3-Tuple Bézier Surface Interpolation Model for Data Visualization

Abd. Fatah Wahab and Mohammad Izat Emir Zulkifly

Abstract—**In this paper, the 3-tuple Bézier surface interpolation model is introduced. The 3-tuple control net relation is defined through intuitionistic fuzzy concept. Later, the control net is blended with Bernstein basis function to obtain surface blending function and to produce 3-tuple Bézier surface. The 3-tuple Bézier surface model is illustrated through the interpolation method by using data point with intuitionistic features. Some numerical example is shown. Lastly, the 3-tuple Bézier surface properties is also discussed.**

Index Terms—**3-tuple, Bézier surface, control net relation, interpolation.**

I. Introduction

Surface and their properties plays an important role in visualizations of data such as from marines, consumer products, medical, geological, physical, design, manufacturing and other natural behavior. The easiest way to build a surface is to sweep a curve through space such that its respective points move along some curves [1]. A tensor product surface or just called surface is defined by a control net that is parameterized in two directions denoted by $u \in [0,1]$ and $v \in [0,1]$ [2]. This control nets consist by set of control points that control the surface. The surface modeling is a mathematical method that used mathematical expression in computer-aided geometric design to visualize an object by using provided data.

One of the surface modeling techniques that often used is the Bézier surface. The geometric properties of Bézier were developed by P. de Casteljau in 1959, [3], [4], and by P. Bézier starting in 1962 as stated in [5]. The Bézier method were used respectively in computer-aided design systems of Renault and Citroën. P. Bézier has derived the mathematical basis of curves and surfaces techniques from geometrical considerations as in [6]–[8]. Later, around 1970s, Forrest in [9] and Gordon and Riesenfeld in [10] found the connection between the work of Bézier and the classical Bernstein polynomials. They discovered that the Bernstein polynomials are in fact the basis functions used for Bézier curves and surfaces.

The uncertainty in the data set is a major problem that exists in the surface design. In geometric modeling, the data set is also called control point for approximation and data points for interpolation methods [11]. Normally, a problematic data (uncertainty data) will be ignored or eliminated from a set of data regardless of its effect on the curve and the resulting surface. Hence, the evaluation and analyzing process will be incomplete. Therefore, if there is an element of uncertainty in a data set, the data set should be filtered so that it can be used for generation of surfaces of a model that want to be investigated. To overcome this matter, intuitionistic fuzzy set (IFS) is used. IFS is a generalization of fuzzy set theory from Zadeh [12] and was introduced by Krassimir T. Atanassov in [13]–[16]. The set consists of three components namely degree of membership, non-membership and uncertainty (non-determinacy).

Surface modeling is a method of mathematical representations construction in the form of geometry while the IFS theory is a mathematical representation that aimed at concepts and techniques to tackles uncertain problems. Therefore, in this paper, a surface model that can handle uncertainty data problems (intuitionistic data) represent by $\langle \mu, \nu, \pi \rangle$ called 3-tuple and its data visualization that focused on Bézier surface interpolation is introduced. The aim of this paper is to visualize data point with intuitionistic features by using Bézier surface function and yield 3-tuple Bézier surface (3-TBS) through interpolation method by using 3-tuple control net relation (3-TCNR). This paper is organized as follow. Section 1 discusses some introduction about this paper. In section 2, some basic definitions and concepts of IFS are shown. Later, section 3 introduces intuitionistic fuzzy point relation (IFPR) with some of its properties. Section 4 introduces 3-TCNR based on IFPR and 3-tuple control point relation (3-TCPR). Section 5 defines and visualized 3-TBS interpolation and some numerical example is shown. In Section 6, the 3-TBS properties are discussed and section 7 concludes this research.

II. Prelimenaries

IFSs have been studied and used in different fields of science and mathematics. Among the works on these sets are as in [17]–[27]. IFS is generally defined by three functions (membership, non-membership and uncertainty) with the constraint that the sum of these three functions must be equal to one [28]. This section shows some basic definition of IFS consists of intuitionistic fuzzy number (IFN), intuitionistic fuzzy relation (IFR) and intuitionistic fuzzy point (IFP).

Definition 1. [14] Let a set X is fixed and let $A \subset X$ be a fixed set. An IFS A^* in X is an object of the following form:

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$$A^* = \left\{ \left\langle x, \mu_A(x), \nu_A(x) \right\rangle \mid x \in X \right\} \quad (1)$$

where functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and non-membership of the element $x \in X$ to the set *A*, respectively and for every $x \in X$, $0 \le \mu_A(x) + \nu_A(x) \le 1$. Obviously, the ordinary fuzzy set has the form $\left\{ \left\langle x, \mu_A(x), 1 - \mu_A(x) \right\rangle \mid x \in X \right\}$. If $\pi_A(x) = 1 - \left(\mu_A(x) + \nu_A(x)\right)$, then $\pi_A(x)$ is the degree of uncertainty or intuitionistic index of the membership of element $x \in X$ to set *A* where $0 \le \pi_A \le 1$.

