Matematika, 2003, Jilid 19, bil. 1, hlm. 47–56

New Nonlinear Four-Step Method for $y'' = f(t, y)$

Nazeeruddin Yaacob & Phang Chang
Department of Mathematics
University Teknologi Malaysia
81310 Skudai,
Johor, Malaysia

Abstract

In this paper, a study is made on the possibility of developing a nonlinear four-step method based on contraharmonic mean. The study is done since the four-step methods always give higher order than popular methods like Numerov and classical Runge-Kutta methods. A detailed study of consistency, stability, convergence and interval of periodicity has been done to convince ourselves of using this new method. The numerical results shows that the method is more accurate than the existing one.

Keywords Contraharmonic mean, interval of periodicity, second order initial value problem.

Abstrak *Dalam kertas kerja ini, satu kajian telah dijalankan tentang kemungkinan untuk membangunkan satu kaedah empat-langkah taklinear berdasarkan min kontraharmonik. Kajian ini telah dijalankan kerana kaedah empat-langkah selalunya memberikan peringkat yang lebih tinggi daripada kaedah terkenal seperti kaedah Numerov dan kaedah Runge-Kutta klasik. Kajian yang terperinci tentang kekonsistenan, kestabilan, penumpuan dan selang berkala telah meyakinkan kita penggunaan kaedah baru ini. Keputusan berangka menunjukkan keputusan yang lebih jitu daripada kaedah yang sedia ada.*

Katakunci *Min kontraharmonik, selang berkala, masalah nilai awal peringkat kedua.*

1 Introduction

For years, much work has been devoted to developing a better formulae to solve second order initial value problems of the type $y'' = f(t, y)$ with solution which is periodic or oscillating with known frequency. In recent years, multistep methods for this problem have been particularly been of concern. It is so because single step methods are inefficient since they do not make full use of the available information. It seems plausible that more accuracy can be obtained if the value of y_{n+1} is made to depend not only on y_n , but also on y_{n-1} and y_{n-2} .The same is true for the more recent method proposed by Simos [4] and Ixaru and Rizea [3].

The new nonlinear method is developed here based on some important hypotheses by Lambert and Watson [2] and Sommeijer, Van der Houwen and Neta [1].

2 Development of Nonlinear Four-Step Method based on Contraharmonic Mean

To develop a nonlinear four-step method, we use the formula of a linear multistep method as (1) and make a slightly modification to suit nonlinear four-step method as (2).

$$\sum_{j=0}^{k} a_j y_{n+j} = h^2 \sum_{j=0}^{k} c_j \psi_j(f), \quad k \leq 2, \tag{1}$$

$$\sum_{j=0}^{k} a_j y_{n+j} = h^2 \sum_{j=0}^{15} c_j \psi_j(f), \quad k = 4, \tag{2}$$

where a_j and c_j are constants and are the function formed by taking nonlinear combination of f at some fixed discrete points of special second order IVP $y'' = f(t, y)$ which is characterized by the polynomials ρ and σ , where

$$\rho(\zeta) = \sum_{j=0}^{k} a_j \xi^j, \tag{3}$$

$$\sigma(\zeta) = \sum_{j=0}^{k} c_j \xi^j. \tag{4}$$

Throughout this paper, we refer to the work of Lambert and Watson [2] for linear multistep methods as a guide. We shall assume that of nonlinear multistep methods satisfy the following hypotheses:

(i) $a_k = 1,\ |a_0 + a_5| \neq 0,\ \sum_{j=1}^{15} |c_j| \neq 0.$

(ii) ρ and σ have no common factors.

(iii) $\rho(1) = \rho'(1) = 0,\ \rho''(1) = 2\sigma(1)$; this is necessary and sufficient for the method to be consistent, that is, to have order at least one.

(iv) The method (ρ, σ) is zero-stable; that is, all the roots of ρ lie in or on the unit circle, and those on the unit circle having multiplicity not greater than two.

