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Research Paper

Numerical solution of hybrid method for third grade flow due to variable accelerated plate in a rotating frame

Shafaruniza Mahadi[1]*, Zainal Abdul Aziz[2], Yeak Su Hoe[3], Faisal Salah[4], Farah Suraya Md Nasrudin[1]

[1]Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 85000 Segamat, Johor, Malaysia
[2]UTM Centre for Industrial and Applied Mathematics (UTM-CIAM), Universiti Teknologi Malaysia, 81310 Johor Bahru, Johor, Malaysia
[3]Mathematics Department, Faculty of Science, Universiti Teknologi Malaysia, 81310 Johor Bahru, Johor, Malaysia
[4]Mathematics Department, Faculty of Science, University of Kordofan, Elobiad, Sudan
**Corresponding author E-mail: shafa88@gmail.com*

Abstract

The aim of this article is to obtain numerical solution for incompressible unsteady flow for third grade fluid induced by variable accelerated plate. Numerical solution is obtained by using Hybrid method which combine between finite difference method (FDM) and asymptotic interpolation method. The influence of difference values of material constant parameters on the velocity flow fluid are discussed and shown graphically.

Keywords: *Asymptotic interpolation method; Finite difference method; Rotating frame; Third grade fluid; Variable accelerated.*

1. Introduction

Fluid can be defined as a substance which includes liquids, gases or plasma and it flows under an applied shear stress (strain rate). The study of the physics of continuous materials is discussed under fluid mechanics where the material will deform when a force is applied onto the material.

There are two types of fluids which are Newtonian fluids and non-Newtonian fluids. However, the discussion is focused on non-Newtonian fluid since the fluid shows vitally important roles in engineering and industry fields. Non-Newtonian fluid can be seen for example in petroleum industry such as heavy crude oil (liquid petroleum with an API gravity less than 20°), food stuff like ketchup and also personal care products like toothpaste in which the motion shows viscoelastic behavior.

Non-Newtonian fluid can be described as a fluid whose viscosity changes rely on slope in flow speed or stress. It is also not proportional to the rate of strain, its higher power and derivatives. The fluid also depends on the kinematics history of its element. These kinds of viscoelastic fluids are accountable for the influences of normal stress which show fluid elasticity. The physical behaviour of non-Newtonian fluid is very complex to describe. Therefore, previous researchers have proposed and developed some models and constitutive equations in order to study and investigate all the properties of fluid as shown in references.

The simplest subclass of non-Newtonian fluid which can only explain the normal stress differences is called second grade fluid. However, the third grade fluid and fourth grade fluid are able to foresee the properties of shear (thinning and thickening) because the fluid attempts to have such characteristics of viscoelastic fluids. Furthermore, the governing equations for the third grade fluid and fourth grade fluid are complicated compared to second grade fluid and thus give additional challenges to the researchers.

This study highlights the parameter of rotating frame. Rotating frame can be defined as rotating relative to an inertial reference frame. The fluid which has rotating in flows can be seen for instance in cosmical process and geophysical fluid dynamics [10]. Recently, there are few research which focused on rotating frame for example solving the problem of MHD rotating flow of fourth grade between two parallel infinite plates [9], solving a constant accelerated flow for a third grade fluid in a rotating frame analytically by using Homotopy Analysis Method [2, 8] and also solving nonlinear Stokes' first problem for the rotating flow of a third grade fluid numerically by using Newton's Method [10].

In this paper, the implicit finite difference method (FDM) will be applied to the nonlinear equations. FDM is the method which was initiated in the early 1950s. This method is very useful to obtain the approximate solution and discretize the problem of nonlinear equation such as in [3-4, 9, 12-14].

The concept of convergence gives an idea to the researchers to introduce an asymptotic interpolation method. In this paper, the study implements an asymptotic type of interpolation technique to handle the infinite size of problem length $z \to \infty$. This asymptotic interpolation method resorts to actual length by introducing special asymptotic function for interpolation. This idea had been applied in solving engineering problem such as in mechanical and chemical hydrodynamics [1]. An asymptotic interpolation method has been proposed to examine the stress intensity factor (SIF) for cracks at the notch root [6]. The modification of the asymptotic interpolation method uses the known of two-sided asymptotic of the solution of the problem and the values of the function chosen are in the middle of region. The asymptotic interpolation method also has been used in chemical hydrodynamics and mass transfer problems which have no exact analytical solution presented. In their research, the special switching functions are suggested for joining the asymptotic solutions [15].