The concept of fuzzy number has been developed through fuzzy set and possibility theory. The notion of fuzzy number was introduced in [29] and [30]. IFN was introduced in [31] and they studied perturbations of IFN and the first properties of the correlation between these numbers. Hence, IFN can be defined as follows:

Definition 2. [32] IFN A^* is defined as an intuitionistic fuzzy subset of the real line, normal i.e. there is any $x_0 \in \mathbb{R}$ such that $\mu_A(x) = 1, \nu_A(x) = 0$, convex for the membership function $\mu_A(x)$ i.e. $\mu_A\left(\lambda x_1 + (1-\lambda) x_2\right) \ge \min\left(\mu_A(x_1), \mu_A(x_2)\right) \forall x_1, x_2 \in \mathbb{R}, \lambda\ [0,1]$ and concave for the non-membership function $\nu_A(x)$ i.e. $\nu_A\left(\lambda x_1 + (1-\lambda) x_2\right) \le \max \left(\nu_A(x_1), \nu_A(x_2)\right) \forall x_1, x_2 \in \mathbb{R}, \lambda[0,1]$.

Definition 3. [33] Triangular IFN is IFS in $\mathbb{R}$ with membership function and non-membership function as follows:

$$\mu_{A^*}(x) = \begin{cases} \dfrac{x-\alpha}{m-\alpha}, & \alpha \le x \le m \\ \dfrac{c-x}{c-b}, & m \le x \le \beta \end{cases} \quad (2)$$

$$\nu_{A^*}(x) = \begin{cases} \dfrac{m-x}{b-\bar{a}}, & \alpha' \le x \le m \\ \dfrac{x-m}{\beta'-m}, & m \le x \le \beta' \end{cases} \quad (3)$$

Where $\alpha' < \alpha < m < \beta < \beta'$ and $\mu_A(x), \nu_A(x) \le 0.5$ for $\mu_A(x) = \nu_A(x) \forall x \in \mathbb{R}$ and value of both membership is 0 and 1 otherwise respectively. The symbolic representation of triangular IFN is $A^*_{IFN} = (\alpha, m, \beta; \alpha', m, \beta')$. α and β are called left and right spreads of membership function $\mu_A(x)$ respectively. α' and β' represented left and right spreads of non-membership function $\nu_A(x)$ respectively.

Fuzzy relations are fuzzy sets defined on universal sets which are Cartesian products of $X \times Y$ that represents the strength of association between elements of the two sets. Fuzzy relation has been studied in [34]–[36]. The concept of IFR is based on the definition of IFS. IFR were introduced in different forms and their approach was from different starting points and independently [37]–[43].

Definition 4. [40] Let $X, Y \subseteq \mathbb{R}$ be universal sets, then

$$R^* = \left\{ \left((x,y), \mu_R(x,y), \nu_R(x,y) \right) \mid (x,y) \in X \times Y \right\} \quad (4)$$

with $\mu_R : X \times Y \to [0,1]$ and $\nu_R : X \times Y \to [0,1]$ where $\pi_A(x,y) = 1 - \left(\mu_A(x,y) + \nu_A(x,y)\right)$ and (4) satisfy the condition $0 \le \mu_R(x,y) + \nu_R(x,y) \le 1, \forall (x,y) \in X \times Y$.

Next, IFP was introduced in [44]–[47] which used the natural generalization of fuzzy point given by [48]. Therefore, IFP is defined as follows:

Definition 5. Let $\alpha, \beta \in (0,1)$ and $\alpha + \beta \le 1$. IFP $p^x_{(\alpha,\beta)}$ of *X* is and IFS of *X* defined by $p^x_{(\alpha,\beta)} = \left\langle x, \mu_p, \gamma_p \right\rangle$ where for $y \in X$

$$\mu_p(y) = \begin{cases} \alpha & \text{if} \quad y = x \\ 0 & \text{if} \quad y \ne x \end{cases} \text{ and } \lambda_p(y) = \begin{cases} \beta & \text{if} \quad y = x \\ 1 & \text{if} \quad y \ne x \end{cases} \quad (5)$$

In this case, x is called the support of $p^x_{(\alpha,\beta)}$. IFP is said to belong to IFS $A^* = \left\langle x, \mu_A, \lambda_A \right\rangle$ of *X*, denoted by $p^x_{(\alpha,\beta)} \in A^*$, if $\alpha \le \mu_A(x)$ and $\beta \ge \lambda_A(x)$. Next section will introduce IFPR and some of its properties.

