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Original Article

Exact solution of boundary value problem describing the convective heat transfer in fully-developed laminar flow through a circular conduit

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Abstract

This paper proposes anexact solution in terms of an infinite series to the classical Graetz problem represented by a nonlinear partial differential equation considering two space variables, two boundary conditions and one initial condition. The mathematical derivation is based on the method of separation of variables whose several stages are elaborated to reach the solution of the Graetz problem. MATLAB was used to compute the eigenvalues of the differential equation as well as the coefficient series. However, both the Nusselt number as an infinite series solution and the Graetz number are based on the heat transfer coefficient and the heat flux from the wall to the fluid. In addition, the analytical solution was compared to the numerical values obtained by the same author using a FORTRAN program, showing that the orthogonal collocation method gave better results. It is important to note that the analytical solution is in good agreement with published numerical data.

Keywords: Graetz problem, Sturm-Liouville problem, partial differential equation, dimensionless variable

1. Introduction

The solutions of one or more partial differential equations (PDEs), which are subjected to relatively simple limits, can be tackled either by analytical or numerical approach. There are two common techniques available to solve PDEs analytically, namely the separation and the combination of variables. The Graetz problem describes the temperature (or concentration) field in fully developed laminar flow in a circular tube where the wall temperature (or concentration) profile is a step-function (Shah & London, 1978). The simplified version of the Graetz problem initially neglected axial diffusion, considering simple wall heating conditions (isothermal and isoflux), using a simple geometric cross-section (either parallel plates or circular channel), and also neglecting fluid flow heating effects; this can be generally labeled as the Classical Graetz Problem (Braga, de Barros, & Sphaier, 2014). Min, Yoo, and Choi (1997) presented an exact solution for a Graetz problem with axial diffusion and flow heating effects in a semi-infinite domain with a given inlet condition. Later, the Graetz series solution was further improved by Brown (1960).

Hsu (1968) studied a Graetz problem with axial diffusion in a circular tube, using a semi- infinite domain formulation with a specified inlet condition. Ou and Cheng (1973) employed separation of variables to study the Graetz problem with finite viscous dissipation. They obtained the solution in the form of a series whose eigenvalues and eigenfunctions satisfy a Sturm–Liouville system. The solution approach is similar to that applicable to the classical Graetz problem, and therefore suffers from the same weakness of poor conver-

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gence near the entrance. The same techniques have been used by other authors to derive analytical solutions, involving the same special functions (Fehrenbach, De Gournay, Pierre, & Plouraboué, 2012; Plouraboue & Pierre, 2007; Plouraboue & Pierre, 2009). Basu and Roy (1985) analyzed the Graetz problem for Newtonian fluid, taking into account viscous dissipation but neglecting the effect of axial conduction. Papoutsakis, Damkrishna, and Lim (1980) presented an analytical solution to the extended Graetz problem with finite and infinite energy or mass exchange sections and prescribed wall energy or mass fluxes, with an arbitrary number of discontinuities. Coelho, Pinho, and Oliveira (2003) studied the entrance thermal flow problem for the case of a fluid obeying the Phan-Thien and Tanner (PTT) constitutive equation. This appears to be the first study of the Graetz problem with a viscoelastic fluid. The solution was obtained by separation of variables and the ensuing Sturm–Liouville system was solved for the eigenvalues by means of a freely available solver, while the ordinary differential equations for the eigenfunctions and their derivatives were calculated numerically with a Runge–Kutta method. In the work of Bilir (1992), a numerical profile based on the finite difference method was developed by using the exact solution of the one-dimensional problem to represent the temperature change in the flow direction.

Ebadian and Zhang (1989) analyzed the convective heat transfer properties of a hydrodynamically fully developed viscous flow in a circular tube. Lahjomri and Oubarra (1999) investigated a new method of analysis and improved the solution of the extended Graetz problem with heat transfer in a conduit. An extensive list of contributions related to this problem may be found in the papers of Papoutsakis, Ramkrishna, Henry, and Lim (1980) and Liou and Wang (1990). In addition, the analytical solution proposed efficiently resolves the singularity, and this methodology allows extension to other problems such as the Hartmann flow (Lahjomri, Oubarra, & Alemany, 2002), conjugated problems (Fithen& Anand, 1988) and other boundary conditions.

Transient heat transfer for laminar pipe or channel flow has been analyzed by many authors.Ates, Darici, and Bilir (2010) investigated the transient conjugated heat transfer in thick-walled pipes for thermally developing laminar flow involving two-dimensional wall and axial fluid conduction. The problem was solved numerically by a finite-difference method for hydrodynamically developed flow in a two-section pipe, initially isothermal in the upstream region that is insulated while the downstream region suddenly applies a uniform heat flux. Darici,Bilir, and Ates (2015) in their work solved a problem in thick-walled pipes by considering axial conduction in the wall. They handled transient conjugated heat transfer in simultaneously developing laminar pipe flow. The numerical strategy used in this work is based on the finite difference method with a thick-walled semi-infinite pipe that is initially isothermal, with hydrodynamically and thermally developing flow, and with a sudden change in the ambient temperature.Darici and Sen (2017) numerically investigated a transient conjugate heat transfer problem in microchannels with the effects of rarefaction and viscous dissipation. They also examined the effects of other parameters on heat transfer, such as the Peclet number, the Knudsen number, the Brinkman number and the wall thickness ratio.

