SOME GRAPHS OF METABELIAN GROUPS OF ORDER 24
AND THEIR ENERGY

AMIRA FADINA BINTI AHMAD FADZIL

A dissertation submitted in partial fulfilment of the
requirements for the award of the degree of
Master of Science

Faculty of Science
Universiti Teknologi Malaysia

FEBRUARY 2017

ACKNOWLEDGEMENT

First of all, I am very grateful to the Almighty Allah, who has given me strength and passion in completing my dissertation. In completion of this work, I have been in contact with many people. They have contributed a lot towards my understanding and ideas in this research.

In particular, I wish to express my sincere appreciation to my supervisor, Prof. Dr. Nor Haniza Sarmin for her encouragement, guidance and continuous suggestions. I would also love to thank my examiners for their comments and suggestions to improve my research.

Next, I would like to thank my parents, Mrs. Suriyani Binti Sadri and Mr. Ahmad Fadzil Bin V M Mohd Abdul Kader for their abundance of love and also a lot of gratitude to my close friends, Fasihah Binti Zulkiflee and Fadhilah Binti Abu Bakar for their encouragement and emotional support throughout the semester.

Last but not least, I would like to acknowledge Ministry of Education Malaysia (MOHE) for my MyBrain15 scholarship.

ABSTRACT

The energy of a graph G is the sum of all absolute values of the eigenvalues of the adjacency matrix. An adjacency matrix is a square matrix where the rows and columns consist of 0 or 1-entry depending on the adjacency of the vertices of the graph. A commuting graph of a group is a graph whose vertex set is the non-central elements of the group and whose edges are pairs of vertices that commute. Meanwhile, a non-commuting graph is a graph whose vertex set is the non-central elements of the group but the edges are the pairs of vertices that do not commute. A conjugacy class graph is a graph with the non-central conjugacy classes vertices. Two vertices are connected if the order of the conjugacy classes have a common prime divisor. Besides, a conjugate graph is a graph whose vertex set is the non-central elements of the group where two distinct vertices are joined if they are conjugate. Furthermore, a group G is said to be metabelian if there exists a normal subgroup H in G such that both H and the factor group G/H are abelian. In this research, the energies of commuting graphs, non-commuting graphs, conjugacy class graphs and conjugate graphs for all nonabelian metabelian group of order 24 are determined. The computations of the graphs and adjacency matrices for the energy of graphs are determined with the assistance of Groups, Algorithms and Programming (GAP) and Maple 2016 softwares. The results show that the energy of graphs of the groups in the study must be an even integer in the case that the energy is rational.

ABSTRAK

Tenaga graf bagi sesuatu G adalah jumlah semua nilai mutlak nilai eigen bagi matriks bersebelahan. Satu matriks bersebelahan adalah matriks persegi di mana baris dan lajur terdiri daripada nilai 0 atau 1 bergantung kepada bucu graf. Graf komutatif bagi kumpulan adalah graf di mana set bucu adalah unsur-unsur bukan pusat kumpulan itu pasangan bucu yang komutatif dihubungkan. Sementara itu, graf tidak komutatif ialah graf di mana bucu set adalah unsur-unsur bukan pusat kumpulan itu tetapi pasangan bucu yang tidak komutatif dihubungkan. Graf kelas konjugat ialah graf dengan bucu kelas konjugat yang bukan pusat. Dua bucu disambungkan jika perintah kelas konjugat mempunyai pembahagi perdana yang sama. Selain itu, graf konjugat ialah graf dengan bucu set merupakan unsur-unsur bukan pusat kumpulan tersebut di mana dua bucu yang berbeza disambungkan jika mereka konjugat. Tambahan pula, kumpulan G dikatakan metabelian jika wujud subkumpulan H normal dalam G dan kedua-dua H dan kumpulan faktor G/H adalah abelian. Dalam kajian ini, tenaga graf komutatif, graf tidak komutatif, graf kelas konjugat dan graf konjugat untuk semua kumpulan metabelian tak abelian perintah 24 ditentukan. Pengiraan bagi graf dan matriks bersebelahan untuk tenaga graf ditentukan dengan bantuan perisian Kumpulan, Algoritma dan Pengaturcaraan (GAP) dan Maple 2016. Hasil kajian telah menunjukkan bahawa tenaga bagi graf kesemua kumpulan dalam skop kajian ini mestilah genap untuk kes tenaga tersebut adalah rasional.

