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Existence conditions for periodic solutions of second-order neutral delay differential equations with piecewise constant arguments

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Abstract: In this paper, we describe a method to solve the problem of finding periodic solutions for second-order neutral delay-differential equations with piecewise constant arguments of the form $x''(t) + px''(t-1) = qx([t]) + f(t)$, where $[\cdot]$ denotes the greatest integer function, p and q are nonzero real or complex constants, and $f(t)$ is complex valued periodic function. The method reduces the problem to a system of algebraic equations. We give explicit formula for the solutions of the equation. We also give counter examples to some previous findings concerning uniqueness of solution.

Keywords: differential equation, piecewise constant argument, periodic solution

MSC 2010: 35-xx, 47-xx, 47A10

1 Introduction

In the study of almost periodic differential equations, many useful methods have been developed in the classical references such as Hale and Lunel [1], Fink [2], Yoshizawa [3], and Hino et al. [4].

Differential equations with piecewise constant arguments are usually referred to as a hybrid system, and could model certain harmonic oscillators with almost periodic forcing. For some excellent works in this field we refer the reader to [5, 6] and references therein, and for a survey of work on differential equations with piecewise constant arguments we refer the reader to [7, 8].

A recently published paper [9] has studied the differential equation of the form

$$x''(t) + px''(t-1) = qx(2[\frac{t+1}{2}]) + f(t), \tag{1}$$

where $[\cdot]$ denotes the greatest integer function, p and q are nonzero constants, and $f(t)$ is a $2n$-periodic continuous function. The $2n$-periodic solvable problem (1) is reduced to the study a system of $n+1$ linear equations. Furthermore, by applying the well-known properties of linear system in algebra, all existence conditions are described for $2n$-periodic solutions that yields explicit formula for the solutions of (1).

In this paper we study certain functional differential equation of neutral delay type with piecewise constant arguments of the form

$$x''(t) + px''(t-1) = qx([t]) + f(t), \tag{2}$$

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where p and q are nonzero real or complex constants, $f(t)$ is a complex valued n-periodic continuous function defined on $\mathbf{R}$.

The papers [5, 6] have investigated the existence of almost periodic solutions of (2), when f is an almost periodic function, while Nth-order differential equation of neutral delay type are studied in [10, 11]. In these works some theorems for the existence and uniqueness of the almost periodic solutions have been obtained. However, there are some incorrectness of uniqueness results given in [5, 10–12].

In the present paper, by adapting the method in [9], we give all exact conditions for the uniqueness, infiniteness and emptiness of n-periodic solutions of (2), in the case when f is n-periodic. We give explicit formula for the exact solutions of the equation. For some functions f, we also show that equation (2) may have infinitely many 3-periodic solutions.

Throughout this paper, we use the notations $\mathbf{R}$ for the set of reals and $\mathbf{Z}$ for the set of integers.

2 Definition of periodic solution

A solution of (2) is defined in [13] as follows

Definition 2.1. *A function x is called a solution of (2) if the following conditions are satisfied:*

(i) *x and x' are continuous on $\mathbf{R}$;*
(ii) *the second-order derivative of $x(t)$ exists everywhere, with possible exception at points $t = n$, $n \in \mathbf{Z}$, where one-sided second-order derivatives of $x(t)$ exist;*
(iii) *x satisfies (2) on each interval $(n, n+1)$ with integer $n \in \mathbf{Z}$.*

Comments: The importance of the differentiability condition imposed on x is given in Definition 13 of [13], since we deal with second-order differential equations. This condition may admit uniqueness condition of periodic solution of (2), however it is a sufficient condition for the uniqueness condition (see Theorem 3.2 (ii) of this paper). Equation (2) may have infinite number of periodic continuous functions, satisfying (2) on each interval $(n, n+1)$ and may not have derivative at each point n, $n \in \mathbf{Z}$ (see Examples 1 and 2 below). Moreover, if we omit the continuity condition of x' on $\mathbf{R}$, then the uniqueness of periodic solution of (2) does not hold, and well-posedness of (2) is not true. In the definition of solution, the continuity condition of x' on $\mathbf{R}$ is omitted in many works (see, for example, [5], [11] and [12]) consequently, the uniqueness of pseudo periodic and hence almost periodic solution does not hold. We illustrate this in the following two examples.

Example 1. Let $f(t) = -3\pi^2\cos\pi t$, $p = 2$, $q = 4$. Then the function

$$x(t) = -15 - 2\alpha + \alpha t - \frac{4}{3}Q(t) + 3 - 3\cos\pi t, \quad t \in [0, 2] \tag{3}$$

satisfies (2) with the function Q defined on $[0, 2]$ as

$$Q(t) = \begin{cases} -\frac{9t^2}{2}, & t \in [0, 1), \\ -\frac{9}{2} - 9(t-1) + (27 + 3\alpha)\frac{(t-1)^2}{2}, & t \in [1, 2], \end{cases}$$

where α is any number. One can easily check that $x(t)$ is continuous on $\mathbf{R}$. But $x'(t)$ is discontinuous at $k \in \mathbf{Z}$ when $\alpha \neq -6$. If $\alpha = -6$, $x(t)$ is the exact solution of (2) (see Figures 1 and 2).

Example 2. Let $f(t) = \frac{1}{\pi^2}\cos\pi t$ and $p^2 \neq 1$, $8 - 8p + q \neq 0$. Then, by the definition of solution given in the papers [5], [10] and [11], the 2-periodic functions

$$x_\alpha(t) = x(0) + \alpha t + \frac{q}{1-p^2}Q(t) + \frac{1 - \cos\pi t}{1-p} \tag{4}$$

are solutions of the equation (2), where α is any number and the function Q defined on $[0, 2]$ as

$$Q(t) = (x(0) - px(1))\frac{t^2}{2} \quad \text{for} \quad t \in [0, 1),$$

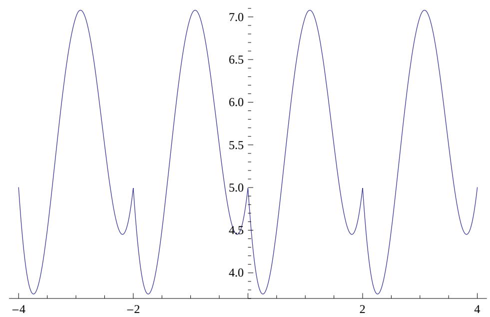

Figure 1: The graph of $x(t)$ when $\alpha = -10$.