Recently, Belhocine (2015) developed a mathematical model to solve the classic problem of Graetz using two numerical approaches, the orthogonal collocation method and the method of Crank-Nicholson.

In this paper, the Graetz problem that consists of two differential partial equations will be solved by separation of variables. The Kummer equation is employed to identify the confluent hypergeometric functions and their properties, in order to determine the eigenvalues of the infinite series that appears in the proposed analytical solution. Also, theoretical expressions for the Nusselt number as a function of the Graetz number were obtained. In addition, the exact analytical solution presented in this work was validated against numerical data previously published by the same author, obtained by the orthogonal collocation method that gave better results.

2. Background of the Problem

As a good model problem, we consider steady state heat transfer to fluid in a fully developed laminar flow through a circular pipe (Figure1). The fluid enters at z=0 at a temperature of T_0 and the pipe walls are maintained at a constant temperature of T_ω. We will write the differential equation for the temperature distribution as a function of r and z, and then express this in a dimensionless form and identify the important dimensionless parameters. Heat generation in the pipe due to the viscous dissipation is neglected, and a Newtonian fluid is assumed. Also, we neglect the dependence of viscosity on temperature. A sketch of the system is shown below.

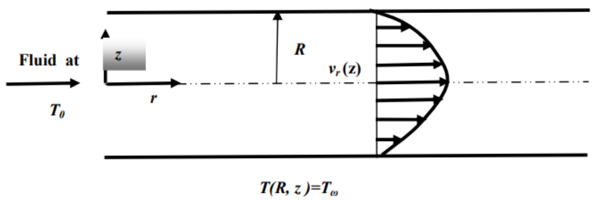

Figure 1. Schematic of the classical Graetz problem and the coordinate system

2.1 The heat equation in cylindrical coordinates

The general equation for heat transfer in cylindrical coordinates, developed by Bird, Stewart, and Lightfoot (1960), is as follows;

$$u_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \nabla^2 T \tag{1}$$

$$\rho C_p\left(\frac{\partial T}{\partial t}+u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial \theta}+u_z\frac{\partial T}{\partial z}\right)=k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right]+$$
$$2\mu\left\{\left(\frac{\partial u_r}{\partial r}\right)^2+\left[\frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta}+u_r\right)\right]^2+\left(\frac{\partial u_z}{\partial z}\right)^2\right\}+\mu\left\{\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)^2+\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)^2\right\}+$$
$$\mu\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta}+r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right]^2 \tag{2}$$

Considering that the flow is steady, laminar and fully developed (Re $\langle$2400), and if the thermal equilibrium has already been established in the flow, then $\frac{\partial T}{\partial t}=0$. The dissipation of energy is also assumed negligible. Other physical properties are also assumed constant (not temperature dependent), including ρ, μ, *Cp*, and *k*. This assumption also implies incompressible Newtonian flow. Axisymmetry of the temperature field is assumed ($\frac{\partial T}{\partial \theta}=0$), where we are using the symbol θ for the polar angle.

By applying the above assumptions, Equation (2) can be written as follows:

$$u_z\frac{\partial T}{\partial z}=\frac{k}{\rho C_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] \tag{3}$$

Given that the flow is fully developed laminar flow (Poiseuille flow), the velocity profile has parabolic distribution across the pipe section, represented by

$$u_z=2\bar{u}\left[1-\left(\frac{r}{R}\right)^2\right] \tag{4}$$

Here $2\bar{u}$ is the maximum velocity at the centerline. Substituting this for the speed in Equation (3), we get:

$$2\bar{u}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z}=\frac{k}{\rho C_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] \tag{5}$$

The boundary conditions as seen in Figure.1 are

$$\left.\begin{array}{l} z=0\ ,\ T=T_0 \\ r=0\ ,\ \frac{\partial T}{\partial r}=0 \\ r=R\ ,\ T=T_\omega \end{array}\right\} \tag{6}$$

It is more practical to study the problem with standardized variables from 0 to 1. For this, new dimensionless variables are introduced, defined as $\theta=\frac{T_\omega-T}{T_\omega-T_0}$, $x=\frac{r}{R}$ and $y=\frac{z}{L}$. The substitution of these variables in Equation (5) gives

$$2\bar{u}\left[1-\frac{x^2R^2}{R^2}\right]\frac{(T_0-T_\omega)}{L}\frac{\partial \theta}{\partial y}=\frac{k}{\rho c_p}\left[\frac{1}{xR}\frac{(T_0-T_\omega)}{R}\frac{\partial \theta}{\partial x}+\frac{(T_0-T_\omega)}{R^2}\frac{\partial^2\theta}{\partial x^2}\right] \tag{7}$$

After simplifications, the following equation is obtained.

$$(1-x^2)\frac{\partial \theta}{\partial y}=\frac{kL}{\rho C_p 2\bar{u}R^2}\left[\frac{1}{x}\frac{\partial \theta}{\partial x}+\frac{\partial^2\theta}{\partial x^2}\right] \tag{8}$$

where the term $\frac{2\bar{u}R\rho C_p}{k}$ is the dimensionless Peclet number (*Pe*), which in fact is Reynolds number divided by Prandtl number.