III. Intuitionistic Fuzzy Point Relation

IFPR is developed and introduced based on the concept of IFS that is discussed in the previous section. Let *V*, *W* be a collection of universal space of points in the Euclidean space and $V, W \in \mathbb{R}^2$, then IFPR is defined as follows:

Definition 6. Let X, Y be a collection of universal space of points with non-empty set and $V, W, I \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, then IFPR is defined as

$$T^* = \left\{ \left\langle (v_i, w_j), \mu_T(v_i, w_j), \nu_T(v_i, w_j), \pi_T(v_i, w_j) \right\rangle \mid \left(\mu_T(v_i, w_j), \nu_T(v_i, w_j), \pi_T(v_i, w_j) \right) \in I \right\} \quad (6)$$

where (v_i, w_j) is an ordered pair of points or coordinate and $(v_i, w_j) \in V \times W$. $\mu_T(v_i, w_j)$, $\nu_T(v_i, w_j)$ and $\pi_T(v_i, w_j)$ are the grade of membership, non-membership and uncertainty of the ordered pair of points respectively in $[0,1] \in I$. Furthermore the condition $0 \le \mu_T(v_i, w_j) + \nu_T(v_i, w_j) \le 1$ is follows and the degree of uncertainty is denoted by

$$\pi_T\left(v_i, w_j\right) = 1 - \left(\mu_T\left(v_i, w_j\right) + \nu_T\left(v_i, w_j\right)\right) \tag{7}$$

The IFPR is based on fuzzy point in the Euclidean space and the IFP is in IFS. Hence, the IFPR is in IFR and denoted by $T^* \in R^*$ and $P^* \times Q^* \in A^* \times B^*$.

Definition 7. Let P^* be an IFP and A^* is the IFN in *V*. Hence, P^* is said to be in A^* and denoted by $P^* \in A^*$ if and only if $\mu_P\left(v_i\right) \le \mu_A\left(v_i\right)$ and $\nu_P\left(v_i\right) \ge \nu_A\left(v_i\right)$ for all $v_i \in V$. Every fuzzy number A^* can be expressed as the union of all IFP that belong to A^* which if $\mu_A\left(v_i\right)$ and $\nu_P\left(v_i\right)$ is non-zero for $v_i \in V$, then $\mu_A\left(v_i\right) =$ $\sup\left\{y : \mu_P\left(v_i\right)\right.$ is IFP (membership) and $\left.0 < y \le \mu_A\left(v_i\right)\right\}$ and $\nu_A\left(v_i\right) = \inf\left\{y : \nu_P\left(v_i\right)\right.$ is IFP (non-membership) and $\left.0 < \nu_A\left(v_i\right) \le y\right\}$ respectively. Therefore, each and every IFP P^* in A^* can be written as $P^* = \{\, P_i^* \mid i = 1, 2, \ldots, n, i \in \mathbb{N}\}$ and $A^* = P_1^* \cup P_2^* \cup \ldots \cup P_n^*$.

Theorem 1. If $A^* = \bigcup_{i \in I} A_i^*$ where $I = \{1, 2, \ldots, n\}$ and *I* is any index, then $P^* \in A^*$ if and only if $P^* \in A_i^*$ for some $i \in I$.

Proof: Let the support for P^* denoted by v_0, then

$$\begin{aligned} \mu_A\left(v_0\right) &= \sup_{i \in I} \mu_{A_i}\left(v_0\right), \\ \nu_A\left(v_0\right) &= \inf_{i \in I} \nu_{A_i}\left(v_0\right) \end{aligned} \tag{8}$$

i) There exists some $i_0 \in I$ such as $\mu_{A_{i_0}}\left(v_0\right) = \mu_A\left(v_0\right)$ and $\nu_{A_{i_0}}\left(v_0\right) = \nu_A\left(v_0\right)$.

ii) $\mu_{A_i}\left(v_0\right) \le \mu_A\left(v_0\right)$ and $\nu_{A_i}\left(v_0\right) \ge \nu_A\left(v_0\right)$ for all $i \in I$.

For (i) $P^* \in A_{i_0}^*$. For (ii) $P^* \in A^*$ implies that $\mu_P\left(v_0\right) \le \mu_A\left(v_0\right)$, $\nu_P\left(v_0\right) \ge \nu_A\left(v_0\right)$ and considering that $\mu_A\left(v_0\right) = \sup_{i \in I} \mu_{A_i}\left(v_0\right)$, $\nu_A\left(v_0\right) = \inf_{i \in I} \nu_{A_i}\left(v_0\right)$, it follows that $\mu_P\left(v_0\right) \le \mu_{A_{i_0}}\left(v_0\right)$, $\nu_P\left(v_0\right) \ge \nu_{A_{i_0}}\left(v_0\right)$ for some i_0. Thus $P^* \in A_{i_0}^*$.