If we intend to have the four-step formula to be based on contraharmonic mean, then we have to form the nonlinear combination as in (5).

$$\sum_{j=0}^{4} a_j y_{n+j} = h^2 \left(\sum_{j=1}^{15} c_j \psi_j \right) \tag{5}$$

where

$$\psi_1 = f_{n+4},\ \psi_2 = f_{n+3},\ \psi_3 = f_{n+2},\ \psi_4 = f_{n+1},\ \psi_5 = f_n,$$
$$\psi_6 = \frac{f_{n+4}^2 + f_{n+3}^2}{f_{n+4} + f_{n+3}},\ \psi_7 = \frac{f_{n+3}^2 + f_{n+2}^2}{f_{n+3} + f_{n+2}},\ \psi_8 = \frac{f_{n+2}^2 + f_{n+1}^2}{f_{n+2} + f_{n+1}},$$
$$\psi_9 = \frac{f_{n+1}^2 + f_n^2}{f_{n+1} + f_n},\quad \psi_{10} = \frac{f_{n+4}^2 + f_n^2}{f_{n+4} + f_n},\ \psi_{11} = \frac{f_{n+3}^2 + f_{n+1}^2}{f_{n+3} + f_{n+1}},$$
$$\psi_{12} = \frac{f_{n+4}^2 + f_{n+1}^2}{f_{n+4} + f_{n+1}},\ \psi_{13} = \frac{f_{n+4}^2 + f_{n+2}^2}{f_{n+4} + f_{n+2}},\ \psi_{14} = \frac{f_{n+3}^2 + f_n^2}{f_{n+3} + f_n},$$
$$\psi_{15} = \frac{f_{n+2}^2 + f_n^2}{f_{n+2} + f_n}.$$

To fulfill the hypothesis (i) and properties of symmetric methods by Sommeijer, Van der Houwen and Neta [1] , we shall let the left hand side of equation (5) as

$$y_{n+4} + q_1 y_{n+3} + q_2 y_{n+1} + y_n.$$

We may then write (5) as

$$\begin{aligned} y_{n+4} + q_1 y_{n+3} + q_2 y_{n+1} + y_n \quad = \quad & h^2 (c_1 f_{n+4} + c_2 f_{n+3} + c_3 f_{n+2} + c_4 f_{n+1} + c_5 f_n \\ & + c_6 \frac{f_{n+4}^2 + f_{n+3}^2}{f_{n+4} + fn+3} + c_7 \frac{f_{n+3}^2 + f_{n+2}^2}{f_{n+3} + fn+2} + c_8 \frac{f_{n+2}^2 + f_{n+1}^2}{f_{n+2} + fn+1} \\ & + c_9 \frac{f_{n+1}^2 + f_n^2}{f_{n+1} + fn} + c_{10} \frac{f_{n+4}^2 + f_n^2}{f_{n+4} + fn} + c_{11} \frac{f_{n+3}^2 + f_{n+1}^2}{f_{n+3} + fn+1} \\ & + c_{12} \frac{f_{n+4}^2 + f_{n+1}^2}{f_{n+4} + fn+1} + c_{13} \frac{f_{n+4}^2 + f_{n+2}^2}{f_{n+4} + fn+2} + c_{14} \frac{f_{n+3}^2 + f_n^2}{f_{n+3} + fn} \\ & + c_{15} \frac{f_{n+2}^2 + f_n^2}{f_{n+2} + fn} \end{aligned} \tag{6}$$