This partial differential equation for steady state in dimensionless form can be written as follows:

$$(1-x^2)\frac{\partial\theta}{\partial y}=\frac{L}{RPe}\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial\theta}{\partial x}\right)\right] \tag{9}$$

The boundary conditions are transformed as follows:

$$z=0,\ \ T=T_0 \Rightarrow\ \ y=0,\ \ \theta=1,$$

$$r=0,\ \ \frac{\partial T}{\partial r}=0 \Rightarrow x=0,\ \ \frac{\partial\theta}{\partial x}=0 \Rightarrow \theta(0,y)\neq 0,$$

$$r=R,\ \ T=T_\omega \Rightarrow\ \ x=1,\ \ \theta=0 \Rightarrow \theta(1,y)=0.$$

It is proposed to apply separation of variables to solve Equation (9) with this set of boundary conditions.

3. Analytical Solution by Separation of Variables

In both analytical and numerical methods, the dependence of solutions on the parameters plays an important role, and there are always more difficulties when there are more parameters. We describe a technique that changes variables so that the new variables are “dimensionless”. This technique will simplify the equation to have fewer parameters. The Graetz problem is given by the following governing equation

$$(1-x^2)\frac{\partial\theta}{\partial y}=\frac{L}{PeR}\left[\frac{1}{x}\frac{\partial\theta}{\partial x}+\frac{\partial^2\theta}{\partial x^2}\right] \tag{10}$$

where *Pe* is the Peclet number, *L* is the tube length and *R* is the tube radius, and the initial and boundary conditions are:

IC : $y=0$, $\theta=1$

BC1 : $x=0$, $\frac{\partial\theta}{\partial x}=0$

BC2 : $x=1$, $\theta=0$

Introducing dimensionless variables as in (Huang, Matloz, Wen, & William, 1984):

$$x=\xi=\frac{r}{R} \tag{11}$$

$$\zeta=\frac{kz}{\rho c_p v_{\max} R^2} \tag{12}$$

$$y=\frac{z}{L} \tag{13}$$

On substituting Equation (13) into Equation (12) it becomes:

$$\zeta=\frac{k\,yL}{\rho c_p v_{\max} R^2} \tag{14}$$

Since $v_{\max}=2\bar{u}$

$$\zeta=\frac{k\,yL}{2\bar{u}\,\rho c_p R^2}=\frac{yL}{\frac{2\bar{u}\,\rho c_p R}{k}.R} \tag{15}$$

Notice that the term $\frac{2\bar{u}\,\rho c_p R}{k}$ in Equation (15) is similar to the Peclet number, *P*.

Thus, Equation (15) can be written as

$$\zeta = \frac{yL}{PeR} \tag{16}$$

Based on Equations (11)-(16) the derivatives transform as

$$\frac{\partial\theta}{\partial x} = \frac{\partial\theta}{\partial\xi} \tag{17}$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial\xi^2} \tag{18}$$

$$\frac{\partial\theta}{\partial y} = \frac{\partial\zeta}{\partial y}.\frac{\partial\theta}{\partial\zeta} = \frac{L}{PeR}\frac{\partial\theta}{\partial\zeta} \tag{19}$$

Now, on substituting Equations (17)-(19) into Equation (10), the governing equation becomes:

$$\frac{L}{PeR}(1-\xi^2)\frac{\partial\theta}{\partial\zeta} = \frac{L}{PeR}\left[\frac{1}{\xi}\frac{\partial\theta}{\partial\xi} + \frac{\partial^2\theta}{\partial\xi^2}\right] \tag{20}$$

This simplifies to:

$$(1-\xi^2)\frac{\partial\theta}{\partial\zeta} = \frac{1}{\xi}\frac{\partial\theta}{\partial\xi} + \frac{\partial^2\theta}{\partial\xi^2} \tag{21}$$

The right hand side can be expressed as

$$\frac{1}{\xi}\frac{\partial\theta}{\partial\xi} + \frac{\partial^2\theta}{\partial\xi^2} = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\theta}{\partial\xi}\right) \tag{22}$$

Finally, the equation that characterizes the Graetz problem has become:

$$(1-\xi^2)\frac{\partial\theta}{\partial\zeta} = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\theta}{\partial\xi}\right) \tag{23}$$

Now, the energy balance in cylindrical coordinates allows decomposing this into two ordinary differential equations. This assumes constant physical properties of the fluid, neglecting axial conduction, and steady state. The associated boundary conditions for the constant-wall-temperature case are as follows:

at entrance $\zeta = 0$, $\theta = 1$

at wall $\xi = 1$, $\theta = 0$

at center $\xi = 0$, $\frac{\partial\theta}{\partial\xi} = 0$

and the dimensionless variables are defined by:

$$\theta = \frac{T_\omega - T}{T_\omega - T_0}, \quad \xi = \frac{r}{r_1} \quad \text{and} \quad \zeta = \frac{kz}{\rho c_p v_{\max} r_1^2}$$

while for the separation of variables we try

$$\theta = Z(\zeta)R(\xi) \tag{24}$$

Finally, Equation (23) can be expressed as follows:

$$\frac{dZ}{Z} = -\beta^2 d\zeta \tag{25}$$

and

$$\xi \frac{d^2R}{d\xi^2}+\frac{dR}{d\xi}+\beta^2\xi(1-\xi^2)R=0 \tag{26}$$

where β^2 is a positive real number intrinsic to the system.