Definition 8. Let P^* and Q^* be an IFP and A^* and B^* is the IFN in *V* and *W* respectively. Hence, IFPR T^* on P^* and Q^*, $P^* \times Q^*$ is said to be in R^*, and denoted by $P^* \times Q^* \in A^* \times B^*$ if and only if $\mu_T\left(v_i, w_j\right) \le \mu_R\left(v_i, w_j\right)$ and $\nu_T\left(v_i, w_i\right) \ge \nu_R\left(v_i, w_i\right)$ for all $\left(v_i, w_j\right) \in V \times W$. Obviously, every R^* can be expressed as the union of all IFPR that belong to R^* which if $\mu_T\left(v_i, w_j\right)$ and $\nu_T\left(v_i, w_j\right)$ is non-zero for $\left(v_i, w_j\right) \in V \times W$, then $\mu_R\left(v_i, w_j\right) = \sup$ $\left\{\mu_{P \times Q}\left(v_i, w_j\right) : \mu_{P \times Q}\left(v_i, w_j\right)\right.$ is IFPR (membership) and $\left.0 < \mu_{P \times Q}\left(v_i, w_j\right) \le \mu_R\left(v_i, w_j\right)\right\}$ and $\nu_R\left(v_i, w_j\right) = \inf$ $\left\{\nu_{P \times Q}\left(v_i, w_j\right) : \nu_{P \times Q}\left(v_i, w_j\right)\right.$ is IFPR (non-membership) and $\left.0 < \nu_R\left(v_i, w_j\right) \le \nu_{P \times Q}\left(v_i, w_j\right)\right\}$ respectively. Therefore, each and every T^* in R^* can be written as $T^* = \{\, T_i^* \mid i = 0, 1, \ldots, n, i \in \mathbb{N}\}$ and $R^* = T_1^* \cup T_2^* \cup \ldots \cup T_n^*$.

Theorem 2. If $R^* = \bigcup_{i \in I} R_i^*$ where $I = \{1, 2, \ldots, n\}$ and *I* is any index, then $T^* \in R^*$ if and only if $T^* \in R_i^*$ for some $i \in I$.

Proof: Let the support for T^* denoted by (v_0, w_0), then

$$\begin{aligned} \mu_R\left(v_0, w_0\right) &= \sup_{i \in I} \mu_{R_i}\left(v_0, w_0\right), \\ \nu_R\left(v_0, w_0\right) &= \inf_{i \in I} \mu_{R_i}\left(v_0, w_0\right) \end{aligned} \tag{9}$$

i) There exists some $i_0 \in I$ such as $\mu_{R_{i_0}}\left(v_0, w_0\right) = \mu_R\left(v_0, w_0\right)$ and $\nu_{R_{i_0}}\left(v_0, w_0\right) = \nu_R\left(v_0, w_0\right)$.

ii) $\mu_{R_i}\left(v_0, w_0\right) \le \mu_R\left(v_0, w_0\right)$ and $\nu_{R_i}\left(v_0, w_0\right) \ge \nu_R\left(v_0, w_0\right)$ for all $i \in I$.

For (i) $T^* \in R_{i_0}^*$. For (ii) $T^* \in R^*$ implies that $\mu_T\left(v_0, w_0\right) \le \mu_R\left(v_0, w_0\right), \nu_T\left(v_0, w_0\right) \ge \nu_R\left(v_0, w_0\right)$ and considering that $\mu_R\left(v_0, w_0\right) = \sup_{i \in I} \mu_{R_i}\left(v_0, w_0\right)$, $\nu_R\left(v_0, w_0\right) =$ $\inf_{i \in I} \nu_{R_i}\left(v_0, w_0\right)$, it follows that $\mu_T\left(v_0, w_0\right) \le \mu_{R_{i_0}}\left(v_0, w_0\right)$, $\nu_T\left(v_0, w_0\right) \ge \nu_{R_{i_0}}\left(v_0, w_0\right)$ for some i_0. Thus $T^* \in R_{i_0}^*$.

IV. 3-Tuple Control Net Relation

The collection of all points or set of points that is used to determine the shape of a spline surface is called control net. The control net plays an important role in the process of generating, controlling and producing smooth surfaces. In this section, 3-TCNR is defined.

Definition 9. Let T^* be an IFPR, then 3-TCPR is defined as a set of points $n+1$ that indicates the positions and coordinates of a location and is used to describe the curve and is denoted by

$$C^* = \left\{C_0^*, C_1^*, \ldots, C_n^*\right\} \tag{10}$$

where *i* is one less than the number of points, *n*. Hence, from (6) and (10) the 3-TCNR can be defined as in **Def. 10**.