Using Taylor's series we expand both sides of equation (5), and compare each coefficient. We obtain the system of equations as follows:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
8 & 6 & 4 & 2 & 0 & 7 & 5 & 3 & 1 & 4 & 4 & 5 & 6 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 16 & 4 & 9 & 4 & 9 & 4 \\
32 & 18 & 8 & 2 & 0 & 25 & 13 & 5 & 1 & 16 & 10 & 17 & 20 & 9 & 4 \\
0 & 0 & 0 & 0 & 0 & 7 & 5 & 3 & 1 & 64 & 16 & 45 & 24 & 27 & 8 \\
128 & 54 & 16 & 2 & 0 & 91 & 35 & 9 & 1 & 64 & 28 & 65 & 72 & 27 & 8 \\
0 & 0 & 0 & 0 & 0 & 123 & 63 & 23 & 3 & 768 & 168 & 603 & 368 & 243 & 48 \\
0 & 0 & 0 & 0 & 0 & 49 & 25 & 9 & 1 & 256 & 64 & 225 & 144 & 81 & 16 \\
0 & 0 & 0 & 0 & 0 & 37 & 19 & 7 & 1 & 256 & 52 & 189 & 112 & 81 & 16 \\
512 & 162 & 32 & 4 & 0 & 337 & 97 & 17 & 1 & 256 & 82 & 257 & 272 & 81 & 16 \\
0 & 0 & 0 & 0 & 0 & 343 & 125 & 27 & 1 & 1024 & 256 & 1125 & 864 & 243 & 32 \\
0 & 0 & 0 & 0 & 0 & 259 & 95 & 21 & 1 & 1024 & 208 & 945 & 672 & 243 & 32 \\
0 & 0 & 0 & 0 & 0 & 231 & 85 & 19 & 1 & 1024 & 192 & 885 & 608 & 243 & 32 \\
0 & 0 & 0 & 0 & 0 & 203 & 75 & 17 & 1 & 1024 & 176 & 825 & 544 & 243 & 32 \\
0 & 0 & 0 & 0 & 0 & 175 & 65 & 15 & 1 & 1024 & 160 & 765 & 480 & 243 & 32 \\
2048 & 486 & 64 & 2 & 0 & 1267 & 275 & 33 & 1 & 1024 & 244 & 1025 & 1056 & 243 & 32
\end{pmatrix}
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \\ c_{12} \\ c_{13} \\ c_{14} \\ c_{15} \end{pmatrix}
=
\begin{pmatrix}
5q_1+2(4+q_2) \\
\frac{4}{3}(16+7q_1+2q_2) \\
0 \\
\frac{1}{3}(41q_1+8(16+q_2)) \\
0 \\
\frac{2}{5}(61q_1+8(32+q_2)) \\
0 \\
0 \\
0 \\
\frac{2}{15}(365q_1+32(64+q_2)) \\
0 \\
0 \\
0 \\
0 \\
0 \\
\frac{4}{21}(547q_1+32(128+q_2))
\end{pmatrix}
$$

Using MATHEMATICA, we get the solution for system of equations in free parameter q_2 as follows:

$$
c_1 = \frac{1700011}{24000000} + \frac{q_2}{480},\ c_2 = \frac{5800009}{6000000} - \frac{q_2}{20},\ c_3 = \frac{924979}{1000000} - \frac{97q_2}{240} c_4 = \frac{1933283}{2000000} - \frac{q_2}{20},
$$
$$
c_5 = \frac{1699931}{24000000} + \frac{q_2}{480},\ c_6 = -\frac{1}{250000},\ c_7 = \frac{1}{1000000}, c_8 = \frac{41}{1000000},
$$
$$
c_9 = \frac{1}{62500},\ c_{10} = \frac{19}{4000000}, c_{11} = \frac{1}{200000}, c_{12} = -\frac{7}{600000},\ c_{13} = \frac{1}{100000},
$$
$$
c_{14} = -\frac{1}{200000},\ c_{15} = -\frac{1}{100000},\ \text{and } q_1 = -1 - \frac{q_2}{2}.
$$

Thus, we obtain a new method for solving $y'' = f(t, y)$. We call it a nonlinear 4-step method based on contraharmonic mean. (We abreviate it as 4-step CoM method).