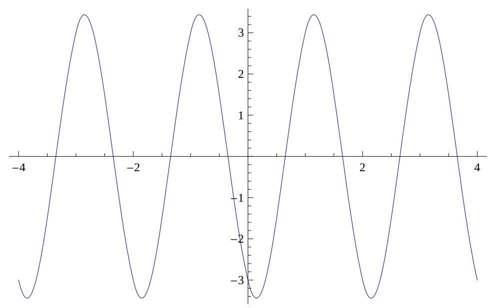

Figure 2: The graph of $x(t)$ when $\alpha = -6$.

$$Q(t) = Q(1-0) + (x(0) - px(1))(t-1) + (x(1) - px(0))\frac{(t-1)^2}{2} \quad \text{for} \quad t \in [1,2),$$
$$x(0) = \frac{2(-6q + 18pq - 4\alpha + 4p^2\alpha - q\alpha + pq\alpha)}{q(8 - 8p + q)},$$
$$x(1) = -\frac{2(18q - 6pq - 4\alpha + 4p^2\alpha + q\alpha - pq\alpha)}{(-8 + 8p - q)q}.$$

Note that $x(0) = x_\alpha(2)$, which gives continuity of x_α on $\mathbf{R}$.

It has been claimed in the papers [5], [10] and [11] that if $|\lambda_i| \neq 1, i = 1,2,3$, then equation (2) has a unique solution, where λ_i, $i = 1,2,3$, are the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 + \frac{q}{2} - p & 1 & p \\ q & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

For the case, when $p = \frac{1}{2}$, $q = 3$, the matrix A has a form

$$A = \begin{pmatrix} 2 & 1 & \frac{1}{2} \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

One can easily check that the eigenvalues $\lambda_i, i = 1,2,3$ are reals and $|\lambda_i| \neq 1, i = 1,2,3$. Moreover, the conditions of the Main results of these papers are satisfied. Example 2 shows incorrectness of the results Theorem 2.11 in [10], Theorem 3.3 in [11] and Theorem 1 in [5], which claim uniqueness of the almost periodic solutions of (2).

3 2 and 3-periodic solutions

In this section we give a method of finding periodic solutions of (2) and their existence conditions. Let f be n-periodic continuous function. We consider two cases $n = 2$ and $n = 3$ in this section before considering the general case n in Section 4.

The case $n = 2$. We seek a function x as a 2-periodic function that solves (2).

Equation (2) is equivalent to

$$x''(t+1) + px''(t) = qx([t+1]) + f(t+1).$$

Since $x''(t+1) = x''(t-1)$, this becomes

$$x''(t-1) = -px''(t) + qx([t+1]) + f(t+1). \tag{5}$$

Substitute this into (2) and assuming $p^2 \neq 1$, we get

$$x''(t) = \frac{q}{1-p^2}(x([t] - px([t+1])) + \frac{1}{1-p^2}(f(t) - pf(t+1)). \tag{6}$$

Integrating (6) twice on $[0, t]$, $t < 2$, we obtain

$$x(t) = x(0) + x'(0)t + \frac{q}{1-p^2}Q(t) + F_2(t), \tag{7}$$

where

$$Q(t) = \int_0^t \int_0^{t_1} (x([s]) - px([s+1])dsdt_1, \quad F_2(t) = \frac{1}{1-p^2}\int_0^t \int_0^{t_1} (f(s) - pf(s+1))dsdt_1. \tag{8}$$

The function Q on $[0, 2)$ can be represented as

$$Q(t) = (x(0) - px(1))\frac{t^2}{2} \quad \text{for} \quad t \in [0, 1)$$

and

$$\begin{aligned} Q(t) &= \int_0^1 \int_0^{t_1} X(s)dsdt_1 + \int_1^t \int_0^1 X(s)dsdt_1 + \int_1^t \int_1^{t_1} X(s)dsdt_1 \\ &= Q(1-0) + (x(0) - px(1))(t-1) + (x(1) - px(0))\tfrac{(t-1)^2}{2} \quad \text{for} \quad t \in [1, 2), \end{aligned}$$

where $X(s) = x([s]) - px([s+1]$ and $Q(a-0)$ denotes a left limit of $Q(t)$ at the point $t = a$.

This shows that the right-hand side of (7) contains only unknown numbers $x(0)$, $x(1)$ and $x'(0)$. Since x and x' are continuous and periodic, they should satisfy $x(0) = x(2)$, $x'(0) = x'(2)$. To find $x(0)$, $x(1)$ and $x'(0)$ we apply (7) to get the system of equations

$$\begin{aligned} x(1) &= x(0) + x'(0) + \tfrac{q}{1-p^2}Q(1) + F_2(1), \\ x(2) &= x(0) + 2x'(0) + \tfrac{q}{1-p^2}Q(2) + F_2(2), \\ x'(0) &= x'(0) + \tfrac{q}{1-p^2}Q'(2) + F_2'(2). \end{aligned}$$

Taking into account

$$Q(1) = \frac{1}{2}(x(0) - px(1)), \quad Q(2) = \frac{3}{2}(x(0) - px(1)) + \frac{1}{2}(x(1) - px(0)),$$

the last system of equations yields

$$\begin{aligned} (1 + \tfrac{1}{2}\tfrac{q}{1-p^2})x(0) - (1 + \tfrac{1}{2}\tfrac{qp}{1-p^2})x(1) + x'(0) &= -F_2(1), \\ \tfrac{1}{2}\tfrac{q}{1-p^2}(3-p)x(0) + \tfrac{1}{2}\tfrac{q}{1-p^2}(1-3p)x(1) + 2x'(0) &= -F_2(2), \\ \tfrac{q}{1+p}x(0) + \tfrac{q}{1+p}x(1) &= -F_2'(2). \end{aligned} \tag{9}$$

The determinant $D(p,q)$ of this system is

$$D(p,q) := \begin{vmatrix} 1+\frac{1}{2}\frac{q}{1-p^2} & -1-\frac{1}{2}\frac{qp}{1-p^2} & 1 \\ \frac{1}{2}\frac{q}{1-p^2}(3-p) & \frac{1}{2}\frac{q}{1-p^2}(1-3p) & 2 \\ \frac{q}{1+p} & \frac{q}{1+p} & 0 \end{vmatrix} = -\frac{4q}{1+p}.$$

Since $D(p,q) \neq 0$, we get

Theorem 3.1. *Let $p^2-1 \neq 0$ and f be 2-periodic continuous function. Then equation (2) has a unique 2-periodic solution having the form (7), where $(x(0), x(1), x'(0))$ is the unique solution of (9).*

Example 3. We consider equation (2) with $p = 3$, $q = 1$ and 2-periodic function $f(t) = \begin{cases} t, & t \in [0,1), \\ 2-t, & t \in [1,2]. \end{cases}$

It can be shown from (8) that

$$F_2(t) = \begin{cases} -\frac{1}{24}t^2(-9+2t), & t \in [0,1), \\ \frac{1}{8}(-\frac{5}{3}+4t+\frac{1}{3}(-1+t)^2(1+2t)), & t \in [1,2], \end{cases}$$

$$Q(t) = \begin{cases} \frac{47t^2}{24}, & t \in [0,1), \\ \frac{1}{24}(2-4t+49t^2), & t \in [1,2]. \end{cases}$$

Solving (9), we obtained $x(0) = -\frac{97}{48}$, $x(1) = -\frac{95}{48}$, $x'(0) = -\frac{1}{192}$. The graph of 2-periodic solution (7) is shown in Figure 3.