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LIST OF SYMBOLS

a_{ij}	-	(i, j)-entry of matrix
$H \lhd G$	-	H is a normal subgroup of G
$A(G)$	-	Adjacency matrix of G
$Z(G)$	-	Center of a Group G
$[a, b]$	-	Commutator of a and b
$[G, G]$	-	Commutator subgroup
G'	-	Commutator subgroup of G
Γ_G^{Comm}	-	Commuting graph of G
K_n	-	Complete graph
$\overline{G}$	-	Complement of a graph
$cl(a)$	-	Conjugacy class of the element a
Γ_G^{CC}	-	Conjugacy class graph of G
Γ_G^{Conj}	-	Conjugate graph of G
aH	-	Coset of H
$\langle a \rangle$	-	Cyclic subgroup generated by a
$d(v)$	-	Degree of vertex
$E(G)$	-	Edge-set of G
λ_n	-	Eigenvalues
$\varepsilon(\Gamma_G)$	-	Energy of graph
G/H	-	Factor group
e	-	Identity
$\|G : H\|$	-	Index of H in G
$H \cap N$	-	Intersection of N and H

Γ_G^{NC}	-	Non-commuting graph of G
N_n	-	Null graph
$\|G\|$	-	Order of group G
$N \rtimes H$	-	Semi direct product of N and H
$V(G)$	-	Vertex-set of G

CHAPTER 1

INTRODUCTION

1.1 Introduction

A group G is called metabelian if there exists a normal subgroup H in G such that both H and the factor group G/H are abelian. A metabelian group is also referred as a group that is close to being abelian by checking its commutator subgroup. Some properties of metabelian groups are that every abelian group is metabelian and every subgroup of a metabelian group is also metabelian. A group is also considered metabelian if and only if its commutator subgroup of its commutator subgroup is equal to one [1].

A graph of a group can be formed by locating the elements of the group as the vertices. First, let G be a group and let Γ_G be a graph of the group G with a set of vertices $V(\Gamma_G) = \{1, \ldots, n\}$ and a set of edges $E(\Gamma_G) = \{e_1, \ldots, e_m\}$. Then, from the graph built, an adjacency matrix of Γ_G denoted by $A(\Gamma_G)$ or simply as A which is the $n \times n$ matrix can be constructed. The adjacency matrix consists of the rows and columns indexed by $V(\Gamma_G)$ with the conditions for the entries of A, if $i \neq j$, then the entries a_{ij} and a_{ji} is 0 for nonadjacent vertices and is 1 for adjacent vertices. Meanwhile, the diagonal entries a_{ii} of A is 0 for $i = 1, \ldots, n$ [2].

Based on the adjacency matrix, the sum of all the eigenvalues can be determined to find the energy of the graph Γ_G of the group G which are denoted by $\varepsilon(\Gamma_G)$. There are a few properties of energy of a graph. For instances, the energy of any graph can never be an odd integer but an even integer given if the energy is rational [2]. This research aimed to compute the energy of some graphs for some metabelian groups of order 24.

1.2 Research Background

This study is motivated from the research done by Abd Rahman and Sarmin [3] in 2012 on the determination of metabelian groups of order at most 24. There are few researchers that have extended the use of metabelian groups such as are Che Mohd [4], Abd Halim [5], Hassan [6] and Sarmin *et al.* [7] but none of them did any research on the graphs of the metabelian groups let alone on the energy of graphs.

Besides, the studies on commuting graphs, non-commuting graphs, conjugacy class graphs and conjugate graphs are usually done in a general scope. For example, Segev [8], Segev *et al.* [9], Iranmanesh *et al.* [10], Parker [11] and Raza *et al.* [12] have studied the commuting graphs of certain groups while Abdollahi *et al.* [13], Darafsheh [14], Abdollahi and Shahverdi [15], Ahanjideh *et al.* [16] and Moradipour *et al.* [17] studied the non-commuting graphs of certain groups. The study on a graph related to conjugacy classes of groups has been done by Bertram *et al.* [18]. Erfanian and Tolue [19] also have done some research but on conjugate graphs of finite groups.

