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The Non-Normal Subgroup Graph of Some Dihedral Groups

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Let G be a finite groupand H is a subgroup of G. The subgroup graph of H in G is defined as a directed simple graph with vertex set G and two distinct elements x and y are adjacent if and only if $xy \in H$. In this paper, the work on subgroup graph is extended by defining a new graph called the non-normal subgroup graph. The subgroup graph is determined for some dihedral groupsof order $2n$ when the subgroup is non-normal.

Keywords: subgroup graph; non-normal subgroup; dihedral group

I. INTRODUCTION

In the recent years, many researches were done on graphs related to groups Zakariya (2016). In Group Theory, the vertices of a graph can be elements of groups or using some characteristics of groups. Let G be a group and Γ_G be a graph of the group G with a set of vertices $V(\Gamma_G) = \{1, \dots, n\}$ and a set of edges $E(\Gamma_G) = \{e_1, \dots, e_m\}$. Graphs related to groups can be directed or undirected graphs. Most of the researches for graphs of groups were on undirected graph including non-commuting graph (Anderson et al., 2012), conjugate graph (Erfanian & Tolue, 2012), and conjugacy class graph (Bertram et al., 1990). Research on directed graph of groups includes Cayley colourdiagram (Cayley, 1878) and directed power graph (Kelarev & Quinn, 2000). Cayley colour diagram which was introduced by Cayley (Cayley, 1878) is a directed graph with coloured edges. The latest directed graph which is thesubgroup graph was firstintroduced by Anderson (2012) and was then formally defined by Kakeri & Erfanian (2015). Anderson investigated the structure of the connected components of $\Gamma_H(G)$ when $|H|$ is either two or three, H is a normal subgroup and G/H is a finite abelian group. Besides, Kakeri and Erfanian (2015) studied on the complement of the subgroup graph. They also discussed some properties and the structure of the complement of subgroup graph. Then, Abdussakir (2017) focused on the energy and detour energy of the subgroup graph and the complement of subgroup graph. The research onthe subgroup graph was mostly done for normal subgroups. In this paper, the subgroup graphfor non-normal subgroups of the dihedral group will be discussed.

This paper is structured as follows: in Section II, some definitions thatrelated to subgroup graph are stated. In the following section, the results are presented on the non-normal subgroup graph of dihedral group.

II. PRELIMINARIES

In this section, some related definitions and properties of groups and graphs are given.

Definition 2.1 (Dummit & Foote, 2004)
The dihedral group, denoted by D_{2n}, is a group of symmetries of a regular polygon, which include rotations and reflections, and its order is $2n$ where n is an integer,$n \geq 3$. The dihedral groups can be presented in a form of generators and relations given as follows:

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$$D_{2n} \cong \langle a, b: a^n = b^2 = 1, bab = a^{-1} \rangle.$$

The subgroup H of a group G can be formed by letting $H$to be the set of all integer powers of x, where is x is any element of G. This subgroup can be written as $H = \langle x \rangle = \{x^n | n \in \mathbb{Z}\}$ and it is said as H is generated by x. From this subgroup, the normal subgroup and non-normal subgroup can be identified by using the following definition.

Proposition 2.1 (Chebolu & Lockridge, 2017)
For $n > 2$ and n odd, the only proper normal subgroups of D_{2n} are the subgroups of $\langle a \rangle$. For$n > 2$ and n even, there are two additional proper normal subgroups, $\langle a^2, b \rangle$ and $\langle a^2, ab \rangle$, both of order n and isomorphic to D_n.

Definition 2.2 (Kakeri & Erfanian, 2015)
Let G be a group and H be a subgroup of G. The subgroup graph $\Gamma_H(G)$ is a directed simple graph with vertex set G; and two distinct elements x and y are adjacent if and only if $xy \in H$.

III. RESULTS AND DISCUSSIONS

In this section, the subgroup graph for non-normal subgroups of a group Gwill be defined. Then, the subgroup graph for somenon-normal subgroups are constructed for dihedral groups of order at most 16.

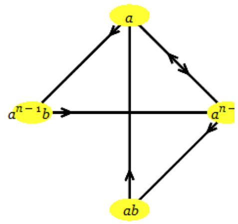
Based on the Proposition 2.1, the group of rotations is a normal subgroup of the dihedral group. Thus, the cyclic non-normal subgroups of D_{2n} are generated by $\langle b \rangle, \langle ab \rangle, .., \langle a^{n-1}b \rangle$ for both n even and odd. Next, the definition of subgroup graph for non-normal subgroups of a group G is shown.

