H_2 = \{1, ab\}$ and $H_3 = \{1, a^2b\}$ are directed graphs as illustrated in the following diagrams:

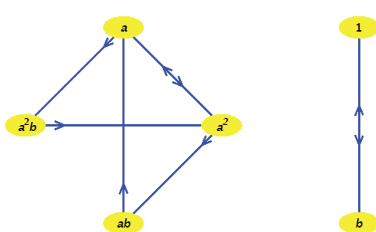

FIGURE 1. $\Gamma_{H_1}(D_6)$

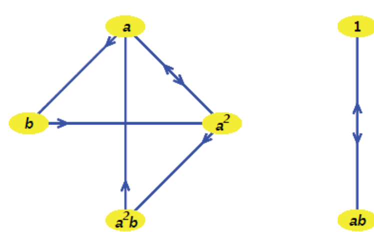

FIGURE 2. $\Gamma_{H_2}(D_6)$

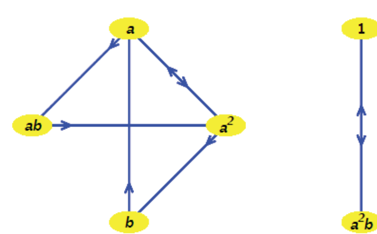

FIGURE 3. $\Gamma_{H_3}(D_6)$

It can be seen that all graphs have the same pattern. Based on the graph in Figure 1, the set of vertices is $V(\Gamma_{H_1}(D_6)) = \left\{1, a, a^2, b, ab, a^2b\right\}$ and the representations of vertices are: a as v_1, a^2 as v_2, a^2b as v_3, ab as v_4, 1 as v_5 and b as v_6. The rows and columns of the incidence matrix are indexed by $V(\Gamma)$ and $E(\Gamma)$, respectively. In the incidence matrix, $Q(\Gamma) = (q_{ij})$ where the (q_{ij})-entry are represented as 1 and -1 according whether edge e_j originates or terminates at vertex i, respectively. Otherwise, it is 0. The graph in Figure 1 can be labelled as illustrated in the following diagram:

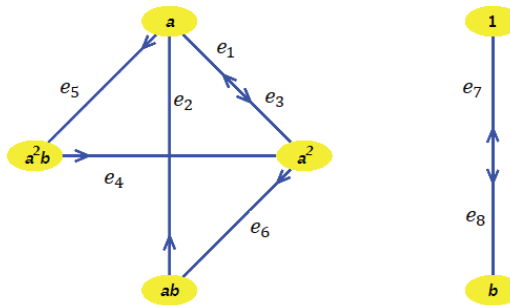

FIGURE 4. detailed graph of $\Gamma_{H_1}(D_6)$

$q_{34} = 1$ since the edge e_4 originates at vertex v_3 and $q_{35} = -1$ since the edge e_5 terminates at vertex v_3 . Meanwhile, $q_{31} = 0$ since the edge e_1 and vertex v_3 are not incident. Therefore, the incidence matrix for this graph is

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	−1	−1	1	0	1	0	0	0
v_2	1	0	−1	−1	0	1	0	0
v_3	0	0	0	1	−1	0	0	0
v_4	0	1	0	0	0	−1	0	0
v_5	0	0	0	0	0	0	−1	1
v_6	0	0	0	0	0	0	1	−1

The same method can also be applied to the graphs in Figure 2 and Figure 3 and will give the same result.

Next, the incidence matrix for the non-normal subgroup graph of the dihedral group of order eight, D_8 is given in the following propositions. There are four non-normal subgroups of D_8, namely $H_1 = \{1, b\}, H_2 = \{1, a^3b\}, H_3 = \{1, ab\}$ and $H_4 = \{1, a^2b\}$. For D_8, we split into two cases, accroding to their graph directions.

Proposition 2 *Let G be the dihedral group of order eight, D_8 and $H_1 = \{1, b\}$, $H_2 = \{1, a^3b\}$ be non-normal subgroups of G. Then the incidence matrices of the non-normal subgroup graphs for H_1 and H_2, $Q(\Gamma^{NN}_{H_i}(G)$ are stated as follows:*

$$\begin{bmatrix} -1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Proof: Assume G is a dihedral group of order eight, D_8 with the elements are $\{1, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$. By Definition 1, the set of vertices for the non-normal subgroup graph is the set of elements of D_8 with with x as an initial vertex and y as a terminal vertex of an edge. These x and y are two distinct points with $xy \in H$. For example, let $x = a$ and $y = a^3$. Since $xy = a \cdot a^3 \in H$, there is a direction from $x = a$ to $y = a^3$.