The study aspires to review the problem of unsteady third grade fluids due to variable accelerated flow in a rotating frame. A new hybrid method which combines between FDM and asymptotic interpolation method will be proposed and highlighted in order to obtain the numerical solution for the fluid flow problems.

2. Mathematical formulation

The velocity field is introduced as $\mathbf{V}=[u(z,t),v(z,t),w(z,t)]$ where the velocity exists in fluid with and are the components that refer to x-axis, y-axis and z-axis respectively. Most of the fluid flow problem mathematically can be described by the continuity equation and momentum equation. Continuity equation is developed by applying the law of conservation of mass to a small volume element within a flowing fluid. Law of conservation gives a formula of rate of mass accumulation is equal to the difference of rate of mass in and rate of mass out. Continuity equation is given as $\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$ with ρ is density. If the fluid density is constant then the continuity equation reduces to $\nabla\cdot\mathbf{V}=0$.

Momentum is defined as mass in motion or mass $\times$ velocity. Momentum equation can be defined from a statement of Newton's second law. Newton's second law stated that the related rate of momentum of a body is equal to the resultant force acting on the body, and takes place in the direction of the force. It also relates to the sum of the forces acting on an element of fluid to its acceleration or rate of change of momentum. Momentum equation involves density of the fluid ρ, material derivatives $\frac{\partial}{\partial t}$, velocity $\mathbf{V}$, body forces ρb, surface forces $div\mathbf{T}$ and Cauchy stress tensor $\mathbf{T}$. Therefore, the momentum equation is given by $\rho\frac{\partial\mathbf{V}}{\partial t}=\rho b+div\mathbf{T}$.

The incompressible fluid of differential type of grade (n) as the simple fluid obeying the constitutive equation where is pressure and is an identity tensor (or can be denoted as the indetermined part of the stress due to the constraint of incompressibility) [3, 7, 9, 11]. The first four tensors $\mathbf{S}_j$ are given by

$$\mathbf{S}_1=\mu\mathbf{A}_1 \tag{1}$$

$$\mathbf{S}_2=\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2 \tag{2}$$

$$\mathbf{S}_3=\beta_1\mathbf{A}_3+\beta_2(\mathbf{A}_2\mathbf{A}_1+\mathbf{A}_1\mathbf{A}_2)+\beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1 \tag{3}$$

$$\begin{aligned}\mathbf{S}_4&=\gamma_1\mathbf{A}_4+\gamma_2(\mathbf{A}_3\mathbf{A}_1+\mathbf{A}_1\mathbf{A}_3)+\gamma_3\mathbf{A}_2^2\\&+\gamma_4\left(\mathbf{A}_2\mathbf{A}_1^2+\mathbf{A}_1^2\mathbf{A}_2\right)+\gamma_5(tr\mathbf{A}_2)\mathbf{A}_2\\&+\gamma_6\left(tr\mathbf{A}_2\mathbf{A}_1^2\right)+\{\gamma_7(tr\mathbf{A}_3)+\gamma_8(tr\mathbf{A}_2\mathbf{A}_1)\}\mathbf{A}_1\end{aligned} \tag{4}$$

where μ is the coefficient of shear viscosity, $\alpha_i(i=1,2),\ \beta_j(j=1,2,3),\ \gamma_k(k=1,2,3,4,...,8)$ are material constants which α_1 and α_2 are the normal stress moduli. $\mathbf{A}_n$ is the Rivlin-Ericson tensors which is defined by the recursion relation [5]

$$\mathbf{A}_n=\frac{d}{dt}(\mathbf{A}_{n-1})+\mathbf{A}_{n-1}(grad\mathbf{V})+(grad\mathbf{V})^T\mathbf{A}_{n-1},\ n>1 \tag{5}$$

$\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematical tensors which are given by

$$\mathbf{A}_1=(grad\mathbf{V})+(grad\mathbf{V})^T \tag{6}$$

and

$$\mathbf{A}_2=\frac{d\mathbf{A}_1}{dt}+(grad\mathbf{V})^T\mathbf{A}_1+\mathbf{A}_1(grad\mathbf{V}) \tag{7}$$

In physics, angular velocity is defined as the related rate of angular displacement and it is a vector quantity which specifies the rotational speed of an object and the axis about which the object is rotating.