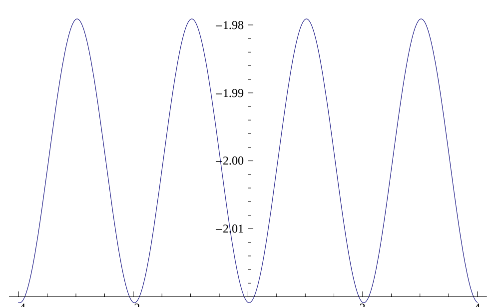

Figure 3: The graph of 2-periodic solution $x(t)$ of equation (2) for Example 3.

Example 4. Let $p = 2i$, $q = 1$ and $f(t) = e^{i\pi t}$. For this case, it can be shown that the 2-periodic solution of equation (2) is

$$x(t) = \begin{cases} -\frac{1}{50\pi^2}\left((10+20i)e^{i\pi t} - (3-4i)(-1+t)t\right), & t \in [0,1), \\ -\frac{1}{50\pi^2}\left((10+20i)e^{i\pi t} + (3-4i)(2-3t+t^2)\right), & t \in [1,2]. \end{cases}$$

The case $n = 3$. Let a function x be a 3-periodic function. From (2) we have

$$\begin{aligned} x''(t) + px''(t-1) &= qx([t]) + f(t), \\ x''(t+1) + px''(t) &= qx([t+1]) + f(t+1), \\ x''(t-1) + px''(t+1) &= qx([t+2]) + f(t+2). \end{aligned} \tag{10}$$

In the last equation we have used the fact that $x''(t+2) = x''(t-1)$. The system of equations (10) of $x''(t-1)$, $x''(t)$ and $x''(t+1)$ is solvable iff

$$\begin{vmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 1 & 0 & p \end{vmatrix} = 1 + p^3 \neq 0.$$

From this system of equations, assuming $p^3 \neq -1$, we get

$$x''(t) = \frac{q}{1+p^3}(p^2x([t+1]) - px([t+2]) + x([t])) + \frac{1}{1+p^3}(p^2f(t+1) - pf(t+2) + f(t)). \tag{11}$$

Integrating (11) two times on $[0, t]$, $t < 3$, we have

$$x(t) = x(0) + x'(0)t + \frac{q}{1+p^3}Q_3(t) + F_3(t), \tag{12}$$

where

$$\begin{aligned} Q_3(t) &= \int_0^t \int_0^{t_1} (p^2x([s+1]) - px([s+2]) + x([s]))ds dt_1, \\ F_3(t) &= \tfrac{1}{1+p^3} \int_0^t \int_0^{t_1} (p^2f(s+1) - pf(s+2) + f(s))ds dt_1. \end{aligned}$$

The function Q_3 on $[0, 3)$ can be presented by

$$\begin{aligned} Q_3(t) &= (p^2x(1) - px(2) + x(0))\tfrac{t^2}{2} \quad \text{for} \quad t \in [0,1), \\ Q_3(t) &= \int_0^1 \int_0^{t_1} Y(s)dsdt_1 + \int_1^t \int_0^1 Y(s)dsdt_1 + \int_1^t \int_1^{t_1} Y(s)dsdt_1 \\ &= Q_3(1-0) + (p^2x(1) - px(2) + x(0))(t-1) + (p^2x(2) - px(0) + x(1))\tfrac{(t-1)^2}{2} \quad \text{for} \quad t \in [1,2), \end{aligned}$$

where $Y(s) = p^2x([s+1]) - px([s+2]) + x([s])$,

$$\begin{aligned} Q_3(t) &= \int_0^2 \int_0^{t_1} Y(s)dsdt_1 + \int_2^t \int_0^1 Y(s)dsdt_1 + \int_2^t \int_1^2 Y(s)dsdt_1 + \int_2^t \int_2^{t_1} Y(s)dsdt_1 \\ &= Q(2-0) + (p^2x(1) - px(2) + x(0))(t-2) + (p^2x(2) - px(0) + x(1))(t-2) + (p^2x(0) - px(1) + x(2))\tfrac{(t-2)^2}{2} \\ &\quad \text{for} \quad t \in [2,3). \end{aligned}$$

Hence the right-hand side of (12) contains only unknown variables $x(0)$, $x(1)$, $x(2)$ and $x'(0)$. Since x and x' are continuous and periodic, they should satisfy

$$x(0) = x(3), \quad x'(0) = x'(3).$$

Using these equations and (12) we have

$$\begin{aligned} x(1) &= x(0) + x'(0) + \tfrac{q}{1+p^3}Q_3(1) + F_3(1), \\ x(2) &= x(0) + 2x'(0) + \tfrac{q}{1+p^3}Q_3(2) + F_3(2), \\ x(3) &= x(0) + 3x'(0) + \tfrac{q}{1+p^3}Q_3(3) + F_3(3), \\ x'(0) &= x'(0) + \tfrac{q}{1+p^3}Q_3'(3) + F_3'(3). \end{aligned} \tag{13}$$

Note that

$$\begin{aligned} Q_3(1) &= \tfrac{1}{2}(p^2x(1) - px(2) + x(0)), \\ Q_3(2) &= \tfrac{3}{2}(p^2x(1) - px(2) + x(0)) + \tfrac{1}{2}(p^2x(2) - px(0) + x(1)), \\ Q_3(3) &= \tfrac{5}{2}(p^2x(1) - px(2) + x(0)) + \tfrac{3}{2}(p^2x(2) - px(0) + x(1)) + \tfrac{1}{2}(p^2x(0) - px(1) + x(2)), \\ Q_3'(3) &= \tfrac{3}{2}(p^2x(1) - px(2) + x(0)) + \tfrac{1}{2}(p^2x(2) - px(0) + x(1)). \end{aligned}$$