Definition 3.1
Let G be a group and H be a non-normal subgroup of G. The subgroup graph $\Gamma_H^{NN}(G)$ is a directed graph with vertex set G; such that x is the initial vertex and y is the terminal vertex of an edge if and only if $x \neq y$ and $xy \in H$.

The results on the subgroup graphs for some dihedral groups are divided to three cases. The first case is for $n = 3$ and $n = 4$, the second case is for $n = 5$ and $n = 6$, and the last case is for$n = 7$ and $n = 8$.

Proposition 3.1
Let G be the dihedral group, D_{2n} with $n = 3$ and 4, which areD_6 and D_8 respectively. Meanwhile, $H_1 = \langle b \rangle = \{b, 1\}$ is a non-normal subgroup of G. Then the subgroup graph for H_1, $\Gamma_{H_1}^{NN}(G)$ is adirected graph as illustrated below.

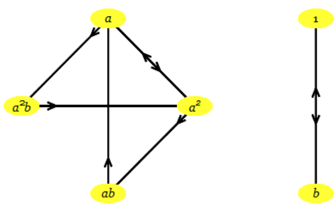
Proof



$$\cup \begin{cases} K_2, & n = 3 \\ K_2 \cup K_2, & n = 4 \end{cases}$$

Let $G = D_6$ where$n = 3$and$H_1 = \{b, 1\}$ is a non- normal subgroup of G. The elements of$D_6 = \{1, a, a^2, b, ab, a^2b\}$.
If $x = 1, y = b$, then $xy = 1(b) = b \in H$ implies a direction from $x = 1$ to $y = b$.
If $x = a, y = a^2$, $y = a^2b$, then$xy = a(a^2) = 1 \in H_1$ implies adirection from $x = a$ to $y = a^2$ and $xy = a(a^2b) = b \in H_1$ implies a direction from $x = a$ to $y = a^2b$.
If $x = a^2, y = a$, $y = ab$then$xy = a^2(a) = 1 \in H_1$ implies a direction from $x = a^2$ to $y = a$ and $xy = a^2(ab) = b \in H_1$ implies a direction from $x = a^2$ to $y = ab$.
If $x = b, y = 1$, then $xy = b(1) = b \in H$ implies a direction from $x = b$ to $y = 1$.
If $x = ab, y = a$, then $xy = ab(a) = b \in H_1$ implies a direction from $x = ab$ to $y = a$.
If $x = a^2b, y = a^2$, then $xy = a^2b(a^2) = b \in H_1$ implies a direction from $x = a^2b$ to$y = a^2$.

Thus, the subgroup graph is illustrated as follows:



There is one complete graph with two vertices, K_2 for the non-normal subgroup graph when $n = 3$. The non-normal subgroup graph for$n = 4$ can be proven using the similar method.

Let $G = D_8$where$n = 4$and$H_1 = \{b, 1\}$ is a non- normal subgroup of G. The elements of $D_8 =$

$\{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$.

If $x = 1, y = b$, then $xy = 1(b) = b \in H$ implies a direction from $x = 1$ to $y = b$.

If $x = a, y = a^3$, $y = a^3b$, then$xy = a(a^3) = 1 \in H_1$ implies a direction from $x = a$ to $y = a^3$ and $xy = a(a^3b) = b \in H_1$ implies a direction from $x = a$ to $y = a^3b$.

If $x = a^2, y = a^2b$ then $xy = a^2(a^2b) = b \in H_1$ implies a direction from $x = a^2$ to$y = a^2b$.

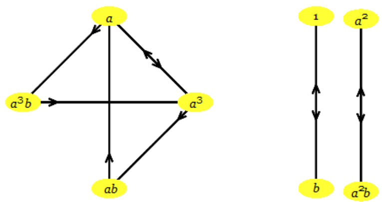
If $x = a^3, y = a$, $y = ab$ then$xy = a^3(a) = 1 \in H_1$ implies a direction from $x = a^3$ to $y = a$and $xy = a^3(ab) = b \in H_1$ implies a direction from $x = a^3$ to $y = ab$.

If $x = b, y = 1$, then $xy = b(1) = b \in H$ implies a direction from $x = b$ to $y = 1$.

If $x = ab, y = a$, then $xy = ab(a) = b \in H_1$ implies a direction from $x = ab$ to $y = a$.

If $x = a^2b, y = a^2$, then $xy = a^2b(a^2) = b \in H_1$implies a direction from $x = a^2b$ to $y = a^2$.