Definition 10. Let C^* be 3-TCPR, then the 3-TCNR can be defined generally as collection of points $n+1$ and $m+1$ for C^* in the direction of *u* and *v* respectively and denoted by $C_{i,j}^*$ that indicates the positions and coordinates of a location to describe the surface and written as

$$C_{i,j}^{*}=\begin{bmatrix} C_{0,0}^{*} & C_{0,1}^{*} & \cdots & C_{0,j}^{*} \\ C_{1,0}^{*} & C_{1,1}^{*} & \cdots & C_{1,j}^{*} \\ \vdots & \vdots & \ddots & \vdots \\ C_{i,0}^{*} & C_{i,1}^{*} & \cdots & C_{i,j}^{*} \end{bmatrix} \tag{11}$$

V. 3-Tuple Bézier Surface Interpolation Model

The surface is a vector value function of parameter *u* and *v*, that represents the mapping of *uv* plane into Euclidean three-dimension space. The tensor product method is basically a bidirectional curve scheme that uses basic functions and geometric coefficients. Therefore, the tensor product of 3-TBS is defined as follows:

Definition 11. Let $C_{i,j}^{*}$ be a 3-TCNR, then the tensor product of 3-TBS is given by

$$B^{*}(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}C_{i,j}^{*}B_{i}^{n}(u)B_{j}^{m}(v) \tag{12}$$

where $B_i^n(u)$ and $B_j^m(v)$ is the Bernstein basic functions in the parametric directions *u* and *v* and written as

$$\begin{aligned} B_i^n(u)&=\binom{n}{i}u^i(1-u)^{n-i} \\ B_j^m(v)&=\binom{m}{j}v^j(1-v)^{m-j} \end{aligned} \quad \text{with} \quad \begin{aligned} \binom{n}{i}&=\frac{n!}{i!(n-i)!} \\ \binom{m}{j}&=\frac{m!}{j!(m-j)!} \end{aligned} \tag{13}$$

The 3-TBS in (12) consists of membership, non-membership and uncertainty surface and denoted as follows:

$$B^{\mu}(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}C_{i,j}^{\mu}B_{i}^{n}(u)B_{j}^{m}(v) \tag{14}$$

$$B^{\nu}(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}C_{i,j}^{\nu}B_{i}^{n}(u)B_{j}^{m}(v) \tag{15}$$

$$B^{\pi}(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}C_{i,j}^{\pi}B_{i}^{n}(u)B_{j}^{m}(v) \tag{16}$$

For the 3-TBS interpolation model, the surface will lie in the data points. Therefore, the interpolation method is given by:

$$\begin{aligned} &\begin{bmatrix} D_{0,0}^{*} & D_{0,1}^{*} & \cdots & D_{0,m}^{*} \\ D_{1,0}^{*} & D_{1,1}^{*} & \cdots & D_{1,m}^{*} \\ \vdots & \vdots & \ddots & \vdots \\ D_{n,0}^{*} & D_{n,1}^{*} & \cdots & D_{n,m}^{*} \end{bmatrix} \\ &=\begin{bmatrix} B^{*}(u_0,v_0) & B^{*}(u_0,v_1) & \cdots & B^{*}(u_0,v_m) \\ B^{*}(u_1,v_0) & B^{*}(u_1,v_1) & \cdots & B^{*}(u_1,v_m) \\ \vdots & \vdots & \ddots & \vdots \\ B^{*}(u_n,v_0) & B^{*}(u_n,v_1) & \cdots & B^{*}(u_n,v_m) \end{bmatrix} \end{aligned} \tag{17}$$

Each $B^{*}(u_i,v_j)$ can be written as a matrix product such as;

$$\begin{aligned} B^{*}(u_i,v_j)&=\begin{bmatrix} B_0^n(u_i) & B_1^n(u_i) & \cdots & B_n^n(u_i) \end{bmatrix}\times \\ &\begin{bmatrix} C_{0,0}^{*} & C_{0,1}^{*} & \cdots & C_{0,m}^{*} \\ C_{1,0}^{*} & C_{1,1}^{*} & \cdots & C_{1,m}^{*} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n,0}^{*} & C_{n,1}^{*} & \cdots & C_{n,m}^{*} \end{bmatrix}\times\begin{bmatrix} B_0^m(v_j) \\ B_1^m(v_j) \\ \vdots \\ B_m^m(v_j) \end{bmatrix} \end{aligned} \tag{18}$$

All the individual equation may be combined into one matrix equation:

$$D^{*}=M^{T}C^{*}N \tag{19}$$

where D^{*} is the given matrix data points as in (17) and C^{*} is the matrix containing the unknown control points $C_{i,j}^{*}$. The matrix M^{T} and N contains the values of the Bernstein polynomials at the given parameters:

$$M^{T}=\begin{bmatrix} B_0^n(u_0) & B_1^n(u_0) & \cdots & B_n^n(u_0) \\ B_0^n(u_1) & B_1^n(u_1) & \cdots & B_n^n(u_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_0^n(u_n) & B_1^n(u_n) & \cdots & B_n^n(u_n) \end{bmatrix} \tag{20}$$

$$N=\begin{bmatrix} B_0^m(v_0) & B_0^m(v_1) & \cdots & B_0^m(v_m) \\ B_1^m(v_0) & B_1^m(v_1) & \cdots & B_1^m(v_m) \\ \vdots & \vdots & \ddots & \vdots \\ B_m^m(v_0) & B_m^m(v_1) & \cdots & B_m^m(v_m) \end{bmatrix} \tag{21}$$