The solution of Equation (25) is

$$Z=c_1 e^{-\beta^2\zeta} \tag{27}$$

where c_1 is an arbitrary constant. In order to solve Equation (26), transformations of dependent and independent variables need to be made:

(I) $v=\beta\xi^2$

(II) $R(v)=e^{-v/2}S(v)$

Equation (26) now becomes

$$v\frac{d^2S}{dv^2}+(1-v)\frac{dS}{dv}-\left(\frac{1}{2}-\frac{\beta}{4}\right)S=0 \tag{28}$$

Equation (28) is of the confluent hypergeometric type as cited in (Slater, 1960), and it is commonly known as the Kummer equation.

3.1 Theorem of Fuchs

A homogeneous linear differential equation of the second order is given by

$$y''+P(Z)y'+Q(Z)y=0 \tag{29}$$

If P(Z) and Q(Z) have a pole at the point $Z=Z_0$, it is possible to find a solution in series form, provided that the limits $\lim_{Z\to Z_0}(Z-Z_0)P(Z)$ and $\lim_{Z\to Z_0}(Z-Z_0)^2Q(Z)$ exist.

The method of Frobenius seeks a solution in the form

$$y(Z)=Z^{\lambda}\sum_{n=0}^{\infty}a_nZ^n \tag{30}$$

where λ is an exponent to be determined. The hypergeometric functions are defined by

$$F(\alpha,\beta,Z)=1+\frac{\alpha}{\beta}Z+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{Z^2}{2!}+\cdots+\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\cdots(\theta+n-1)}\frac{Z^n}{n!}+\cdots+$$

$$\text{and } F(\alpha,\beta,\gamma) \text{ converges } \forall Z \tag{31}$$

On differentiating with respect to Z

$$\frac{d}{dZ}[F(\alpha,\beta,Z)]=\frac{\alpha}{\beta}\left[1++\frac{(\alpha+1)}{(\beta+1)}Z+\frac{(\alpha+1)(\alpha+2)}{(\beta+1)(\beta+2)}\frac{Z^2}{2!}\right.$$
$$\left.+\cdots+\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\cdots(\alpha+n-1)}\frac{Z^n}{n!}+\cdots+-\right]$$
$$=\frac{\alpha}{\beta}F(\alpha+1,\beta+1,Z) \tag{32}$$

This gives {left hand side should have derivative, not q}

$$\frac{d}{q\xi}F\left(\frac{1}{2}-\frac{\beta_n}{4},1,\beta_n\xi^2\right)=(-\beta_n{}^2)\xi\left(\frac{1}{2}-\frac{1}{\beta_n}\right)F\left(\frac{3}{2}-\frac{\beta_n}{4},2,\beta_n\xi^2\right) \tag{33}$$

Thus, a solution of Equation (26) is given by:

$$R=c_2 e^{-\beta\xi^2/2}F\left(\frac{1}{2}-\frac{\beta}{4},1,\beta\xi^2\right) \tag{34}$$

$$F\left(\frac{1}{2}-\frac{\beta_n}{4},1,\beta_n\right)=0 \tag{35}$$

where n = 1, 2, 3 ... and the eigenvalues β_n are the roots of Equation (35). These can be readily computed in MATLAB since it has a built-in hypergeometric function calculator. The solutions of our equation are the eigenfunctions of the Graetz problem. It can be shown by series expansion that these eigenfunctions are:

$$G_n(\xi)=e^{-\beta_n\xi^2/2}F\left(\frac{1}{2}-\frac{\beta_n}{4},1,\beta_n\xi^2\right) \tag{36}$$

where F is the confluent hypergeometric function or the Kummer function. These functions have power series in ξ resembling the exponential function (Abramowitz & Stegun, 1965). The functions above have symmetry properties since they are even functions. Hence the boundary condition at ξ=0 is satisfied. Since the systemislinear, the general solutionis a superposition:

$$\theta=\sum_{n=1}^{\infty}C_n e^{-\beta_n^2\zeta}.e^{-\beta_n\xi^2/2}.F\left(\frac{1}{2}-\frac{\beta_n}{4},1,\beta_n\xi^2\right) \tag{37}$$

The constants in Equation (37) can be sought using the orthogonality of Sturm-Liouville systems after the initial condition is applied

$$c_n=\frac{\left(\frac{1}{2}-\frac{1}{\beta_n}\right)e^{-\beta_n/2}.F\left(\frac{3}{2}-\frac{\beta_n}{4},2,\beta_n\right)}{\int_0^1(\xi-\xi^3)e^{-\beta_n\xi^2}.\left[F\left(\frac{1}{2}-\frac{\beta_n}{4},1,\beta_n\xi^2\right)\right]^2 d\xi} \tag{38}$$

The integral in the denominator of Equation (38) can be evaluated using numerical integration.