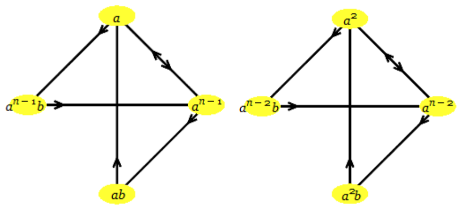
If $x = a^3b, y = a^3$, then $xy = a^3b(a^3) = b \in H_1$ impliesa direction from $x = a^3b$to $y = a^3$.Thus, the subgroup graph is illustratedas follows:



There are two complete graphs with two vertices, K_2 for the non-normal subgroup graph when $n = 4$.

Proposition 3.2

Let G be the dihedral group D_{2n} with $n = 5$ and 6, which are D_{10} and D_{12} respectively. Meanwhile, $H_1 = \langle b \rangle = \{b, 1\}$ is a non-normal subgroup of G. Then the subgroup graph for H_1, $\Gamma_{H_1}^{NN}(G)$ for $n = 5$ and 6 is a directed graph as illustrated below.



$$\cup \begin{cases} K_2, & n = 5 \\ K_2 \cup K_2, & n = 6 \end{cases}$$

Proof

Let $G = D_{10}$ where$n = 5$ and$H_1 = \{b, 1\}$ is a non-normal subgroup of G. The elements of $D_{10} = \{1, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$.

If $x = 1, y = b$, then $xy = 1(b) = b \in H$ implies a direction from $x = 1$ to $y = b$.

If $x = a, y = a^4b, y = a^4$ then$xy = a(a^4b) = b \in H_1$ implies a direction from $x = a$ to $y = a^4b$ and $xy = a(a^4) = 1 \in H_1$ implies a direction from $x = a$ to $y = a^4$.

If $x = a^2, y = a^3b, y = a^3$, then$xy = a^2(a^3b) = b \in H_1$implies a direction from $x = a^2$ to $y = a^3b$ and$xy = a^2(a^3) = 1 \in H_1$implies a direction from $x = a^2$ to $y = a^3$.

If $x = a^3, y = a^2b$,$y = a^2$, then$xy = a^3(a^2b) = b \in H_1$ implies a direction from $x = a^3$ to $y = a^2b$ and $xy = a^3(a^2) = 1 \in H_1$implies a direction from $x = a^3$ to $y = a^2$.

If $x = a^4, y = ab$,$y = a$ then$xy = a^4(ab) = b \in H_1$implies a direction from $x = a^2b$ to $y = a^2$ and$xy = a^4(a) = 1 \in H_1$ implies a direction from $x = a^4$ to $y = a$.

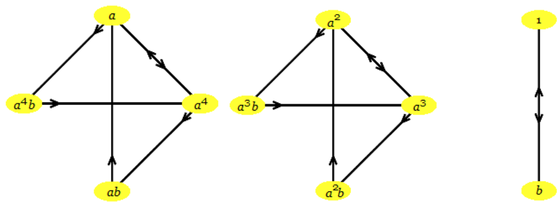
If $x = b, y = 1$, then $xy = b(1) = b \in H$ implies a direction from $x = b$ to $y = 1$.

If $x = ab, y = a$, then $xy = ab.a = b \in H_1$ implies a direction from $x = ab$ to $y = a$.

If $x = a^2b, y = a^2$, then $xy = a^2b(a^2) = b \in H_1$ which implies a direction from $x = a^2b$ to $y = a^2$.

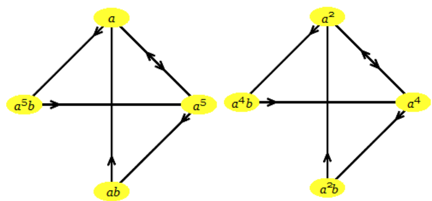
If $x = a^3b, y = a^3$, then $xy = a^3b(a^3) = b \in H_1$ implies a direction from $x = a^3b$ to $y = a^3$.

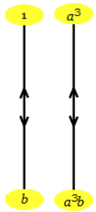
If $x = a^4b, y = a^4$, then $xy = a^4b(a^4) = b \in H_1$ implies a direction from $x = a^4b$ to $y = a^4$.Thus, the non-normal subgroup graph is illustrated as follows:



There is one complete graph with two vertices, K_2 for the non-normal subgroup graph when $n = 5$. The non-normal subgroup graph for$n = 6$ can be proven using the similar method.