Hence, the non-normal subgroup graph of $H_1 = \{1, b\}$ and $H_2 = \{1, a^3b\}$ are directed graphs as illustrated in the following diagrams:

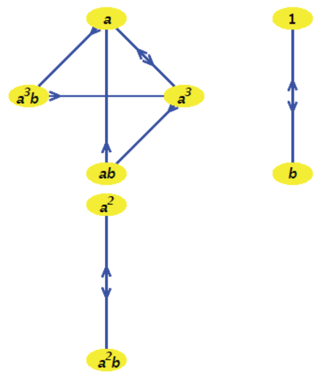

FIGURE 5. $\Gamma_{H_1}(D_8)$

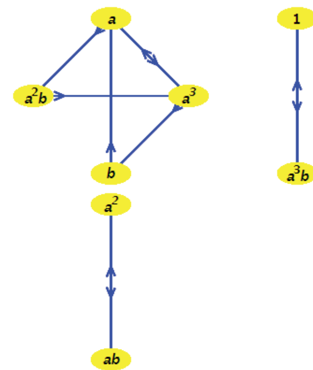

FIGURE 6. $\Gamma_{H_2}(D_8)$

For $\Gamma_{H_1}(D_8)$ and $\Gamma_{H_2}(D_8)$ and by using the same method, the incidence matrix is

$$\begin{bmatrix} -1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Proposition 3 *Let G be the dihedral group of order eight, D_8 and $H_3 = \{1, ab\}, H_4 = \{1, a^2b\}$ be non-normal subgroups of G. Then the incidence matrices of the non-normal subgroup graphs for H_3 and H_4, $Q(\Gamma_{H_i}^{NN}(G)$ are stated as follows:*

$$\begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Proof: Assume G is a dihedral group of order eight, D_8 with the elements are $\{1, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$. By Definition 1, the set of vertices for the non-normal subgroup graph is the set of elements of D_8 with x as an initial vertex and y as a terminal vertex of an edge. These x and y are two distinct points with $xy \in H$. For example, let $x = a$

and $y = a^3$. Since $xy = a \cdot a^3 \in H$, there is a direction from $x = a$ to $y = a^3$. Hence, the non-normal subgroup graph of $H_3 = \{1, ab\}$ and $H_4 = \{1, a^2b\}$ are directed graphs as illustrated in the following diagrams:

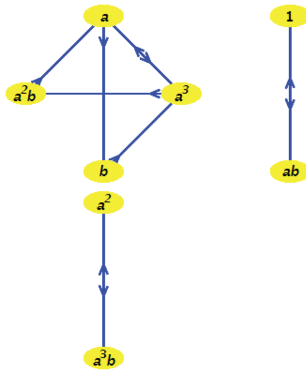

FIGURE 7. $\Gamma_{H_3}(D_8)$

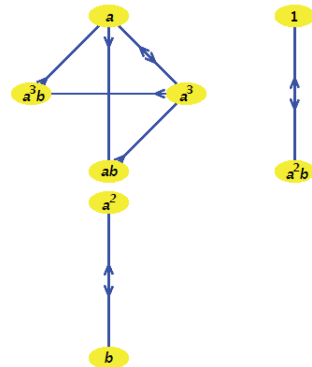

FIGURE 8. $\Gamma_{H_4}(D_8)$

For $\Gamma_{H_3}(D_8)$ and $\Gamma_{H_4}(D_8)$ and by using the same method, the incidence matrix is

$$\begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Proposition 4 *Let G be the dihedral group of order ten, D_{10} and H be a non-normal subgroup of G. Then the incidence matrix of the non-normal subgroup graphs for H, $Q(\Gamma)_H^{NN}(G))$ is stated as follows:*