The study on the energy of general simple graphs was first defined by Gutman in 1978 in motivation from the Hückel Molecular Orbital Theory proposed in 1930s by Hückel [20]. Study on energy graphs then continued by many such as Zhou [21], Balakrishnan [22], Bapat and Pati [23], Yu *et al.* [24] and many others. Although there are many researches on graph energy in the area, it is noticeable that there is none of them that have studied specifically on the energy graphs of metabelian groups. Hence, it encourages this research study.

1.3 Problem Statement

There are no research done specifically on the energy of graphs of groups, in particular on metabelian groups. Therefore, this study is to determine the energy of some graphs of metabelian groups of order 24.

1.4 Research Questions

The research questions are as the following:

1. What are the energy of the commuting graphs of some metabelian groups of order 24?
2. What are the energy of the non-commuting graphs of some metabelian groups of order 24?
3. What are the energy of the conjugacy class graphs of some metabelian groups of order 24?
4. What are the energy of the conjugate graphs of some metabelian groups of order 24?

1.5 Research Objectives

The objectives of this study are as the following:

1. To find the energy of the commuting graphs of some metabelian groups of order 24.

2. To determine the energy of the non-commuting graphs of some metabelian groups of order 24.
3. To compute the energy of the conjugacy class graphs of some metabelian groups of order 24.
4. To obtain the energy of the conjugate graphs of some metabelian groups of order 24.

1.6 Scope of the Study

This study aimed to obtain the energy of some graphs of some metabelian groups. The groups considered in this study are all nonabelian metabelian groups of order 24 given in [3]. There are a total of ten of them. Furthermore, four types of graphs are focused in this study namely commuting graphs, non-commuting graphs, conjugacy class graphs and conjugate graphs.

1.7 Significance of Findings

This study contributed in finding the energy of graphs of metabelian groups. The results on the energy of graphs can be applied to the molecular graphs in chemistry. Besides, the commuting graphs, non-commuting graphs, conjugacy class graphs and conjugate graphs found can be a great use for other researchers who studied graphs. This research also contributed as some applications to the metabelian groups.

1.8 Research Methodology

This research begins with the review on some previous works that have been done on metabelian groups, some graphs related to finite groups and energy of graphs. Then, the works started by listing the nonabelian metabelian groups of order 24 which have been determined by Abd Rahman and Sarmin in 2012 [3]. Firstly, the group presentations given by Abd Rahman and Sarmin in [3] are verified using Groups, Algorithms, and Programming (GAP) software [25] as in Appendix B to make sure that the presentations are correct. The verified group presentations are then implemented into the Maple 2016 software [26] to generate their 24×24 Cayley table. There are two types of Cayley table generated by Maple 2016 software which are the commuting Cayley table which differentiate the colours of the elements according to their commutativity and the conjugacy Cayley table which differentiate the colours of the elements according to their conjugacy classes. The graphs needed in this research are the commuting graphs, non-commuting graphs, conjugacy class graphs and conjugate graphs. In order to find each graphs, the elements are analyzed from the related Cayley table. All graphs are found by using the definitions defined by previous works. Firstly, for the commuting graphs, each non-central elements are put as the vertices and the commuting elements are connected with an edge. The same procedure is done for the non-commuting graphs but the edges are the connected non-commuting elements. Next, for the conjugacy class graphs, the conjugacy class of each elements are listed down together with their order. The vertices are now the non-central conjugacy classes and they are connected if the order of the vertices have common divisor. Since the conjugacy classes are listed down, the next step are to build the conjugate graphs where the non-central distinct vertices are joined by an edge if they are conjugate. Furthermore, in order to calculate the energy of the graphs, the steps needed are to arrange the adjacency matrices for each graphs and to find their eigenvalues from their characteristic polynomials. So, according to the definition of energy of graphs which are the summation of the positive values of the eigenvalues, the energy of each graph is obtained. Some of the research works have been assisted by Maple 2016 software. The research methodology are summarized in the Figure 1.1.