Let $G = D_{12}$ where$n = 6$ and$H_1 = \{b, 1\}$ is a non-normal subgroup of G. The set of elements is $\{1, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$.By using the similar method, the subgroup graph is illustrated as follows:

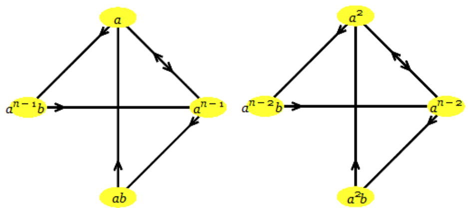


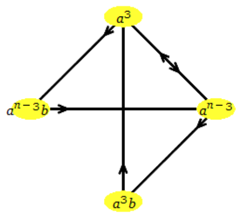


There are two complete graphs with two vertices, K_2 for the non-normal subgroup graph when $n = 6$.

Proposition 3.3

Let G be the dihedral group D_{2n} with $n = 7$ and8, which are D_{14} and D_{16} respectively. Meanwhile $H_1 = \langle b \rangle = \{b, 1\}$ is a non-normal subgroup of G. Then the subgroup graph for H_1, $\Gamma_{H_1}^{NN}(G)$for $n = 7$ and 8 is a directed graph which is illustrated as follows:





$$\cup \begin{cases} K_2, & n = 7 \\ K_2 \cup K_2, & n = 8 \end{cases}$$

Proof

Let $G = D_{14}$ where$n = 7$ and$H_1 = \{b, 1\}$ is a non-normal subgroup of G.The set of elements is $\{1, a, a^2, a^3, a^4, a^5, a^6, b, ab, a^2b, a^3b, a^4b, a^5b,$ $a^6b\}$.If $x = 1, y = b$, then $xy = 1(b) = b \in H$ which implies a direction from $x = 1$ to $y = b$.

If $x = a, y = a^6b, y = a^6$, then$xy = a(a^6b) = b \in H_1$which implies a direction from $x = a$ to $y = a^6b$ and $xy = a(a^6) = 1 \in H_1$implies a direction from $x = a$ to $y = a^6$.

If $x = a^2, y = a^5b, y = a^5$ then$xy = a^2(a^5b) = b \in H_1$ which implies a direction from $x = a^2$ to $y = a^5b$ and $xy = a^2(a^5) = 1 \in H_1$ implies a direction from $x = a^2$ to $y = a^5$.

If $x = a^3, y = a^4b, y = a^4$, then$xy = a^3(a^4b) = b \in H_1$ which implies a direction from $x = a^3$ to $y = a^4b$ and $xy = a^3(a^4) = 1 \in H_1$ implies a direction from $x = a^3$ to $y = a^4$.

If $x = a^4, y = a^3b, y = a^3$, then$xy = a^4(a^3b) = b \in H_1$ which implies a direction from $x = a^4$ to $y = a^3b$ and $xy = a^4(a^3) = 1 \in H_1$ implies a direction from $x = a^4$ to $y = a^3$.

If $x = a^5, y = a^2b, y = a^2$, then$xy = a^5(a^2b) = b \in H_1$ which implies a direction from $x = a^5$ to $y = a^2b$ and $xy = a^5(a^2) = 1 \in H_1$ implies a direction from $x = a^5$ to $y = a^2$.

If $x = a^6, y = a^2b, y = a$, then$xy = a^6(a^2b) = b \in H_1$ which implies a direction from $x = a^6$ to $y = a^2b$ and$xy = a^6(a) = 1 \in H_1$ implies a direction from $x = a^6$ to $y = a$.

If $x = b, y = 1$, then $xy = b(1) = b \in H$ which implies a direction from $x = b$ to $y = 1$.

If $x = ab, y = a$, then $xy = ab(a) = b \in H_1$ which implies a direction from $x = ab$ to $y = a$.

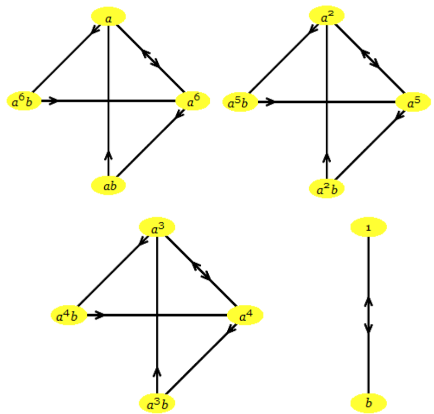
If $x = a^2b, y = a^2$, then $xy = a^2b(a^2) = b \in H_1$ which implies a direction from $x = a^2b$ to$y = a^2$.