Equation (19) is conveniently decomposed into a sequence of linear systems. First $C=M^{T}C^{*}$ is defined and later reduced to $D^{*}=CN$ and simplified to

$$C^{*}=\left(M^{T}\right)^{-1}D^{*}N^{-1} \tag{22}$$

To illustrate the 3-TBS interpolation, let considered 3-TCNR with degree of membership, non-membership and uncertainty with $m=3, n=3$ as follows:

$$\begin{bmatrix} C_{0,0}^{*} & C_{0,1}^{*} & C_{0,2}^{*} & C_{0,3}^{*} \\ C_{1,0}^{*} & C_{1,1}^{*} & C_{1,2}^{*} & C_{1,3}^{*} \\ C_{2,0}^{*} & C_{2,1}^{*} & C_{2,2}^{*} & C_{2,3}^{*} \\ C_{3,0}^{*} & C_{3,1}^{*} & C_{3,2}^{*} & C_{3,3}^{*} \end{bmatrix}$$

where each column with their respective value and 3-tuple degree $\langle \mu,\nu,\pi \rangle$ is given as;

$$\begin{bmatrix} C_{0,0}^{*} \\ C_{1,0}^{*} \\ C_{2,0}^{*} \\ C_{3,0}^{*} \end{bmatrix}=\begin{bmatrix} \langle(-17,17);0.3,0.6,0.1\rangle \\ \langle(-7,17);0.5,0.3,0.2\rangle \\ \langle(7,17);0.5,0.1,0.4\rangle \\ \langle(17,17);0.6,0.2,0.2\rangle \end{bmatrix}$$

$$\begin{bmatrix} C_{0,1}^* \\ C_{1,1}^* \\ C_{2,1}^* \\ C_{3,1}^* \end{bmatrix} = \begin{bmatrix} \langle(-17,7);0.8,0.2,0\rangle \\ \langle(-7,7);0.7,0.1,0.2\rangle \\ \langle(7,7);0.7,0.3,0\rangle \\ \langle(17,7);0.3,0.5,0.2\rangle \end{bmatrix}$$

$$\begin{bmatrix} C_{0,2}^* \\ C_{1,2}^* \\ C_{2,2}^* \\ C_{3,2}^* \end{bmatrix} = \begin{bmatrix} \langle(-17,-7);0.3,0.3,0.4\rangle \\ \langle(-7,-7);0.4,0.4,0.2\rangle \\ \langle(7,-7);0.4,0.6,0\rangle \\ \langle(17,-7);0.3,0.5,0.2\rangle \end{bmatrix}$$

$$\begin{bmatrix} C_{0,3}^* \\ C_{1,3}^* \\ C_{2,3}^* \\ C_{3,3}^* \end{bmatrix} = \begin{bmatrix} \langle(-17,-17);0.5,0.4,0.1\rangle \\ \langle(-7,-17);0.6,0.3,0.1\rangle \\ \langle(7,-17);0.4,0.2,0.4\rangle \\ \langle(17,-17);0.6,0.3,0.1\rangle \end{bmatrix}$$

Through (22), the unknown control points $C_{i,j}^*$ is obtained. After that, the desired surface is illustrated by blending the control net for membership, non-membership and uncertainty with the Bernstein polynomials using (12)–(16) as in Fig. 1 until Fig. 3. The red star in Fig. 1 until Fig. 3 represent control points $C_{i,j}^*$ and the line connecting the control points is called the intuitionistic control polygon. Fig. 1 until Fig. 3 shows the 3-TBS interpolation for membership, non-membership, and uncertainty surface, respectively. The figures give a surface that follows the condition $0 \le \mu_C\left(C_{i,j}^*\right)+\nu_C\left(C_{i,j}^*\right) \le 1$ and $\pi_T\left(C_{i,j}^*\right)=1-\left(\mu_C\left(C_{i,j}^*\right)+\nu_C\left(C_{i,j}^*\right)\right)$.