$$\begin{bmatrix} -1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Proof: Assume G is a dihedral group of order ten, D_{10} with the elements are $\{1, a, a^2, a^3, a^4, a^5b, b, ab, a^2b, a^3b, a^4b, a^5b\}$. The non-normal subgroups of D_{10} are $H_1 = \{1, b\}$, $H_2 = \{1, ab\}$, $H_3 = \{1, a^2b\}$, $H_4 = \{1, a^3b\}$ and $H_5 = \{1, a^4b\}$. By Definition 1, the set of vertices for the non-normal subgroup graph is the set of elements of D_{10} with x as an initial vertex and y as a terminal vertex of an edge. These x and y are two distinct points with $xy \in H$. For example, let $x = a$ and $y = a^2$. Since $xy = a \cdot a^2 \in H$, there is a direction from $x = a$ to $y = a^2$. Hence, the non-normal subgroup graph of $H_1 = \{1, b\}$, $H_2 = \{1, ab\}$, $H_3 = \{1, a^2b\}$, $H_4 = \{1, a^3b\}$ and $H_5 = \{1, a^4b\}$ are directed graphs as illustrated in the following diagrams:

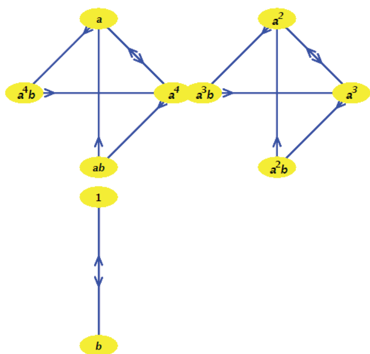

FIGURE 9. $\Gamma_{H_1}(D_{10})$

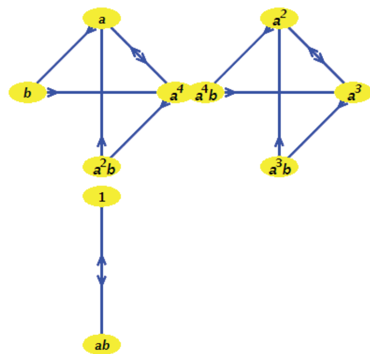

FIGURE 10. $\Gamma_{H_2}(D_{10})$

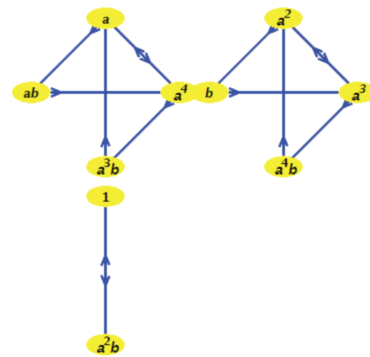

FIGURE 11. $\Gamma_{H_3}(D_{10})$

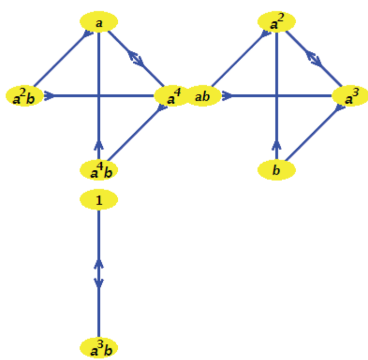

FIGURE 12. $\Gamma_{H_4}(D_{10})$

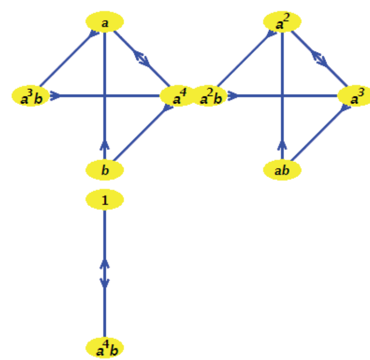

FIGURE 13. $\Gamma_{H_5}(D_{10})$

It can be seen that all graphs have the same pattern. In the incidence matrix, the (i, j)-entry are represented as 1 and -1 according whether edge e_j originates or terminates at vertex i, respectively. Otherwise, it is 0.

CONCLUSION

In this paper, the incidence matrix of the non-normal subgroup graph for some dihedral groups have been found. The incidence matrices for dihedral group of order six and ten are the same for each subgroup. However, for dihedral group of order eight, there are two distinct incidence matrices because of the differences in graph directions. The incidence matrices can be applied in the graph-theoretic method for finding the Moore-Penrose inverse matrix of the non-normal subgroup graph.

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