Therefore, we have

$$\begin{aligned} (1 + \tfrac{1}{2}\tfrac{q}{1+p^3})x(0) + (\tfrac{1}{2}\tfrac{q}{1+p^3} - 1)x(1) - \tfrac{1}{2}\tfrac{qp}{1+p^3}x(2) + x'(0) &= -F_2(1), \\ (1 + \tfrac{q}{1+p^3}\tfrac{3-p}{2})x(0) + \tfrac{q}{1+p^3}\tfrac{3p^2+1}{2}x(1) + (\tfrac{q}{1+p^3}\tfrac{-3p+p^2}{2} - 1)x(2) + 2x'(0) &= -F_2(2), \\ \tfrac{q}{1+p^3}\tfrac{5-3p+p^2}{2}x(0) + \tfrac{q}{1+p^3}\tfrac{5p^2+3-p}{2}x(1) + \tfrac{q}{1+p^3}\tfrac{-5p+3p^2+1}{2}x(2) + 3x'(0) &= -F_2(3), \\ \tfrac{q}{1+p^3}(1 - p + p^2)x(0) + \tfrac{q}{1+p^3}(1 - p + p^2)x(1) + \tfrac{q}{1+p^3}(1 - p + p^2)x(2) &= -F_2'(3). \end{aligned} \tag{14}$$

Let $D_3(p, q)$ be the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 + \frac{1}{2}\frac{q}{1+p^3} & \frac{1}{2}\frac{q}{1+p^3} - 1 & -\frac{1}{2}\frac{qp}{1+p^3} & 1 \\ 1 + \frac{q}{1+p^3}\frac{3-p}{2} & \frac{q}{1+p^3}\frac{3p^2+1}{2} & \frac{q}{1+p^3}\frac{-3p+p^2}{2} - 1 & 2 \\ \frac{q}{1+p^3}\frac{5-3p+p^2}{2} & \frac{q}{1+p^3}\frac{5p^2+3-p}{2} & \frac{q}{1+p^3}\frac{-5p+3p^2+1}{2} & 3 \\ \frac{q}{1+p^3}(1-p+p^2) & \frac{q}{1+p^3}(1-p+p^2) & \frac{q}{1+p^3}(1-p+p^2) & 0 \end{pmatrix}.$$

It can be shown that

$$D_3(p,q) = -\frac{q(36 + 36p^2 + 6p(-6+q) + 6q + q^2)}{4(1+p^3)}.$$

Summarizing these results, we get

Theorem 3.2. *Let $p^3 + 1 \neq 0$ and f be a 3-periodic continuous function.*

(i) If $D_3(p,q) \neq 0$, then equation (2) has a unique 3-periodic solution having the form (12), where $(x(0), x(1), x(2), x'(0))$ is the unique solution of (14).

(ii) If $D_3(p,q) = 0$ and $F_3(1) = F_3(2) = F_3(3) = F_3'(3) = 0$, then equation (2) has infinite number of 3-periodic solutions having the form

$$x(t) = \alpha\big(x(0) + x'(0)t + \frac{q}{1+p^3} Q_3(t)\big) + F_3(t),$$

where $(x(0), x(1), x(2), x'(0))$ is an eigenvector of $\mathbf{A}$ corresponding to 0, α is any number.

(iii) If $D_3(p,q) = 0$ and the rank$[\mathbf{A}]$ < rank$[\mathbf{A}, \mathbf{F}^T]$, $\mathbf{F} = (F_3(1), F_3(2), F_3(3), F_3'(3))$, then equation (2) does not have any 3-periodic solution.

Remark 3.1. *We emphasize again that in the definition of solution, it is important to have the continuity condition of its derivative. If we omit this condition, we must remove the equality $x'(0) = x'(3)$ in (13). Let $p = 1$. Then the equation (13) is equivalent to 3×3 system*

$$\begin{aligned} -F_2(1) - \alpha &= (1 + \tfrac{1}{2}\tfrac{q}{2})x(0) + (\tfrac{1}{2}\tfrac{q}{2} - 1)x(1) - \tfrac{1}{2}\tfrac{q}{2}x(2), \\ -F_2(2) - 2\alpha &= (1 + \tfrac{q}{2}\tfrac{2}{2})x(0) + \tfrac{q}{2}\tfrac{4}{2}x(1) + (\tfrac{q}{2}\tfrac{-1}{2} - 1)x(2), \\ -F_2(3) - 3\alpha &= \tfrac{q}{2}\tfrac{3}{2}x(0) + \tfrac{q}{2}\tfrac{7}{2}x(1) + \tfrac{q}{2}\tfrac{-1}{2}x(2), \end{aligned} \tag{15}$$

where α is any number. Since the determinant $D(q)$ of the system of equations (15) is

$$D(q) = \frac{1}{4}q(9 + 3q + \frac{q^2}{4}),$$

(15) has a unique solution $(x(0), x(1), x(2))$ when $D(q) \neq 0$. Therefore, for any continuous 3-periodic f and q with $q(9 + 3q + \frac{q^2}{4}) \neq 0$ the continuous function

$$x_\alpha(t) = x(0) + \alpha t + \frac{q}{2}Q_3(t) + F_3(t)$$

satisfies (2). In particular, the functions $\{x_\alpha\}$ are well-defined for q and 3- periodic functions f satisfying the conditions of the Theorem 2.1 in [12], which claim on uniqueness of almost periodic solutions. This example shows that the Theorem 2.1 in [12] is incorrect.