$$
\begin{aligned}
& y_{n+4} - (1 + \frac{q_2}{2})y_{n+3} + q_2 y_{n+2} - (1 + \frac{q_2}{2})y_{n+1} + y_n \\
& = h^2((\frac{1700011}{24000000} + \frac{q_2}{480})f_{n+4} + (\frac{5800009}{6000000} - \frac{q_2}{20})f_{n+3} + (\frac{924979}{1000000} - \frac{97q_2}{240})f_{n+2} \\
& + (\frac{1933283}{2000000} - \frac{q_2}{20})f_{n+1} + (\frac{1699931}{24000000} + \frac{q_2}{480})f_n - (\frac{1}{250000})\left(\frac{f_{n+4}^2 + f_{n+3}^2}{f_{n+4} + f_{n+3}}\right) \\
& + (\frac{1}{1000000})\left(\frac{f_{n+3}^2 + f_{n+2}^2}{f_{n+3} + f_{n+2}}\right) + (\frac{41}{1000000})\left(\frac{f_{n+2}^2 + f_{n+1}^2}{f_{n+2} + f_{n+1}}\right) \\
& + (\frac{1}{62500})\left(\frac{f_{n+1}^2 + f_n^2}{f_{n+1} + f_n}\right) + (\frac{19}{4000000})\left(\frac{f_{n+4}^2 + f_n^2}{f_{n+4} + f_n}\right) + (\frac{1}{200000})\left(\frac{f_{n+3}^2 + f_{n+1}^2}{f_{n+3} + f_{n+1}}\right) \\
& - (\frac{7}{600000})\left(\frac{f_{n+4}^2 + f_{n+1}^2}{f_{n+4} + f_{n+1}}\right) + (\frac{1}{100000})\left(\frac{f_{n+4}^2 + f_{n+2}^2}{f_{n+4} + f_{n+2}}\right) \\
& + (\frac{1}{200000})\left(\frac{f_{n+3}^2 + f_n^2}{f_{n+3} + f_n}\right) - (\frac{1}{100000})\left(\frac{f_{n+2}^2 + f_n^2}{f_{n+2} + f_n}\right))
\end{aligned} \tag{7}
$$

3 Determination of Interval of Periodicity for 4-step CoM Method

The first and the second characteristic polynomial for 4-step CoM method may be written as follows

$$\rho(\zeta) = \zeta^4 - \left(1 + \frac{q_2}{2}\right)\zeta^3 + q_2\zeta^2 - \left(1 + \frac{q_2}{2}\right)\zeta + 1 \tag{8}$$

$$\begin{aligned}\sigma(\zeta) = &\left(\frac{1700011}{24000000} + \frac{q_2}{480}\right)\zeta^4 + \left(\frac{5800009}{6000000} - \frac{q_2}{20}\right)\zeta^3 + \left(\frac{924979}{1000000} - \frac{97q_2}{240}\right)\zeta^2 \\ &+ \left(\frac{1933283}{2000000} - \frac{q_2}{20}\right)\zeta + \left(\frac{1699931}{24000000} + \frac{q_2}{480}\right) - \left(\frac{1}{250000}\right)\left(\frac{\zeta^8 + \zeta^6}{\zeta^4 + \zeta^3}\right) \\ &+ \left(\frac{1}{1000000}\right)\left(\frac{\zeta^6 + \zeta^4}{\zeta^3 + \zeta^2}\right) + \left(\frac{41}{1000000}\right)\left(\frac{\zeta^4 + \zeta^2}{\zeta^2 + \zeta}\right) + \left(\frac{1}{62500}\right)\left(\frac{\zeta^2 + 1}{\zeta + 1}\right) \\ &+ \left(\frac{19}{4000000}\right)\left(\frac{\zeta^8 + 1}{\zeta^4 + 1}\right) + \left(\frac{1}{200000}\right)\left(\frac{\zeta^6 + \zeta^2}{\zeta^3 + \zeta}\right) - \left(\frac{7}{600000}\right)\left(\frac{\zeta^8 + \zeta^2}{\zeta^4 + \zeta}\right) \\ &+ \left(\frac{1}{100000}\right)\left(\frac{\zeta^8 + \zeta^4}{\zeta^4 + \zeta^2}\right) + \left(\frac{1}{200000}\right)\left(\frac{\zeta^6 + 1}{\zeta^3 + 1}\right) - \left(\frac{1}{100000}\right)\left(\frac{\zeta^4 + 1}{\zeta^2 + 1}\right)\end{aligned} \tag{9}$$

Next, we consider the equation

$$\phi(\zeta; H^2) = \rho(\zeta) + H^2\sigma(\zeta). \tag{10}$$

The objective is to find H_0^2 such that $|\zeta_j| = 1$ where ζ_j are zeroes of the equation $\phi(\zeta; H^2) = 0$, and $H^2 \in (0, H_0^2)$. This is done by fixing some values of H^2 which will give values of $\phi = 0$ in the equation (10). We do this until the required H_0^2 is found. We present some of the values obtained in Table 1 using $q_2 = 2$ (where the method is consistent).