If $x = a^3b, y = a^3$, then $xy = a^3b(a^3) = b \in H_1$ which implies a direction from $x = a^3b$ to$y = a^3$.

If $x = a^4b, y = a^4$, then $xy = a^4b(a^4) = b \in H_1$ which implies a direction from $x = a^4b$ to$y = a^4$.

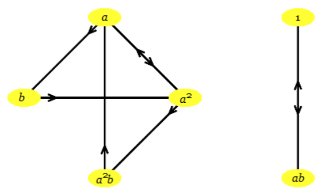
If $x = a^5b, y = a^5$, then $xy = a^5b(a^5) = b \in H_1$ which implies a direction from $x = a^5b$ to $y = a^5$.

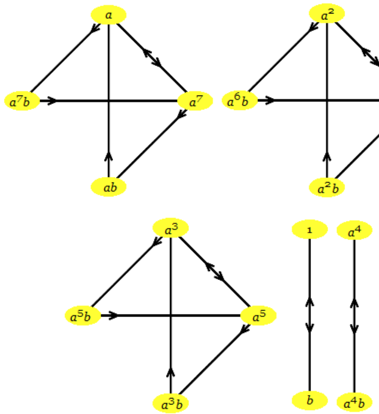
If $x = a^6b, y = a^6$, then $xy = a^6b(a^6) = b \in H_1$ which implies a direction from $x = a^6b$ to$y = a^6$.Thus, the subgroup graph is illustrated as follows:



There is one complete graph with two vertices, K_2 for the non-normal subgroup graph when $n = 7$. The non-normal subgroup graphfor $n = 8$can alsobe proven using the similar method.

Let $G = D_{16}$ where$n = 8$ and$H_1 = \{b, 1\}$ is a non-normal subgroup of G.The set of elements is $\{1, a, a^2, a^3, a^4, a^5, a^6,$ $a^7, b, ab,\ a^2b,\ a^3b, a^4b,$ $a^5b, a^6b, a^7b\}$. By using similar method before, the subgroup graph is illustrated as follows:



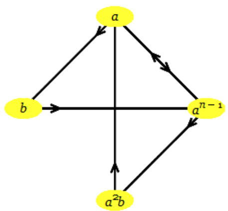


There are two complete graphs with two vertices, K_2 for the non-normal subgroup graph when $n = 8$.

The similar method can be used to obtain the subgroup graph of other dihedral group of order $2n$ when the subgroups are non-normal. The following propositions are the non-normal subgroup graphs for $H_2 = \{ab, 1\}$.

Proposition 3.4

Let G be the dihedral group, D_{2n} with $n = 3$ and 4, which are D_6 and D_8 respectively. Meanwhile $H_2 = \langle ab \rangle = \{ab, 1\}$ is a non-normal subgroup of G. Then the non-normal subgroup graph for H_2,$\Gamma_{H_2}^{NN}(G)$for $n = 3$ and 4 is a directed graph which is illustrated as follows:

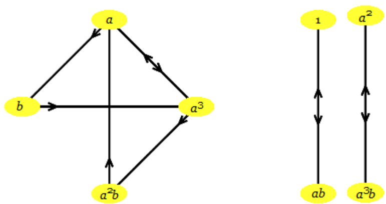


$$\cup \begin{cases} K_2, & n = 3 \\ K_2 \cup K_2, & n = 4 \end{cases}$$

Proof

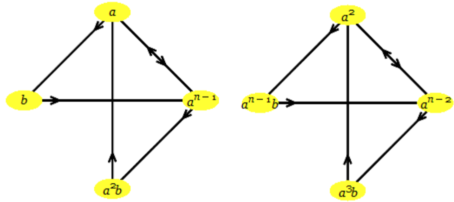
Let $G = D_6$ where $n = 3$ and$H_2 = \{ab, 1\}$ is a non-normal subgroup of G.By using the similar method in the proposition 3.1, the non-normal subgroup graph for $n = 3$ is illustrated as follows:

The non-normal subgroup graph for $n = 4$is illustrated as follows:



Proposition 3.5

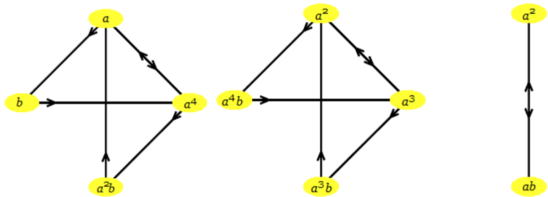
Let G be the dihedral group D_{2n} with $n = 5$ and 6, which areD_{10} and D_{12} respectively. Meanwhile $H_2 = \langle ab \rangle = \{ab, 1\}$ is a non-normal subgroup of G. Then the non-normal subgroup graph for H_2, $\Gamma_{H_2}^{NN}(G)$for $n = 5$ and 6 is a directed graph which is illustrated as follows:



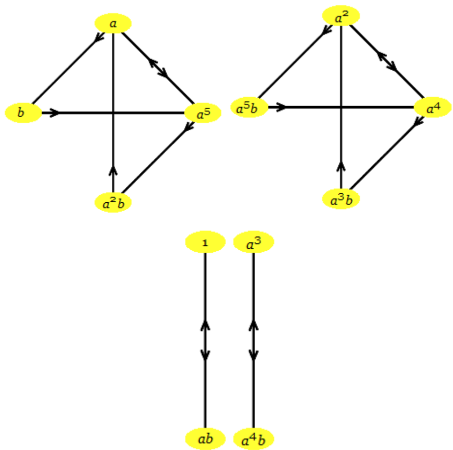
$$\cup \begin{cases} K_2, & n = 5 \\ K_2 \cup K_2, & n = 6 \end{cases}$$

Proof

Let $G = D_{10}$ where$n = 5$ and$H_2 = \{ab, 1\}$ is anon-normal subgroup of G.By using the similar method in the proposition 3.1, the non-normal subgroup graph for $n = 5$ is illustrated as follows:

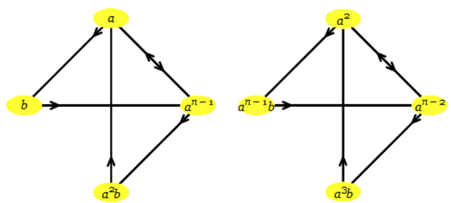


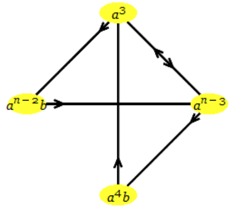
The non-normal subgroup graph for $n = 6$is illustrated as follows:



Proposition 3.6

Let G be the dihedral group D_{2n} with $n = 7$ and 8, which areD_{14} and D_{16} respectively. Meanwhile $H_2 = \langle ab \rangle = \{ab, 1\}$ is a non-normal subgroup of G. Then the non-normal subgroup graph for H_2, $\Gamma_{H_2}^{NN}(G)$for $n = 7$ and8 is a directed graph which is illustrated as follows:

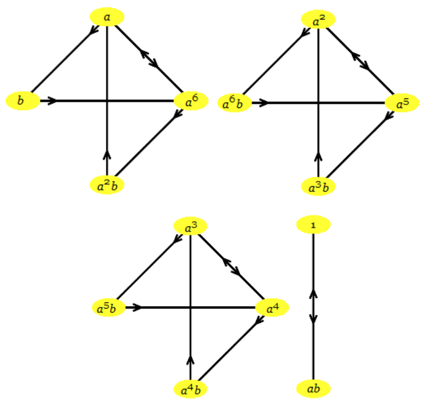




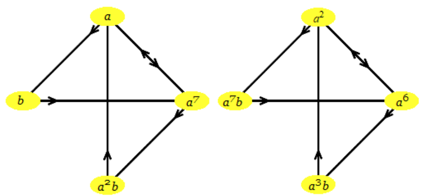
$$\cup \begin{cases} K_2, & n = 7 \\ K_2 \cup K_2, & n = 8 \end{cases}$$

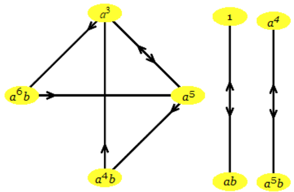
Proof

Let $G = D_{14}$ where$n = 7$ and$H_2 = \{ab, 1\}$ is a non-normal subgroup of G.By using the similar method in the proposition 3.1, the non-normal subgroup graph for $n = 7$ is illustrated as follows:



The non-normal subgroup graph for $n = 8$is illustrated as follows:





This method can be applied to get thesubgroup graphs for any non-normal subgroup for dihedral group.

IV. SUMMARY

In this study, the non-normal subgroup graph for some dihedral groups are determined. Only two non-normal subgroups are discussed in this paper. Furthermore, the subgroup graphs for other non-normal subgroups also can be obtained by using the similar method. It can be concluded that for all dihedral groups D_{2n}, the non-normal subgroup graphis a union of directed graphs and complete graphs.

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