We denote by $\mathcal{L}_0$ the class of continuous 3-periodic functions f with $F_3(1) = F_3(2) = F_3(3) = F_3'(3) = 0$, where

$$F_3(t) = \frac{1}{1+p^3}\int_0^t\int_0^{t_1}(p^2 f(s+1) - pf(s+2) + f(s))\,ds\,dt_1.$$

Remark 3.2. *Theorem (3.2) (ii) shows that the differentiability of a solution of (2) does not ensure uniqueness of the 3-periodic and hence almost periodic solutions of this equation. We can observe the existence of infinite number of 3-periodic solutions for the case when f belongs to $\mathcal{L}_0$. Let $f(t) = f_k(t)$, $f_k(t) = \cos 2k\pi t$, $k = 1, 2, \dots$ and $p = 1$, $q = -6$. Then $F_3(t) = \frac{1}{1+p^2}\frac{1}{(2k\pi)^2}(1 - \cos 2k\pi t)$ and hence $f_k \in \mathcal{L}_0$. The matrix $\mathbf{A}$ is represented by*

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{5}{2} & \frac{3}{2} & 1 \\ -2 & -6 & 2 & 2 \\ -\frac{9}{2} & -\frac{21}{2} & \frac{3}{2} & 3 \\ -3 & -3 & -3 & 0 \end{pmatrix}.$$

One can cheek that, $\det(\mathbf{A}) = D_3(1,-6) = 0$ *and the number* 0 *is an eigenvalue of* $\mathbf{A}$ *with multiplicity* 2. *The corresponding eigenvectors are* $\mathbf{x}_1 = (-\frac{1}{2}, \frac{1}{2}, 0, 1)$, $\mathbf{x}_2 = (-2, 1, 1, 0)$. *By Theorem 3.2 the functions*

$$x_1(t) = (-\frac{1}{2} + t - 3Q_{31}(t))\alpha + \frac{1}{8}\frac{1}{(k\pi)^2}(1 - \cos 2k\pi t), \tag{16}$$

$$x_2(t) = (-2 - 3Q_{32}(t))\alpha + \frac{1}{8}\frac{1}{(k\pi)^2}(1 - \cos 2k\pi t), \tag{17}$$

are 3*-periodic solutions of (2), where* α *is any nonzero constant,*

$$Q_{31}(t) = \begin{cases} 0 \quad as \quad t \in [0,1), \\ \frac{(t-1)^2}{2} \quad as \quad t \in [1,2), \\ -\frac{3}{2} + t - \frac{(t-2)^2}{2} \quad as \quad t \in [2,3], \end{cases}$$

$$Q_{32}(t) = \begin{cases} -t^2 \quad as \quad t \in [0,1), \\ 1 - 2t + 2(t-1)^2 \quad as \quad t \in [1,2), \\ -5 + 2t - (t-2)^2 \quad as \quad t \in [2,3]. \end{cases}$$

4 *n*-periodic solutions

We next solve equation (2), where f is periodic with positive integer period $n \geq 3$. It is clear that to seek a function x as a periodic function, we assume that $x(t) = x(t+n)$.

It follows from (2) and periodicity of $x(t)$ that

$$\begin{array}{lll} (x(t) + px(t-1))'' & = & qx([t]) + f(t), \\ (x(t+1) + px(t))'' & = & qx([t+1]) + f(t+1), \\ \dots & \dots & \dots \\ (x(t+n-2) + px(t+n-3))'' & = & qx([t+n-2]) + f(t+n-2), \\ (x(t-1) + px(t+n-2))'' & = & qx([t+n-1]) + f(t+n-1). \end{array} \tag{18}$$

Assuming the right-hand sides of (18) are known, we consider this system of equations with respect to

$$x''(t-1), x''(t), \dots, x''(t+n-1).$$

It is solvable if and only if $\Delta(p) \neq 0$, where $\Delta(p) = \det \mathbf{P}$ and $\mathbf{P}$ is the $n \times n$ matrix

$$\mathbf{P} = \begin{pmatrix} p & 1 & 0 & \dots & 0 & 0 \\ 0 & p & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & p & 1 \\ 1 & 0 & 0 & \dots & 0 & p \end{pmatrix}.$$

Observe that

$$\Delta(p) = \det \mathbf{P} = p^n - (-1)^n.$$

Assuming $p^n \neq (-1)^n$, we can find $x''(t)$ from (18), i.e.,

$$x''(t) = \frac{\Delta(p;t)}{\Delta(p)}, \tag{19}$$

where $\Delta(p;t) = \det \mathbf{Q}$ and $\mathbf{Q}$ is the $n \times n$ matrix

$$\mathbf{Q} = \begin{pmatrix} p & Q_1(t) & 0 & \dots & 0 & 0 \\ 0 & Q_2(t) & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & Q_{n-1}(t) & 0 & \dots & p & 1 \\ 1 & Q_n & 0 & \dots & 0 & p \end{pmatrix},$$

$$Q_k(t) = qx([t+k-1]) + f(t+k-1),\ k = 1, 2, \dots, n.$$

Using the properties of determinant, we have

$$\begin{aligned}
\det \mathbf{Q} =& Q_1(t)\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0\\ 0 & p & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & p & 1\\ 1 & 0 & 0 & \dots & 0 & p\end{pmatrix} + Q_2(t)\begin{pmatrix} p & 0 & 0 & \dots & 0 & 0\\ 0 & p & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & p & 1\\ 1 & 0 & 0 & \dots & 0 & p\end{pmatrix} + \cdots\\
&+ Q_{n-1}(t)(-1)^{n-1}\begin{pmatrix} p & 0 & 0 & \dots & 0 & 0\\ 0 & 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & 1 & 0\\ 1 & 0 & 0 & \dots & 0 & p\end{pmatrix} + Q_n(t)(-1)^n\begin{pmatrix} p & 0 & 0 & \dots & 0 & 0\\ 0 & 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & 1 & 0\\ 1 & 0 & 0 & \dots & 0 & 1\end{pmatrix}\\
=& (-1)^{n+1}Q_1(t) + Q_2(t)p^{n-1} + \cdots + (-1)^{n-1}Q_{n-1}(t)p^2 + (-1)^n Q_n(t)p).
\end{aligned}$$

Taking into $Q_1(t) = Q_{n+1}(t)$, the equation (19) represents

$$\begin{aligned}
x''(t) &= \tfrac{1}{\Delta(p)}\textstyle\sum_{k=1}^n (-1)^{k+1}Q_{k+1}(t)p^{n-k}\\
&= \tfrac{q}{\Delta(p)}\textstyle\sum_{k=1}^n (-1)^{k+1}x([t+k])p^{n-k} + \tfrac{1}{\Delta(p)}\sum_{k=1}^n (-1)^{k+1}f(t+k)p^{n-k}.
\end{aligned} \tag{20}$$

Integrating (20) we get

$$x(t) = x(0) + x'(0)t + \frac{q}{\Delta(p)}\widetilde{Q}_n(t) + F_n(t), \tag{21}$$

where

$$\begin{aligned}
\widetilde{Q}_n(t) &= \int_0^t\int_0^{t_1} R(s)\,ds\,dt_1,\\
R(s) &= \textstyle\sum_{k=1}^n (-1)^{k+1}x([s+k])p^{n-k},\\
F_n(t) &= \tfrac{1}{\Delta(p)}\int_0^t\int_0^{t_1}\textstyle\sum_{k=1}^n (-1)^{k+1}f(s+k)p^{n-k}\,ds\,dt_1.
\end{aligned}$$