From the Table 1, it is observed that the value of H_0^2 is approximately 0.55. This is true for $H^2 \in (0, 0.55)$, we have $|\zeta_j| = 1$. Hence, the interval of periodicity for 4-step CoM method is (0,0.55).

4 Numerical Results

We shall test the new method using two different values of free parameter for two different problems. We compare our results to those of Method V developed by Lambert and Watson [2] given as follows:

$$\begin{aligned} &y_{n+4} - (2+q)y_{n+3} + (2+2q)y_{n+2} - (2+2q)y_{n+1} + y_n \\ &= h^2\left[\left(\frac{18+q}{24}\right)f_{n+4} + \left(\frac{26+3q}{30}\right)f_{n+3} + \left(\frac{14-97q}{120}\right)f_{n+2} \right. \\ &\left. + \left(\frac{26-3q}{30}\right)f_{n+1} + \left(\frac{18+q}{240}\right)f_n\right]\end{aligned} \tag{11}$$

Table 1: Determination of H_0^2

H^2	ζ_j	$\lvert\zeta_j\rvert$
0.50	-0.999998	0.999998
	$-0.707107 \pm 0.707106i$	1.
	$-0.504766 \pm 0.863261i$	1.
	$-6.93165 \pm 0.999997i$	0.999997
	$0.500002 \pm 0.866025i$	1.
	$0.707106 \pm 0.707107i$	1.
	$0.877584 \pm 0.479422i$	1.
0.55	-0.999997	0.999997
	$-0.707107 \pm 0.707106i$	1.
	$-0.505909 \pm 0.862593i$	1.
	$-8.58288 \pm 0.999997i$	0.999997
	$0.500002 \pm 0.866025i$	1.
	$0.707106 \pm 0.707107i$	1.
	$0.852528 \pm 0.522681i$	1.
0.56	-0.999997	0.999997
	$-0.707107 \pm 0.707106i$	1.
	$-0.506155 \pm 0.862449i$	1.000001
	$-8.94191 \pm 0.999996i$	0.999996
	$0.500002 \pm 0.866025i$	1.
	$0.707106 \pm 0.707108i$	1.
	$0.84726 \pm 0.531179i$	1.

Problem 1 Consider the problem of solving

$$y'' = -4y, \ \ y(0) = 0, \ \ y'(0) = 2$$

where the exact solution of this problem is $y(t) = \sin(2t)$.

Using the 4-step CoM formula as in (7), we solve Problem 1 using $q_2 = 1/1000$ with step size $h = 0.1$. The absolute errors for the numerical solution near the zeroes of $y(t)$ are given in Table 2.

Table 2: Absolute errors in Method V and 4-step CoM method using step size $h = 0.1$ for Problem 1

t	Exact solution $y(t)$ near zero	Absolute errors (Method V)	Absolute errors (4-step CoM method)
π	−8.3089402817e-02	4.6211810859e-07	6.5705635603e-06
4π	6.7208072525e-02	1.8615250752e-06	2.1496595322e-07
16π	7.2493920780e-02	7.6815485629e-06	3.5145162423e-06
25π	−7.9548542875e-02	1.1634989843e-05	6.0222911229e-06
32π	−6.1890250722e-02	1.4912883813e-05	4.4873629581e-07
36π	5.3289163091e-03	1.6806657479e-05	4.1541678249e-06
40π	7.2523985782e-02	1.8628661728e-05	1.8771784017e-06
50π	4.0723376706e-02	2.3326315267e-05	8.2667880686e-06