For the Graetz problem, it is noted that:

$$(\xi-\xi^3)\frac{\partial\theta}{\partial\zeta}=\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\theta}{\partial\xi}\right) \tag{39}$$

where $(\xi-\xi^3)$ is the function of the weight $/\beta_n$ eigenvalues

B.C $\xi=1,\theta=0$

B.C $\xi=0,\frac{\partial\theta}{\partial\xi}=0$

IC $\zeta=0,\theta=1$

$$\theta=\sum_{n=1}^{\infty}C_n e^{-\beta_n^2\zeta}G_n(\xi)=\sum_{n=1}^{\infty}C_n e^{-\beta_n^2\zeta}e^{-\beta_n\xi^2/2}F\left(\frac{1}{2}-\frac{\beta_n}{4},1,\beta_n\xi^2\right) \tag{40}$$

$G_n(\xi)=e^{-\beta_n\xi^2/2}F\left(\frac{1}{2}-\frac{\beta_n}{4},1,\beta_n\xi^2\right)$ is the weight function in the Sturm-Liouville problem.

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dG_n}{d\xi}\right)-(1-\xi^2)\beta_n^2G_n=0 \tag{41}$$

$$\frac{dG_n}{d\xi}=0 \text{ for } \xi=0 \text{ , } G_n=0 \text{ for } \xi=1 \tag{42}$$

IC $\zeta = 0, \theta = 1$

$$\theta(\zeta = 0) = 1 = \sum_{n=1}^{\infty} C_n e^{-\beta_n \xi^2/2} F\left(\frac{1}{2} - \frac{\beta_n}{4}, 1, \beta_n \xi^2\right) \tag{43}$$

Relation of orthogonality

$$\int_0^1 W(x) Y_i(x) Y_j(x) = 0, \; (i \neq j) \tag{44}$$

$$\int_0^1 (\xi - \xi^3) e^{-\beta_n \xi^2/2} F\left(\frac{1}{2} - \frac{\beta_n}{4}, 1, \beta_n \xi^2\right) e^{-\beta_m \xi^2/2} F\left(\frac{1}{2} - \frac{\beta_m}{4}, 1, \beta_m \xi^2\right) d\xi = 0 \;, (\forall n \neq m) \tag{45}$$

$$\begin{aligned} \frac{\partial \theta}{\partial \xi} &= \sum_{n=1}^{\infty} C_n e^{-\beta_n^2 \zeta} (-\beta_n \xi) e^{-\beta_n \xi^2/2} F\left(\frac{1}{2} - \frac{\beta_n}{4}, 1, \beta_n \xi^2\right) \\ &+ \sum_{n=1}^{\infty} C_n e^{-\beta_n^2 \zeta} (-\beta_n{}^2) \xi \left(\frac{1}{2} - \frac{1}{\beta_n}\right) e^{-\beta_n \xi^2/2} F\left(\frac{3}{2} - \frac{\beta_n}{4}, 2, \beta_n \xi^2\right) \end{aligned} \tag{46}$$

$$\frac{\partial \theta}{\partial \zeta} = \sum_{n=1}^{\infty} C_n e^{-\beta_n^2 \zeta} (-\beta_n{}^2) e^{-\beta_n \xi^2/2} F\left(\frac{1}{2} - \frac{\beta_n}{4}, 1, \beta_n \xi^2\right) \tag{47}$$

By considering $F\left(\frac{1}{2} - \frac{\beta_n}{4}, 1, \beta_n\right)$, Equation (39) and onwards can be rewritten as

$$\int_0^1 (\xi - \xi^3) \frac{\partial \theta}{\partial \zeta} d\xi = \int_0^1 \frac{\partial}{\partial \xi}\left(\xi \frac{\partial \theta}{\partial \xi}\right) = \xi \frac{\partial \theta}{\partial \xi}\bigg|_0^1 \tag{48}$$

$$\int_0^1 (\xi - \xi^3) \frac{\partial \theta}{\partial \zeta} d\xi = \frac{\partial \theta}{\partial \xi}\bigg|_{\xi=1} \tag{49}$$

$$\begin{aligned} &\int_0^1 (\xi - \xi^3) \left[\sum_{n=1}^{\infty} C_n e^{-\beta_n^2 \zeta} (-\beta_n{}^2) e^{-\beta_n \xi^2/2} F\left(\frac{1}{2} - \frac{\beta_n}{4}, 1, \beta_n \xi^2\right)\right] d\xi \\ &= \sum_{n=1}^{\infty} C_n e^{-\beta_n^2 \zeta} (-\beta_n{}^2) e^{-\beta_n \;/2} F\left(\frac{1}{2} - \frac{\beta_n}{4}, 1, \beta_n\right) \\ &+ \sum_{n=1}^{\infty} C_n e^{-\beta_n^2 \zeta} (-\beta_n{}^2) \left(\frac{1}{2} - \frac{1}{\beta_n}\right) e^{-\beta_n \;/2} F\left(\frac{3}{2} - \frac{\beta_n}{4}, 2, \beta_n\right) \end{aligned} \tag{50}$$