We set

$$X[s] = \sum_{k=1}^n (-1)^{k+1}x([s+k])p^{n-k}.$$

Then the function $\widetilde{Q}_n$ on $[0, n)$ can be represented by

$$\begin{aligned}
\widetilde{Q}_n(t) =& \ X[0]\tfrac{t^2}{2},\quad t\in[0,1),\\
\widetilde{Q}_n(t) =& \int_0^1\int_0^{t_1} R(s)\,ds\,dt_1 + \int_1^t\int_0^1 R(s)\,ds\,dt_1 + \int_1^t\int_1^{t_1} R(s)\,ds\,dt_1\\
=& \ \widetilde{Q}_n(1-0) + X[0](t-1) + X[1]\tfrac{(t-1)^2}{2},\quad t\in[1,2),\\
\widetilde{Q}_n(t) =& \int_0^2\int_0^{t_1} R(s)\,ds\,dt_1 + \int_2^t\int_0^1 R(s)\,ds\,dt_1 + \int_2^t\int_1^2 R(s)\,ds\,dt_1 + \int_2^t\int_2^{t_1} R(s)\,ds\,dt_1\\
=& \ \widetilde{Q}_n(2-0) + (X[0]+X[1])(t-2) + X[2]\tfrac{(t-2)^2}{2},\quad t\in[2,3),\\
\dots& \qquad \dots\dots\dots\\
\widetilde{Q}_n(t) =& \int_0^{n-1}\int_0^{t_1} R(s)\,ds\,dt_1 + \int_{n-1}^t \textstyle\sum_{l=1}^{n-1}\int_{l-1}^{l} R(s)\,ds\,dt_1 + \int_{n-1}^t\int_{n-1}^t R(s)\,ds\,dt_1\\
=& \ \widetilde{Q}_n(n-1-0) + \textstyle\sum_{l=1}^{n-1} X[l-1](t-n+1) + X[n-1]\tfrac{(t-n+1)^2}{2},\quad t\in[n-1,n).
\end{aligned}$$

We remark that the functions $\widetilde{Q}_n$ and F_n are twice differentiable on $(n-1, n)$ and there exist one-sided second derivatives at $t = n-1$ and $t = n$.

The right-hand side of (21) contains only unknown numbers $x(0), \dots, x(n-1), x'(0)$. Using the periodicity conditions x and continuity x' from (21) we have $n+1$ system of equations

$$\begin{aligned} x(1) &= x(0) + x'(0) + \tfrac{q}{\Delta(p)} Q_n(1) + F_n(1), \\ \dots &\quad \dots\dots\dots \\ x(n-1) &= x(0) + (n-1)x'(0) + \tfrac{q}{\Delta(p)} Q_n(n-1) + F_n(n-1), \\ x(n) &= x(0) + nx'(0) + \tfrac{q}{\Delta(p)} Q_n(n) + F_n(n), \\ x'(n) &= x'(0) + \tfrac{q}{\Delta(p)} Q'_n(n) + F'_n(n). \end{aligned} \tag{22}$$

The last system of equations can be written as

$$\begin{aligned} x(1) &= x(0) + x'(0) + \tfrac{1}{2}\tfrac{q}{\Delta(p)} X(0) + F_n(1), \\ x(2) &= x(0) + 2x'(0) + \tfrac{q}{\Delta(p)} \sum_{k=1}^{2} \Big(\sum_{r=1}^{k-1} X(r-1) + \tfrac{1}{2} X(k-1) \Big) + F_n(2), \\ \dots &\quad \dots \quad \dots \\ x(n-1) &= x(0) + (n-1)x'(0) + \tfrac{q}{\Delta(p)} \sum_{k=1}^{n-1} \Big(\sum_{r=1}^{k-1} X(r-1) + \tfrac{1}{2} X(k-1) \Big) + F_n(n-1), \\ 0 &= nx'(0) + \tfrac{q}{\Delta(p)} \sum_{k=1}^{n} \Big(\sum_{r=1}^{k-1} X(r-1) + \tfrac{1}{2} X(k-1) \Big) + F_n(n), \\ 0 &= \tfrac{q}{\Delta(p)} \sum_{l=1}^{n} X[l-1] + F'_n(n). \end{aligned} \tag{23}$$

Since

$$\sum_{k=1}^{m} \Big(\sum_{r=1}^{k-1} X(r-1) + \frac{1}{2} X(k-1) \Big) = \sum_{k=0}^{m-1} (m-k-\frac{1}{2}) X(k),$$

we have

$$\begin{aligned} x(1) &= x(0) + x'(0) + \tfrac{1}{2}\tfrac{q}{\Delta(p)} X(0) + F_n(1), \\ x(2) &= x(0) + 2x'(0) + \tfrac{q}{\Delta(p)} \sum_{k=0}^{1} (2-k-\tfrac{1}{2}) X(k) + F_n(2), \\ \dots &\quad \dots \quad \dots \\ x(n-1) &= x(0) + (n-1)x'(0) + \tfrac{q}{\Delta(p)} \sum_{k=0}^{n-2} (n-1-k-\tfrac{1}{2}) X(k) + F_n(n-1), \\ 0 &= nx'(0) + \tfrac{q}{\Delta(p)} \sum_{k=0}^{n-1} (n-k-\tfrac{1}{2}) X(k) + F_n(n), \\ 0 &= \tfrac{q}{\Delta(p)} \sum_{l=1}^{n} X[l-1] + F'_n(n). \end{aligned} \tag{24}$$

To evaluate the above summation, we apply the following lemma.