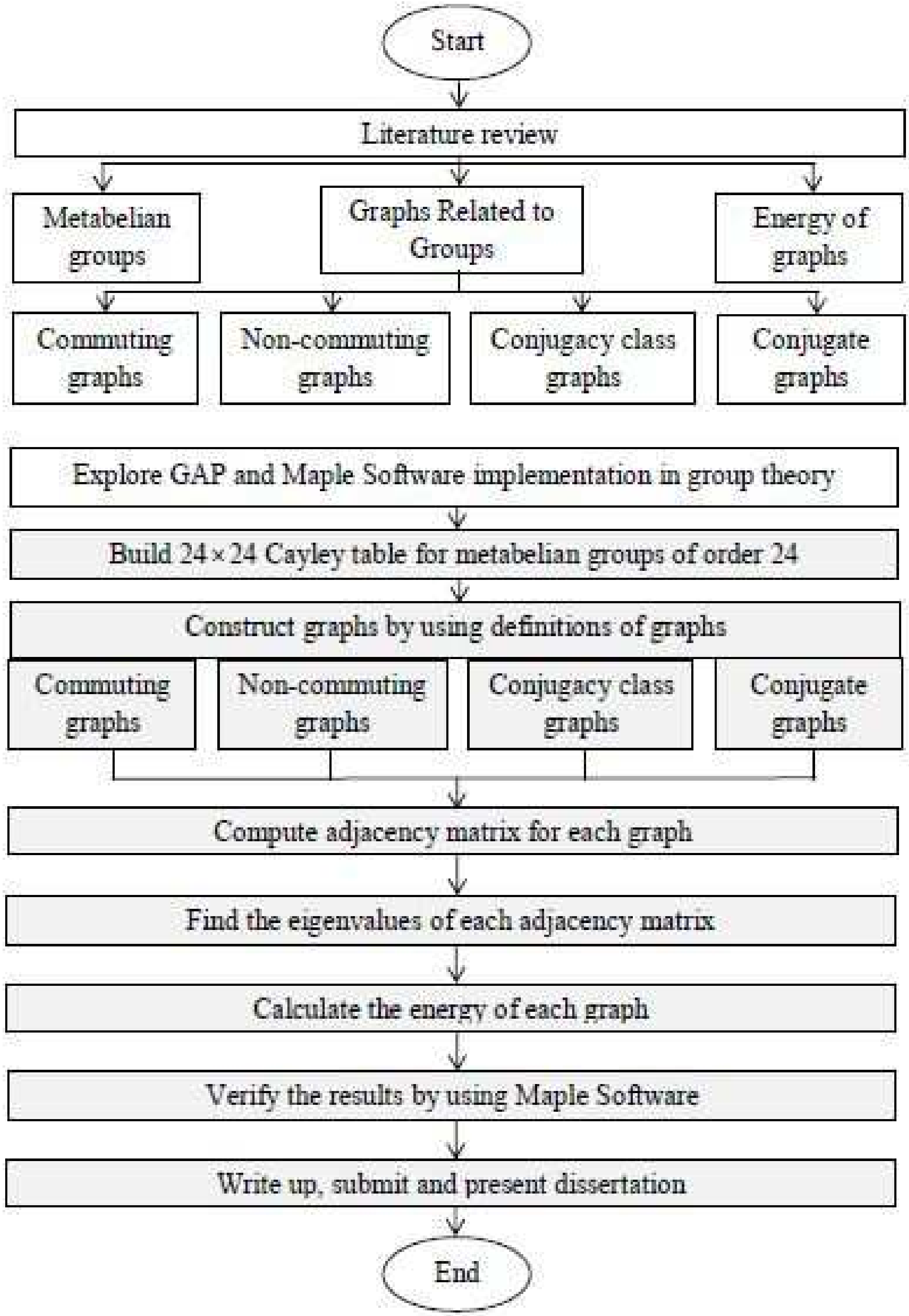

Figure 1.1 Research Methodology

1.9 Thesis Organization

This thesis is organized into seven chapters. First, Chapter 1 gives a brief overview on metabelian groups and energy of graphs. Statements of the problem have been stated and the research questions have been pointed out. The objectives, the scope, the significance and the methodology of the research have also been specified.

In Chapter 2, the literature review of the study is discussed. This chapter focuses on introducing some basic concepts and properties on group, graph and linear algebra that were used throughout the study. Some definitions and theorems that are related on metabelian groups are also included. In addition, this chapter also take account of some previous works that have been done associated to the topic.

Next, Chapter 3, 4, 5 and 6 focused on finding the commuting graphs, non-commuting graphs, conjugacy class graphs and conjugate graphs of the metabelian groups and the computations of the energy of the graphs respectively. The energy of the graphs are done by finding their adjacency matrices, characteristic polynomials and eigenvalues for each graphs.