$$\begin{aligned} &\sum_{n=1}^{\infty} C_n e^{-\beta_n^2 \zeta} (-\beta_n{}^2) \int_0^1 (\xi - \xi^3) e^{-\beta_n \xi^2/2} F\left(\frac{1}{2} - \frac{\beta_n}{4}, 1, \beta_n \xi^2\right) d\xi \\ &= \sum_{n=1}^{\infty} C_n e^{-\beta_n^2 \zeta} (-\beta_n{}^2) \left(\frac{1}{2} - \frac{1}{\beta_n}\right) e^{-\beta_n \;/2} F\left(\frac{3}{2} - \frac{\beta_n}{4}, 2, \beta_n\right) \end{aligned} \tag{51}$$

On combining Equations (49), (50) and (51) this reduces to

$$\int_0^1 (\xi - \xi^3) e^{-\beta_n \xi^2/2} F\left(\frac{1}{2} - \frac{\beta_n}{4}, 1, \beta_n \xi^2\right) d\xi = \left(\frac{1}{2} - \frac{1}{\beta_n}\right) e^{-\beta_n \;/2} F\left(\frac{3}{2} - \frac{\beta_n}{4}, 2, \beta_n\right) \tag{52}$$

Let's multiply Equation (43) by Equation (53) and then integrate Equation (54),

$$(\xi - \xi^3) e^{-\beta_m \xi^2/2} F\left(\frac{1}{2} - \frac{\beta_m}{4}, 1, \beta_m \xi^2\right) \tag{53}$$

$$\begin{aligned} &\int_0^1 (\xi - \xi^3) e^{-\beta_m \xi^2/2} F\left(\frac{1}{2} - \frac{\beta_m}{4}, 1, \beta_m \xi^2\right) d\xi \\ &= \sum_{n=1}^{\infty} C_n \int_0^1 (\xi - \xi^3) e^{-\beta_m \xi^2/2} F\left(\frac{1}{2} - \frac{\beta_m}{4}, 1, \beta_m \xi^2\right). \\ &\quad e^{-\beta_n \xi^2/2} F\left(\frac{1}{2} - \frac{\beta_n}{4}, 1, \beta_n \xi^2\right) d\xi \end{aligned} \tag{54}$$

This shows that:

(i) If ($n \neq m$)the result is equal to zero (0)

(ii) If ($n = m$) the result is

$$\int_0^1 (\xi-\xi^3)e^{-\beta_n\xi^2/2}F\left(\frac{1}{2}-\frac{\beta_n}{4},1,\beta_n\xi^2\right)d\xi$$

$$= C_n\int_0^1 (\xi-\xi^3)e^{-\beta_n\xi^2}\left[F\left(\frac{1}{2}-\frac{\beta_n}{4},1,\beta_n\xi^2\right)\right]^2 d\xi \tag{55}$$

Substituting Equation (52) into Equation (55) gives

$$\left(\frac{1}{2}-\frac{1}{\beta_n}\right)e^{-\beta_n/2}F\left(\frac{3}{2}-\frac{\beta_n}{4},2,\beta_n\right) = C_n\int_0^1 (\xi-\xi^3)e^{-\beta_n\xi^2}\left[F\left(\frac{1}{2}-\frac{\beta_n}{4},1,\beta_n\xi^2\right)\right]^2 d\xi \tag{56}$$

The constants C_n are given by

$$C_n = \frac{\left(\frac{1}{2}-\frac{1}{\beta_n}\right)e^{-\beta_n/2}F\left(\frac{3}{2}-\frac{\beta_n}{4},2,\beta_n\right)}{\int_0^1 (\xi-\xi^3)e^{-\beta_n\xi^2}\left[F\left(\frac{1}{2}-\frac{\beta_n}{4},1,\beta_n\xi^2\right)\right]^2 d\xi} \tag{57}$$

4. Results and Discussion

4.1 Evaluation of the first four eigenvalues and the constant C_n

A few coefficients values of the series are given in Table 1 together with the corresponding eigenvalues. The calculated central temperature as a function of the axial coordinate ζ is also summarized in Table 2.