3. Governing equations

The problem from previous research is reviewed [2, 8]. The continuity equation is given by $div\mathbf{V}=0$ and momentum equation is given by

$$\rho\left(\frac{\partial\mathbf{V}}{\partial t}+2\mathbf{\Omega}\times\mathbf{V}+\mathbf{\Omega}\times(\mathbf{\Omega}\times r)\right)=-\nabla p+div\mathbf{T} \tag{8}$$

where $\mathbf{V}=(u(z,t),v(z,t),0)$ is a velocity field, ρ is the fluid density, p is the hydrostatic pressure, $\mathbf{T}$ is stress tensor, Ω is the angular velocity and r is radial coordinate $r=\sqrt{x^2+y^2}$ [2, 8, 10]. The constitutive equation for third grade fluid is

$$\begin{aligned}T&=-pI+\mu A_1+\alpha_1A_2+\alpha_2A_1^2+\beta_1A_3\\&+\beta_2(A_2A_1+A_1A_2)+\beta_3\left(trA_1^2\right)A_1\\&=-pI+\left[\mu+\beta_3\left(trA_1^2\right)\right]A_1+\alpha_1A_2+\alpha_2A_1^2\end{aligned} \tag{9}$$

where μ represents the co-efficient of shear viscosity and $\alpha_i(i=1,2),\ \beta_j(j=1,2,3)$ are material constants for second grade and third grade fluids. The nonlinear equation are given by

$$\rho\left[\frac{\partial u}{\partial t}-2\Omega v\right]=\mu\frac{\partial^2u}{\partial z^2}+\alpha_1\frac{\partial^3u}{\partial z^2\partial t}+2\beta_3\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)\right) \tag{10}$$

and

$$\rho\left[\frac{\partial v}{\partial t}+2\Omega u\right]=\mu\frac{\partial^2v}{\partial z^2}+\alpha_1\frac{\partial^3v}{\partial z^2\partial t}+2\beta_3\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right)\right) \tag{11}$$

Following [8], the initial and boundary conditions corresponding to variable accelerated plate are

$u(z,0)=0$, $v(z,0)=0$ for $z>0$

$u(0,t)=Bt^2$, $v(0,t)=0$ for $t>0$

$u(z,t)\to0$, $v(z,t)\to0$ as $z\to\infty$ for every t

From both equations, multiplied equation (11) with i (imaginary, $i=\sqrt{-1}$). Then, adding it with first equation (10). The nonlinear equations become

$$\frac{\partial F}{\partial t}+2i\Omega F=\upsilon\frac{\partial^2F}{\partial z^2}+\frac{\alpha_1}{\rho}\frac{\partial^3F}{\partial z^2\partial t}+\frac{2\beta_3}{\rho}\frac{\partial}{\partial z}\left(\frac{\partial F}{\partial z}\right)^2\left(\frac{\partial\overline{F}}{\partial z}\right) \tag{12}$$

where $F=u+iv$, $\overline{F}=u-iv$. The initial and boundary conditions now are $F(z,0)=0$, $F(0,t)=Bt^2$, $F(z,t)\to0$ as $z\to\infty$. Introducing the non-dimensional parameters

$$g=\frac{F}{\left(v^2B\right)^{\frac{1}{5}}},\xi=z\left(\frac{B}{v^3}\right)^{\frac{1}{5}},\tau=t\left(\frac{B^2}{v}\right)^{\frac{1}{5}},\omega=\Omega\left(\frac{v^2}{B^2}\right)^{\frac{1}{5}}$$

where g represents the function, ξ is length, τ is time and ω represents rotation. Therefore, the equation become

$$\frac{\partial g}{\partial \tau}+2i\omega g=\frac{\partial^2 g}{\partial \xi^2}+\alpha_2\frac{\partial^3 G}{\partial \xi^2 \partial \tau}+2\beta_2\frac{\partial}{\partial \xi}\left(\left(\frac{\partial g}{\partial \xi}\right)^2\left(\frac{\partial \bar{g}}{\partial \xi}\right)\right) \quad (13)$$

with $g(\xi,0)=0$, $g(0,\tau)=B\tau^2$, $f(\xi,\tau)\to 0$ as $\xi\to\infty$ and $a=\frac{\alpha_1}{\rho}\left(\frac{B^2}{v^6}\right)^{\frac{1}{5}}, b=\frac{\beta_3}{\rho}\left(\frac{B^4}{v^7}\right)^{\frac{1}{5}}$.