Problem 2: Undamped Duffing problem

We use 4-step CoM method with $q_2 = 2$ and Method V to find the numerical solution of the following initial value problem, which is the nonlinear undamped Duffing equation

$$y'' + y + y^3 = B \cos \Omega t$$

forced by a harmonic function where $B = 0.002$ and $\Omega = 1.01$. The exact solution computed by the Galerkin method with a precision 10^{-12} of the coefficients is given by

$$y(t) = A_1 \cos \Omega t + A_3 \cos 3\Omega t + A_5 \cos 5\Omega t + A_7 \cos 7\Omega t + A_9 \cos 9\Omega t$$

where

$$\begin{aligned} A_1 &= 0.200179477536, \\ A_3 &= 0.000246946143, \\ A_5 &= 0.000000304014, \\ A_7 &= 0.000000000374, \\ A_9 &= 0.000000000000. \end{aligned}$$

(This problem are also solved by U.Anantha Krishnaiah [6]). We present the results in Table 3.

Table 3: Absolute error in Method V and 4-step CoM method using step size $h = \pi/10$ for Problem 2

t	Exact solution near zero	Absolute errors (Method V)	Absolute errors (4-step CoM method)
4π	1.9883085347e-01	3.0174707690e-03	9.0133968507e-04
4.2π	1.4489947717e-01	3.1834901583e-03	8.4771739226e-03
4.8π	−1.7782049510e-01	4.0003985992e-03	7.7119748608e-04
7.9π	1.9998640851e-01	4.6872008091e-03	1.4612857199e-03
11.8π	1.9375073508e-01	5.9496054289e-03	4.8996022141e-03
12.7π	−1.7131986058e-01	5.4529363933e-03	6.6239472623e-03
15.8π	1.9866753458e-01	1.0080427819e-02	1.1368402356e-03
26.1π	8.4424476916e-02	2.0912004348e-02	4.6788378483e-03
27.4π	1.0381052847e-01	2.2980506875e-02	2.1040142289e-05

5 Consistency

The 4-step CoM method is said to be consistent if and only if $\rho'(1) = \rho(1) = 0$ and $\rho''(1) = 2\sigma(1)$.

Proof:

$$\begin{aligned}
&\rho(\zeta) = \zeta^4 - (1+\frac{q_2}{2})\zeta^3 + q_2\zeta^2 - (1+\frac{q_2}{2})\zeta + 1 \\
\Longrightarrow\ &\rho(1) = (1)^4 - (1+\frac{q_2}{2})(1)^3 + q_2(1)^2 - (1+\frac{q_2}{2})(1) + 1 = 0 \\
\text{and}\ &\rho'(\zeta) = 4\zeta^3 - 3(1+\frac{q_2}{2})\zeta^2 + 2q_2\zeta - (1+\frac{q_2}{2}) \\
&\rho'(1) = 4(1)^3 - 3(1+\frac{q_2}{2})(1)^2 + 2q_2(1) - (1+\frac{q_2}{2}) = 0. \\
\text{So}\ &\rho'(1) = \rho(1) = 0.
\end{aligned}$$

Next, we have

$$\begin{aligned}
&\rho''(\zeta) = 12\zeta^2 - 6(1+\frac{q_2}{2})\zeta + 2q_2 \\
\Longrightarrow\ &\rho''(1) = 12(1)^2 - 6(1+\frac{q_2}{2})(1) + 2q_2 = 4 - 2q_2
\end{aligned}$$