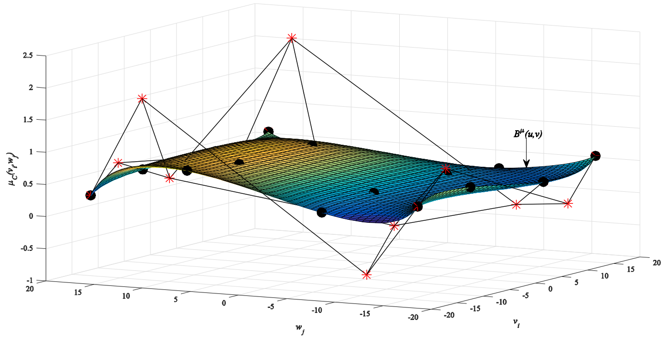

Fig. 1. 3-TBS interpolation with its respective data points, control points and control polygon (membership).

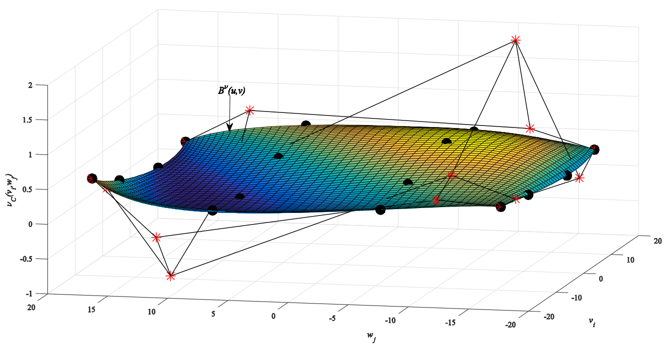

Fig. 2. 3-TBS interpolation with its respective data points, control points and control polygon (non-membership).

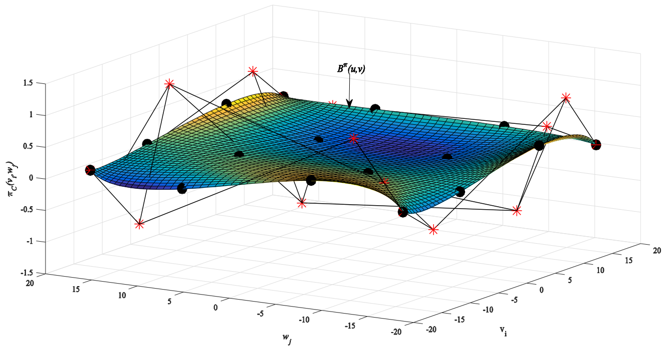

Fig. 3. 3-TBS interpolation with its respective data points, control points and control polygon (uncertainty).

The black dot in Fig. 1 until Fig. 3 represent the data points where the surfaces interpolate. The 3-TBS interpolation and its respective data points is shown separately in a better view as in Fig. 4 until Fig. 6.

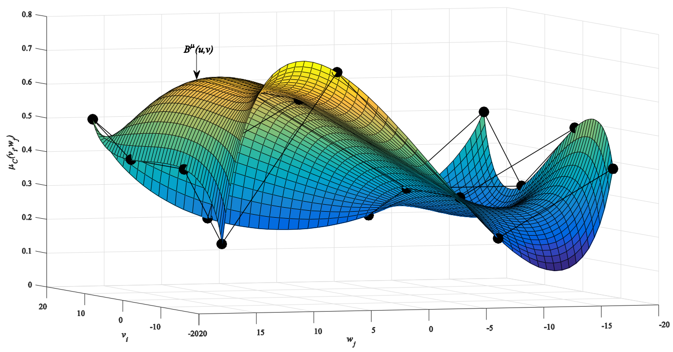

Fig. 4. 3-TBS interpolation with its respective data points (membership).

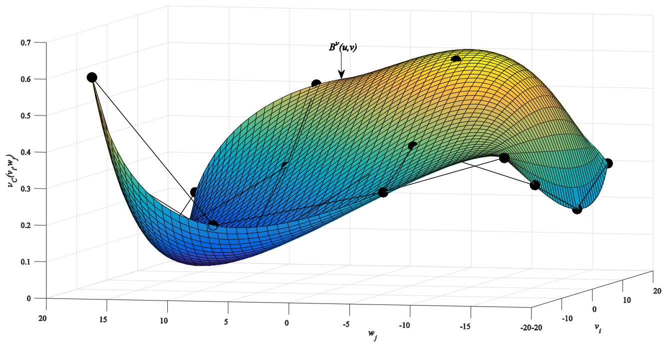

Fig. 5. 3-TBS interpolation with its respective data points (non-membership).

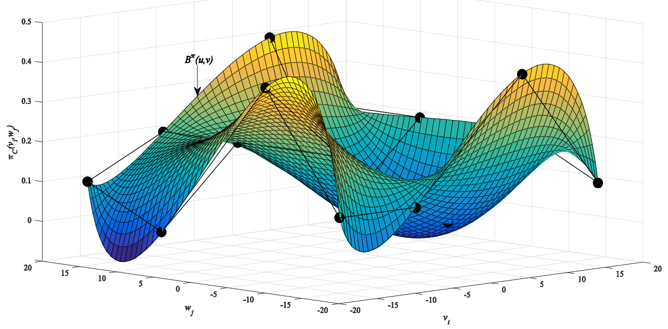

Fig. 6. 3-TBS interpolation with its respective data points (uncertainty).