Table 1. Eigenvalues and constants for the Graetz's problem.

n	Eigenvalue β_n	Coefficient C_n	$G_n(\xi=0)$
1	2.7044	0.9774	1.5106
2	6.6790	0.3858	-2.0895
3	10.6733	-0.2351	-2.5045
4	14.6710	0.1674	-2.8426
5	18.6698	-0.1292	-3.1338

Table 2. The central temperature θ (ζ)

ζ	Temperature (θ)	$\theta(\zeta,0)$
0	1.0000000	1.0000000
0,05	0,93957337	1,02424798
0,1	0,70123412	0,71053981
0,15	0,49191377	0,49291463
0,25	0,23720134	0,2372129
0,5	0,03811139	0,03811139
0,75	0,0061231	0,0061231
0,8	0,00424771	0,00424771
0,85	0,00294671	0,00294671
0,9	0,00204419	0,00204419
0,95	0,00141809	0,00141809
0,96	0,00131808	0,00131808
0,97	0,00122512	0,00122512
0,98	0,00113871	0,00113871
0,99	0,0010584	0,0010584
1	0,00098376	0,00098376

The leading term in the solution for the central temperature is

$$\theta(\zeta,0) \approx 0.9774\, e^{-2.704\,\zeta} G(0) \tag{58}$$

4.2 Graphical representation of the exact solution of the Gratez problem

The central temperature profile is shown in Figure 2, obtained by truncating the series to five terms. As seen in this figure, the dimensionless temperature (θ) decreases with the dimensionless axial position (ζ). Note that the five-term series solution is not accurate for ζ<0.05.

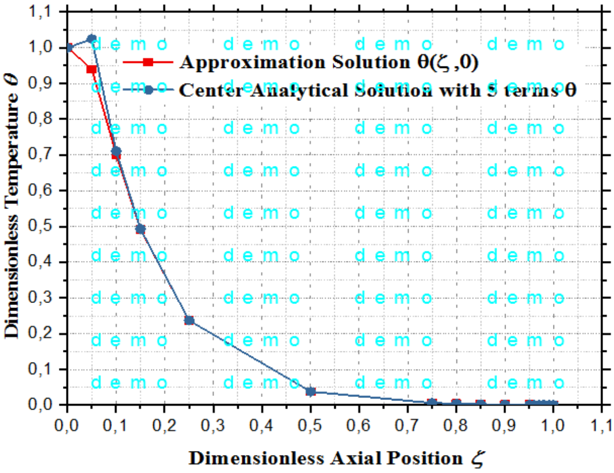

Figure 2. Axial temperature profile in the dimensionless variables temperature (θ) and axial distance (ζ)

4.3 Comparison between the analytical model and prior numerical simulation results

In order to compare with prior numerical results in Belhocine (2015) for our heat transfer problem, we chose the results of orthogonal collocation that gives the best results. Figure 3 shows the comparison with clearly good agreement between the numerical results and the analytical solution of the Graetz problem, along the centerline.

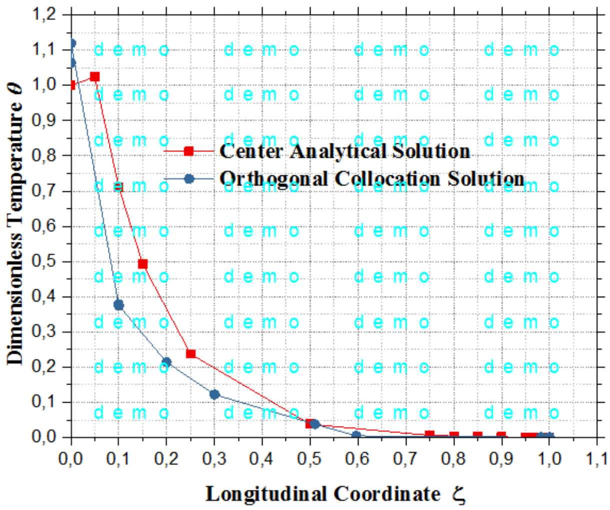

Figure 3. A comparison of the present analytical results with prior numerical results from orthogonal collocation (Belhocine, 2015)

4.4 Heat transfer coefficient correlation

The heat flux from the wall to the fluid $q_{\omega}(z)$ is a function of axial position. It can be calculated directly from

$$q_{\omega}(z) = k\frac{\partial T}{\partial r}(R,z) \tag{59}$$

but as we noted earlier, it is customary to define the heat transfer coefficient $h(z)$ via

$$q_{\omega}(z) = h(z)\,(T_{\omega} - T_b) \tag{60}$$

where the bulk or cup-mixing average temperature T_b is introduced. The way to experimentally determine the bulk average temperature is to collect the fluid coming out of the system at a given axial location, mix it completely, and measure its temperature. The mathematical definition of the bulk average temperature is

$$T_b = \frac{\int_0^R 2\pi r v(r) T(r,z)dr}{\int_0^R 2\pi r v(r)dr} \tag{61}$$

where the velocity field $v(r) = v_0(1 - r^2/R^2)$. You can see from the definition of the heat transfer coefficient that it is related to the temperature gradient at the tube wall in a simple manner:

$$h(z) = \frac{k\frac{\partial T}{\partial r}(R,z)}{(T_{\omega} - T_b)} \tag{62}$$

We can define a dimensionless heat transfer coefficient known as the Nusselt number.

$$Nu(\zeta) = \frac{2hR}{k} = -2\frac{\frac{\partial \theta}{\partial \xi}(\zeta,1)}{\theta_b(\zeta)} \tag{63}$$

where θ_b is the dimensionless bulk average temperature.