Lemma 4.1. *For any positive integer number $l \geq 2$ the equality*

$$\sum_{k=0}^{l-1} (l-k-\frac{1}{2}) X(k) = \sum_{r=1}^{n} P_{lr}(p) x(r)$$

holds, where

$$P_{lr}(p) = \sum_{k=0}^{r-1} (l-k-\frac{1}{2})(-1)^{r-k+1} p^{n-r+k} + \sum_{k=r}^{l-1} (l-k-\frac{1}{2})(-1)^{n+r+1-k} p^{k-r}.$$

Proof. Let $\alpha_{lk} = l - k - \frac{1}{2}$.

$$
\begin{aligned}
\frac{\Delta(p)}{q}\sum_{k=0}^{l-1}\alpha_{lk}X(k) &= \sum_{k=0}^{l-1}\alpha_{lk}\sum_{r=1}^{n}(-1)^{r+1}x(k+r)p^{n-r} = \sum_{k=0}^{l-1}\alpha_{lk}\sum_{s=k+1}^{k+n}(-1)^{s-k+1}x(s)p^{n-s+k}\\
&= \sum_{k=0}^{l-1}\alpha_{lk}\Big(\sum_{s=k+1}^{n}(-1)^{s-k+1}x(s)p^{n-s+k} + \sum_{s=n+1}^{k+n}(-1)^{s-k+1}x(s)p^{n-s+k}\Big)\\
&= \sum_{k=0}^{l-1}\alpha_{lk}\Big(\sum_{s=k+1}^{n}(-1)^{s-k+1}x(s)p^{n-s+k} + \sum_{r=1}^{k}(-1)^{r+n-k+1}x(r+n)p^{-r+k}\Big)\\
&= \sum_{k=0}^{l-1}\alpha_{lk}\Big(\sum_{r=1}^{k}(-1)^{r+n-k+1}x(r)p^{-r+k} + \sum_{r=k+1}^{n}(-1)^{r-k+1}x(r)p^{n-r+k}\Big)\\
&= \alpha_{l0}\sum_{r=1}^{n}(-1)^{r+1}x(r)p^{n-r} + \alpha_{l1}\Big(\sum_{r=1}^{1}(-1)^{r+n}x(r)p^{1-r} + \sum_{r=2}^{n}(-1)^{r}x(r)p^{n-r+1}\Big)\\
&\quad + \cdots + \alpha_{l,l-1}\Big(\sum_{r=1}^{l-1}(-1)^{r+n-l+2}x(r)p^{-r+l-1} + \sum_{r=l}^{n}(-1)^{r-l+2}x(r)p^{n-r+l-1}\Big)\\
&= \big(\alpha_{l0}p^{n-1} + \alpha_{l1}(-1)^{1+n}p^{0} + \cdots + \alpha_{l,l-1}(-1)^{n-l+3}p^{l-2}\big)x(1)\\
&\quad + \cdots + \big(\alpha_{l0}(-1)^{l}p^{n-l+1} + \alpha_{l1}(-1)^{l-1}p^{n-l+2} + \cdots + \alpha_{l,l-2}(-1)^{2}p^{n-1} + \alpha_{l,l-1}(-1)^{n+1}p^{0}\big)x(l-1)\\
&\quad + \big(\alpha_{l0}(-1)^{l+1}p^{n-l} + \alpha_{l1}(-1)^{l}p^{n-l+1} + \cdots + \alpha_{l,l-1}(-1)^{2}p^{n-1}\big)x(l)\\
&\quad + \cdots + \big(\alpha_{l0}(-1)^{n+1}p^{0} + \alpha_{l1}(-1)^{n}p + \cdots + \alpha_{l,l-1}(-1)^{n-l+2}p^{l-1}\big)x(n)\\
&= \sum_{r=1}^{n}x(r)\Big(\sum_{k=0}^{r-1}\alpha_{lk}(-1)^{r-k+1}p^{n-r+k} + \sum_{k=r}^{l-1}\alpha_{lk}(-1)^{n+r+1-k}p^{k-r}\Big).
\end{aligned}
$$

By Lemma 4.1, equation (24) can be written as

$$
\begin{aligned}
x(1) &= x(0) + x'(0) + \tfrac{q}{\Delta(p)}\sum_{r=1}^{n}P_{1r}(p)x(r) + F_n(1),\\
x(2) &= x(0) + 2x'(0) + \tfrac{q}{\Delta(p)}\sum_{r=1}^{n}P_{2r}(p)x(r) + F_n(2),\\
&\cdots \qquad \cdots \qquad \cdots\\
x(n-1) &= x(0) + (n-1)x'(0) + \tfrac{q}{\Delta(p)}\sum_{r=1}^{n}P_{n-1,r}(p)x(r) + F_n(n-1),\\
0 &= nx'(0) + \tfrac{q}{\Delta(p)}\sum_{r=1}^{n}P_{nr}(p)x(r) + F_n(n),\\
0 &= \tfrac{q}{\Delta(p)}\sum_{r=1}^{n}P_{0}(p)x(r) - F_n'(n),
\end{aligned}
\tag{25}
$$

where

$$P_0(p) = \sum_{k=1}^{n}(-1)^{k+1}p^{n-k}.$$

We rewrite (25) as

$$
\begin{aligned}
(1 + \tfrac{q}{\Delta(p)}P_{1n}(p))x(0) + (\tfrac{q}{\Delta(p)}P_{11}(p) - 1)x(1) + \tfrac{q}{\Delta(p)}\sum_{r=2}^{n-1}P_{1r}(p)x(r) + x'(0) &= -F(1),\\
(1 + \tfrac{q}{\Delta(p)}P_{2n}(p))x(0) + (\tfrac{q}{\Delta(p)}P_{22}(p) - 1)x(2) + \tfrac{q}{\Delta(p)}\sum_{r\neq 2}^{n-1}P_{1r}(p)x(r) + 2x'(0) &= -F(1),\\
\cdots \qquad \cdots \qquad & \cdots\\
(1 + \tfrac{q}{\Delta(p)}P_{n-1,n}(p))x(0) + (\tfrac{q}{\Delta(p)}P_{n-1,n-1}(p) - 1)x(n-1) &\\
+ \tfrac{q}{\Delta(p)}\sum_{r=1}^{n-2}P_{1r}(p)x(r) + (n-1)x'(0) &= -F(1),\\
\tfrac{q}{\Delta(p)}P_{nn}(p)x(0) + \tfrac{q}{\Delta(p)}\sum_{r=1}^{n-1}P_{nr}(p)x(r) + nx'(0) &= -F_n(n),\\
\tfrac{q}{\Delta(p)}P_0(p)\sum_{r=1}^{n}x(r) &= -F_n'(n).
\end{aligned}
\tag{26}
$$