Finally, the 3-tuple Bézier surface interpolation is illustrated in Fig. 7 until Fig. 9 with different view and perspective.

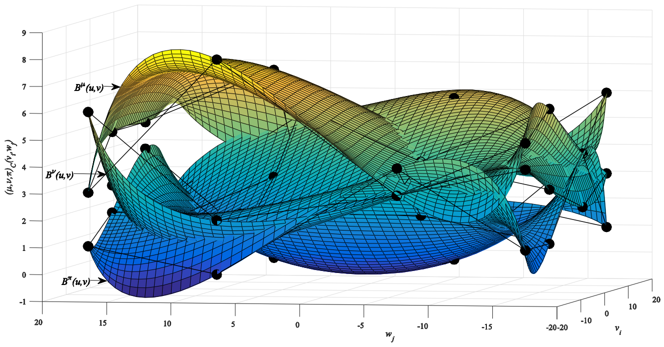

Fig. 7. 3-TBS interpolation with its respective data points.

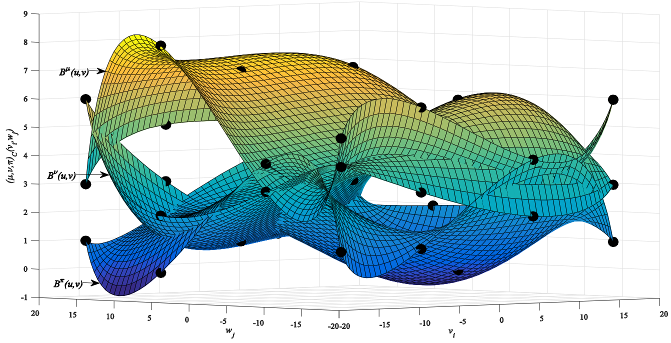

Fig. 8. 3-TBS interpolation with its respective data points (without control polygon).

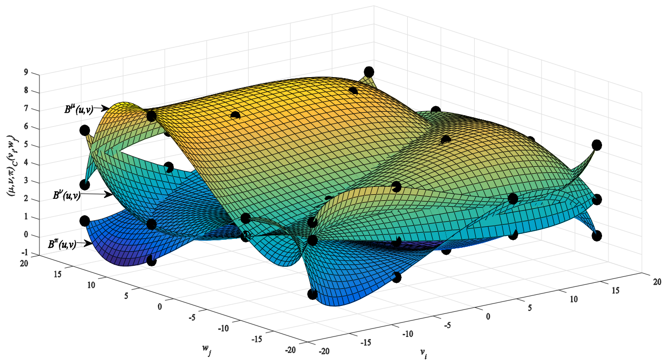

Fig. 9. 3-tuple Bézier surface interpolation with its respective data points (different view).

In this paper, we only considered relation of point from two universal sets where the third universe is the strength of relation or membership function.

VI. 3-Tuple Bézier Surface Properties

3-TCNR is a geometric coefficient of surface that are set in bidirectional of a net with $n \times m$. The 3-TCNR can gives many valuable information for early analysis as the shape and properties of 3-TBS depends on 3-TCNR. Therefore, some of the 3-TCNR and its respective surface properties are listed as follow:

i) The degree of 3-TBS in each parametric direction is one less than the number of 3-TCNR vertices in that direction.
ii) The 3-TBS generally follows the shape of 3-TCNR.
iii) The corner point of 3-TCNR and the resulting 3-TBS are coincident.
iv) The 3-TBS generally follows the shape of the 3-TCNR.
v) The 3-TBS is contained within the convex hull of 3-TCNR for $(u,v) \in [0,1] \times [0,1]$.
vi) The continuity of the 3-TBS in each parametric direction is two less than the number of 3-TCNR vertices in that direction.
vii) The 3-TBS is invariant under an affine transformation.
viii) The 3-TBS does not exhibit the variation-diminishing property. The variation-diminishing property for bivariant 3-TBS is both undefined and unknown.

VII. Conclusion

This paper has introduced 3-TBS interpolation model by defining 3-TCNR. This research can be extended by using B-spline and non-uniform rational B-splines (NURBS) function for surfaces to obtain better results. The introduced model has potentials in overcome surface data visualization problems such as modelling spatial regions with indeterminate boundary in geoinformation systems (GIS), remote sensing, object reconstruction from airborne laser scanner, bathymetric data visualization and much more. By using 3-TBS interpolation model, problems with uncertainty characteristic can be handled and overcame. The 3-TCNR and 3-TBS interpolation model can give complete analysis and description of modelling problem where each surface is visualized consists of membership, non-membership and uncertainty that have its own meaning and reasoning.

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