and

$$\begin{aligned}\sigma(1) &= \left(\frac{1700011}{24000000} + \frac{q_2}{480}\right)(1)^4 + \left(\frac{5800009}{6000000} - \frac{q_2}{20}\right)(1)^3 + \left(\frac{924979}{1000000} - \frac{97q_2}{240}\right)(1)^2 \\ &+ \left(\frac{1933283}{2000000} - \frac{q_2}{20}\right)(1) + \left(\frac{1699931}{24000000} + \frac{q_2}{480}\right) - \left(\frac{1}{250000}\right)\left(\frac{(1)^8+(1)^6}{(1)^4+(1)^3}\right) \\ &+ \left(\frac{1}{1000000}\right)\left(\frac{(1)^6+(1)^4}{(1)^3+(1)^2}\right) + \left(\frac{41}{1000000}\right)\left(\frac{(1)^4+(1)^2}{(1)^2+(1)}\right) + \left(\frac{1}{62500}\right)\left(\frac{(1)^2+1}{(1)+1}\right) \\ &+ \left(\frac{19}{4000000}\right)\left(\frac{(1)^8+1}{(1)^4+1}\right) + \left(\frac{1}{200000}\right)\left(\frac{(1)^6+(1)^2}{(1)^3+1}\right) - \left(\frac{7}{600000}\right)\left(\frac{(1)^8+(1)^2}{(1)^4+1}\right) \\ &+ \left(\frac{1}{100000}\right)\left(\frac{(1)^8+(1)^4}{(1)^4+(1)^2}\right) - \left(\frac{1}{200000}\right)\left(\frac{(1)^6+1}{(1)^3+1}\right) - \left(\frac{1}{100000}\right)\left(\frac{(1)^4+1}{(1)^2+1}\right)) \\ &= 2 - q_2 \end{aligned} \tag{12}$$

Since $\rho''(1) = 2\sigma(1)$, the 4-step CoM method is consistent.

6 Zero-stability

The multistep method is said to be zero-stable if no root of the first characteristic polynomial $\rho(\zeta)$ has modulus greater than one, and every root of modulus one has multiplicity not greater than two.
Proof:
Since

$$\rho(\zeta) = \zeta^4 - 2\zeta^3 + 2\zeta^2 - 2\zeta + 1 = 0,$$

we get $\zeta = i, -i, 1$ and 1. Thus $|\zeta| = 1$. So

(i) No modulus of root > 1, and

(ii) Every root of modulus one has multiplicity exactly 2.

Hence the 4-step CoM method is zero-stable.

7 Convergence

The necessary and sufficient conditions for a multistep method to be convergent are that it be consistent and zero-stable.

Proof:

The nonlinear multistep method based on contraharmonic mean is shown to be

(i) consistent, and

(ii) zero-stable.

Thus, the 4-step CoM method is convergent.

8 Conclusion

In this paper, we have pioneered in using nonlinear mean to develop the nonlinear 4-step method for $y'' = f(t, y)$. Finally, we shall determine the agreement of the hypotheses mentioned in Section 1.

(i) $a_k = 1,\ |a_0| + |c_5| \neq 0, \sum_{j=1}^{15} |c_j| \neq 0$

(ii) ρ and σ have no common factors.

(iii) $\rho(1) = \rho'(1) = 0, \rho''(1) = 2\sigma(1)$;

This is a necessary and sufficient conditions for the method (ρ, σ) to be consistent, that is, to have order at least one.

(iv) The method (ρ, σ) is zero-stable;

That is all the roots of ρ lie in or on the unit circle, and those on the unit circle having multiplicity not greater than two.

Proof:

(i) $a_4 = 1,\ |a_0| = 1,\ c_5 = \dfrac{1699931}{24000000} + \dfrac{q_2}{480}$.

So $|a_0| + |c_5| \neq 0$ and

$$
\begin{aligned}
\sigma(1) = &(\frac{1700011}{24000000} + \frac{2}{480}) + (\frac{5800009}{6000000} - \frac{2}{20}) + (\frac{924979}{1000000} - \frac{97(2)}{240}) \\
&+ (\frac{1933283}{2000000} - \frac{2}{20}) + (\frac{1699931}{24000000} + \frac{2}{480}) - \frac{1}{250000} \\
&+ \frac{1}{1000000} + \frac{41}{1000000} + \frac{1}{62500} \\
&+ \frac{19}{4000000} + \frac{1}{200000} - \frac{7}{600000} \\
&+ \frac{1}{100000} - \frac{1}{200000} - \frac{1}{100000} \\
&= 2 \neq 0
\end{aligned}
$$

(ii), (iii) and (iv) are verified in Sections 5 to 7.

Furthermore, from the Table 2 and 3, we notice that this new nonlinear four-step method is generally more accurate than the Method V.

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