By substituting the infinite series solution for both the numerator and the denominator, the Nusselt number becomes

$$Nu(\zeta) = \frac{2hR}{k} = -2\frac{\sum_{n=1}^{\infty} C_n e^{-\beta_n^2\zeta} \frac{\partial G_n(\xi)}{\partial \xi}}{4\sum_{n=1}^{\infty} C_n e^{-\beta_n^2\zeta} \int_0^1 (\xi - \xi^3)\, G_n(\xi)\, d\xi} \tag{64}$$

The denominator can be simplified by using the governing differential equation for $G_n(\xi)$, along with the boundary conditions, to finally yield the following result.

$$Nu = \frac{\sum_{n=1}^{\infty} C_n e^{-\beta_n^2\zeta} \frac{\partial G_n(1)}{\partial \xi}}{2\sum_{n=1}^{\infty} C_n \frac{e^{-\beta_n^2\zeta}}{\beta_n^2} e^{-\beta_n^2\zeta} \frac{\partial G_n(1)}{\partial \xi}} \tag{65}$$

We can see that, for large ζ , only the first term in the infinite series in the numerator, and likewise the first term in the infinite series in the denominator are important. Therefore, as $\zeta \to \infty$, $Nu \to \frac{\beta_1^2}{2} = 3.656$. Also:

$$N_{Nu,a} = \frac{2N_{Gr}\sum_{n=1}^{\infty} C_n e^{-\beta_n/2}\left(1-\frac{2}{\beta_n}\right) F\left(\frac{3}{2}-\frac{\beta_n}{4},2,\beta_n\right).\left[1-e^{-\frac{\pi\beta_n^2}{2N_{Gr}}}\right]}{\pi\sum_{n=1}^{\infty}\frac{1}{2}+C_n e^{-\beta_n/2}\left(1-\frac{2}{\beta_n}\right) F\left(\frac{3}{2}-\frac{\beta_n}{4},2,\beta_n\right).e^{-\frac{\pi\beta_n^2}{2N_{Gr}}}} \tag{66}$$

where $N_{Gr} = \frac{\pi\zeta c_p v_{\max} R^2}{2kL}$ is the Graetz number.

Figure 4 shows the Nusselt number against the dimensionless length along the tube with uniform heat flux. As expected, the Nusselt number is very high at the beginning in the entrance region and thereafter decreases exponentially to the fully developed Nusselt number.

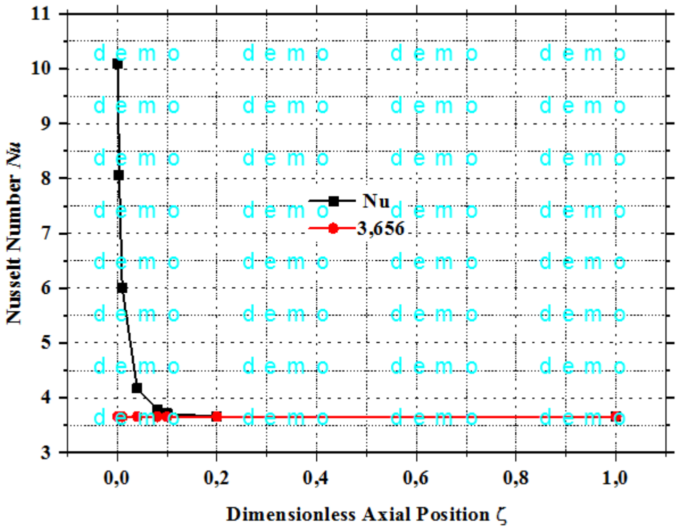

Figure 4. Nusselt number versus dimensionless axial coordinate

4. Conclusions

In this paper, an exact solution of the Graetz problem was successfully obtained by separation of variables. The hypergeometric functions were employed in order to determine the eigenvalues and constants C_n, and later on to find a solution for the Graetz problem. The mathematical approach in this study can be applied to predicting the temperature distribution in steady state laminar flow with heat transfer, based on the fully developed velocity for fluid flow through a circular tube. In future work, we may pursue Graetz solutions by separation of variables for a variety of cases, including non-Newtonian flow, turbulent flow, and other geometries besides a circular tube. It is important to note that the present analytical solutions of the Graetz problem are in good agreement with previously published numerical results of the author. It will be also interesting to compare actual experimental data with the proposed exact solution.

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Appendix

Nomenclature

a	:parameter of confluent hypergeometric function
b	: parameter of confluent hypergeometric function
c_p	:heat capacityf
C_n	: coefficient of solution defined in Equation (38)
$F\ (a;b;x)$	: standard confluent hypergeometric function
K	: thermal conductivity
L	: length of the circular tube
N_G	: Graetz number
Nu	: Nusselt number
Pe	: Peclet number
R	: radial cylindrical coordinate
r_1	: radius of the circular tube
T	: temperature of the fluid inside a circular tube
T_0	: temperature of the fluid entering the tube
T_ω	: temperature of the fluid at the wall of the tube
υ_{max}	: maximum axial velocity of the fluid

Greek letters

β_n	: eigenvalues
ζ	: dimensionless axial direction
θ	: dimensionless temperature
ξ	: dimensionless radial direction
ρ	: density of the fluid
μ	: viscosity of the fluid