We denote by $D_n(p,q)$ the determinant of the equation (26). Then, $D_n(p,q)$ is the determinant of the matrix

$$\mathbf{D} = \begin{pmatrix} 1+\frac{q}{\Delta(p)}P_{1n}(p) & \frac{q}{\Delta(p)}P_{11}(p)-1 & \frac{q}{\Delta(p)}P_{12}(p) & \dots & \frac{q}{\Delta(p)}P_{1,n-1}(p) & 1 \\ 1+\frac{q}{\Delta(p)}P_{2n}(p) & \frac{q}{\Delta(p)}P_{11}(p) & \frac{q}{\Delta(p)}P_{22}(p)-1 & \dots & \frac{q}{\Delta(p)}P_{2,n-1}(p) & 2 \\ \dots & \dots & \dots & \dots\dots & & \\ 1+\frac{q}{\Delta(p)}P_{n-1,n}(p) & \frac{q}{\Delta(p)}P_{n-1,1}(p) & \frac{q}{\Delta(p)}P_{n-1,2}(p) & \dots & \frac{q}{\Delta(p)}P_{n-1,n-1}(p)-1 & n-1 \\ \frac{q}{\Delta(p)}P_{nn}(p) & \frac{q}{\Delta(p)}P_{n,1}(p) & \frac{q}{\Delta(p)}P_{n,2}(p) & \dots & \frac{q}{\Delta(p)}P_{n,n-1}(p) & n \\ \frac{q}{\Delta(p)}P_0(p) & \frac{q}{\Delta(p)}P_0(p) & \frac{q}{\Delta(p)}P_0(p) & \dots & \frac{q}{\Delta(p)}P_0(p) & 0 \end{pmatrix}$$

Summarizing these results, we get the following main result of the paper.

Theorem 4.1. *Let $p^n-(-1)^n \neq 0$ and f be n-periodic continuous function.*

(i) If $D_n(p,q) \neq 0$, then equation (2) has a unique n-periodic solution having the form (21), where $(x(0),\dots,x(n-1),x'(0))$ is the unique solution of (26).

(ii) If $D_n(p,q)=0$ and $F_n(1)=\dots=F_n(n)=F_n'(n)=0$, then equation (2) has infinite number of n-periodic solutions having the form

$$x(t)=\alpha\Big(x(0)+x'(0)t+\frac{q}{\Delta(p)}\widetilde{Q}_n(t)\Big)+F_n(t),$$

where $(x(0),\dots,x(n-1),x'(0))$ is an eigenvector of $\mathbf{D}$ corresponding to 0, α is any number.

(iii) if $D_n(p,q)=0$ and rank$[\mathbf{D}]$ < rank$[\mathbf{D},\mathbf{F}^T]$, $\mathbf{F}=(F_n(1),\dots,F_n(n),F_n'(n))$, then equation (2) does not have any n-periodic solution.

Example 5. Let $p=2i$, $q=1$ and $f(t)=\cos(\pi t)+i\sin(\frac{\pi}{2}t)$. It can be shown that the unique 4-periodic solution x of equation (2) has the form

$$\begin{aligned} x(t) &= -\tfrac{1}{\pi^2}(\tfrac{19}{2550}+\tfrac{4i}{1275})\Big((8080i)+(131-188i)t-(147-20i)t^2+ \\ &\quad +(304-128i)\cos(\tfrac{\pi}{2}t)+(42+36i)\cos(\pi t)-(64+152i)\sin(\tfrac{\pi}{2}t)\Big) \quad \text{for} \quad t\in[0,1), \\ x(t) &= -\tfrac{1}{\pi^2}(\tfrac{19}{2550}+\tfrac{4i}{1275})\Big((246+216i)-(201+460i)t+(19+156i)t^2 \\ &\quad +(304-128i)\cos(\tfrac{\pi}{2}t)+(42+36i)\cos(\pi t)-(64+152i)\sin(\tfrac{\pi}{2}t)\Big) \quad \text{for} \quad t\in[1,2), \\ x(t) &= -\tfrac{1}{\pi^2}(\tfrac{19}{2550}+\tfrac{4i}{1275})\Big((734-392i)-(689-148i)t+(141+4i)t^2 \\ &\quad +(304-128i)\cos(\tfrac{\pi}{2}t)+(42+36i)\cos(\pi t)-(64+152i)\sin(\tfrac{\pi}{2}t)\Big) \quad \text{for} \quad t\in[2,3), \\ x(t) &= \tfrac{1}{\pi^2}(\tfrac{19}{2550}+\tfrac{4i}{1275})\Big((652+2048i)-(235+1252i)t+(13+180i)t^2 \\ &\quad -(304-128i)\cos(\tfrac{\pi}{2}t)-(42+36i)\cos(\pi t)+(64+152i)\sin(\tfrac{\pi}{2}t)\Big) \quad \text{for} \quad t\in[3,4]. \end{aligned}$$

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References

[1] Jack K. Hale and Sjoerd M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer, New York, 1993.
[2] Arlington M. Fink, *Almost Periodic Differential Equation*, Springer, Berlin, 1974.
[3] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, Springer, New York, 1975.
[4] Yoshiyuki Hino, Toshiki Naito, Nguyen VanMinh, and Jong Son Shin, *Almost Periodic Solutions of Differential Equations in Banach Spaces*, Stability and Control: Theory, Method and Applications, vol. 15, Taylor & Francis, London, 2002.
[5] George Seifert, *Second-order neutral delay-differential equations with piecewise constant time dependence*, J. Math. Anal. Appl. **281** (2003), no. 1, 1–9.

[6] Rong Yuan, *Pseudo-almost periodic solutions of second-order neutral delay differential equations with piecewise constant argument*, Nonlinear Anal. **41** (2000), 871–890.

[7] Kenneth L. Cooke and Joseph Wiener, *A survey of differential equations with piecewise continuous arguments*, Delay Differential Equations and Dynamical Systems, Lecture Notes in Math., vol. 1475, Springer, Berlin, Heidelberg, 1991.

[8] Joseph Wiener, *Generalized Solutions of Functional Differential Equations*, World Scientific, Singapore, 1993.

[9] Mukhiddin I. Muminov, *On the method of finding periodic solutions of second-order neutral differential equations with piecewise constant arguments*, Adv. Difference Equ. (2017), 2017:336, DOI 10.1186/s13662-017-1396-7.

[10] Rong-Kun Zhuang, *Existence of almost periodic solutions to n-th order neutral differential equation with piecewise constant arguments*, Abstr. Appl. Anal. (2012), Article ID 186361, DOI 10.1155/2012/186361.

[11] Rong-Kun Zhuang and Rong Yuan, *Weighted pseudo almost periodic solutions of N-th order neutral differential equations with piecewise constant arguments*, Acta Math. Sin. (Engl. Ser.) **30** (2014), no. 7, 1259–1272.

[12] Hong-Xu Li, *Almost periodic solutions of second-order neutral delay-differential equations with piecewise constant argument*, J. Math. Anal. Appl. **298** (2004), 693–709.

[13] Elhadi Ait Dads and Lahcen Lhachimi, *New approach for the existence of pseudo almost periodic solutions for some second-order differential equation with piecewise constant argument*, Nonlinear Anal. **64** (2006), no. 6, 1307–1324.