The last chapter gives summary and conclusion of the whole research. Some recommendations for future research which are related to metabelian groups and energy graphs are also included. The thesis organization are summarized in the Figure 1.2.

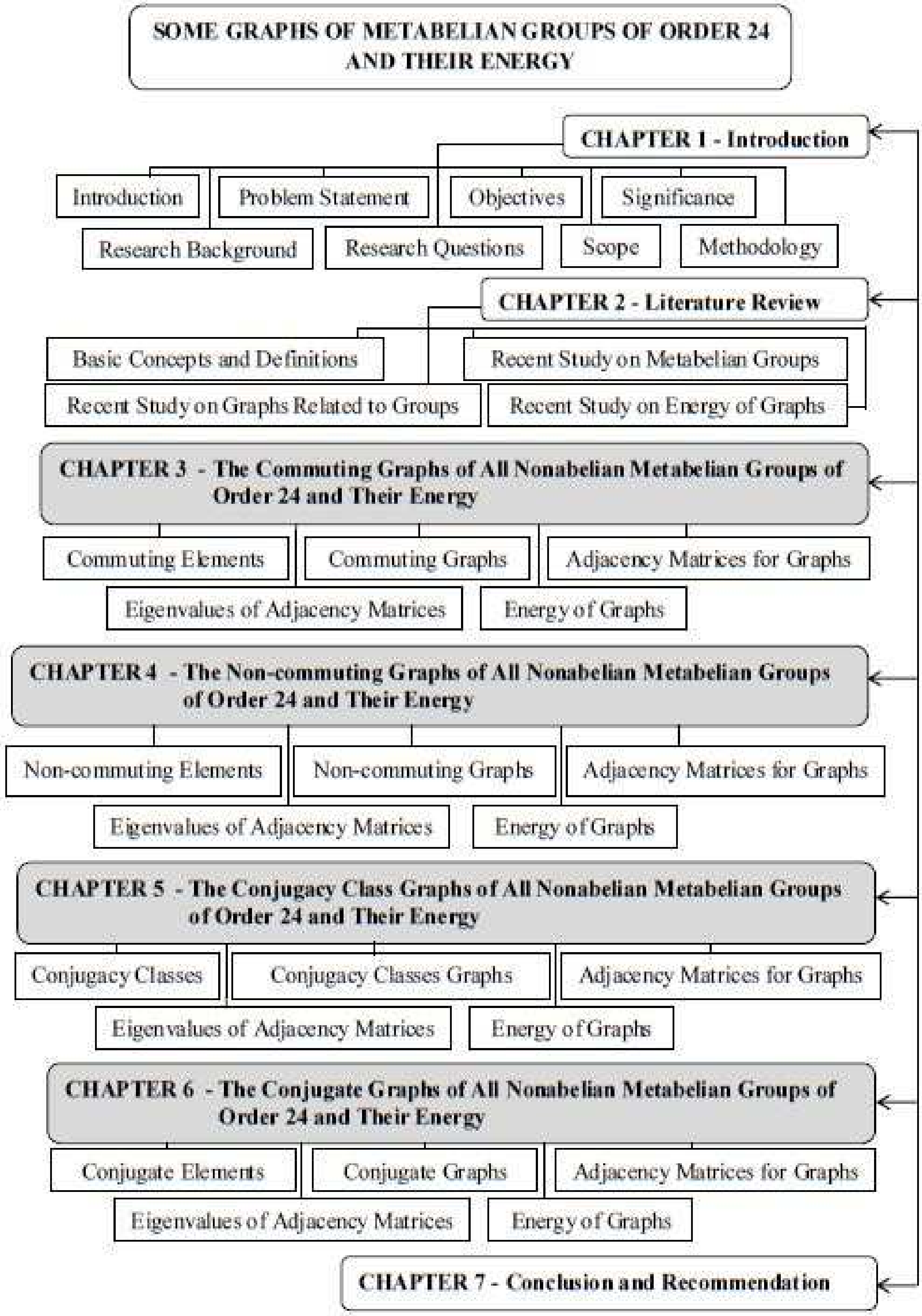

Figure 1.2 Thesis organization

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