4. Finite difference method and asymptotic interpolation method

The nonlinear equation is discretized by using FDM. Rearrange the equation until it becomes $n+1$ on the left side and n on the right side as follow,

$$\begin{aligned}&\left(\frac{a}{2h^2k}\right)g_{i-1}^{n+1}+\left(-\frac{2a}{2h^2k}-\frac{1}{k}\right)g_i^{n+1}+\left(\frac{a}{2h^2k}\right)g_{i+1}^{n+1}\\&=-\frac{1}{k}g_i^n+2i\omega_1 g_i^n-\frac{1}{h^2}\left(g_{i+1}^n-2g_i^n+g_{i-1}^n\right)\\&-\frac{b}{h^4}\left(g_{i+1}^n-2g_i^n+g_{i-1}^n\right)\left(g_{i+1}^n+g_{i-1}^n\right)\left(\bar{g}_{i+1}^n+\bar{g}_{i-1}^n\right)\\&-\frac{b}{2h^4}\left(g_{i+1}^n+g_{i-1}^n\right)^2\left(\bar{g}_{i+1}^n-2\bar{g}_i^n+\bar{g}_{i-1}^n\right)\\&+\frac{a}{2h^2k}\left(g_{i+1}^{n-1}-2g_i^{n-1}+g_{i-1}^{n-1}\right)\end{aligned} \quad (14)$$

with $g_{i,1}=0$, $g_{1,j}=B(j)^2$, $g_{i,j}\to 0$ as $i\to\infty$.

The equation (18) will be solved by using the matrix inversion. Due to the infinite problem length, the asymptotic type interpolation technique is introduced. The method resorts to actual length by introducing a special asymptotic function for interpolation. To achieve the convergence property, the interpolation technique will use the function $y=m+ne^{-q^2L}$ where m is an asymptote line as show in the concept of limit,

$$\begin{aligned}\lim_{L\to\infty} y&=\lim_{L\to\infty}\left(m+ne^{-q^2L}\right)\\&=\lim_{L\to\infty}\left(m+\frac{n}{e^{q^2L}}\right)\\&=m+\frac{n}{e^{q^2(\infty)}}\\&=m+0=m.\end{aligned} \quad (15)$$

This method wills use difference of length to fulfill the boundary condition which tends to infinity, for example $L=6,12,18$.

5. Results and discussion

The results by using hybrid method are compared well with the previous research [8] as follow.

Table 1: Comparison of the velocity profile between the present method and HAM

η	$a=0.5, b=0, \omega=1, t=1$		
	HAM	Present	Difference
0.00	1.0000	1.0000	0
0.25	0.7788	0.7446	0.0342
0.50	0.6065	0.5751	0.0314
0.75	0.4724	0.4440	0.0284
1.00	0.3679	0.3425	0.0254
1.25	0.2865	0.2640	0.0225
1.50	0.2231	0.2034	0.0197
1.75	0.1738	0.1567	0.0171
2.00	0.1353	0.1206	0.0147

Figure 1 and Table 1 show the comparison results between present study by using numerical method which is hybrid method (finite difference method and asymptotic interpolation method) and previous study by using analytical method, HAM. From Table 1, it shows that the difference points are less than 0.05 and it is clearly observed that this new hybrid method has the ability to give the accurate result as presented in Figure 1.

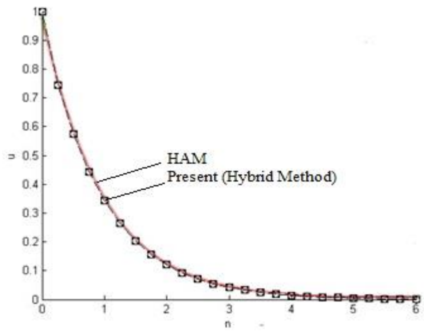

Fig. 1: The comparison of numerical solution (Hybrid method) and HAM solution at $a=0.5, b=0, \omega=\tau=1$

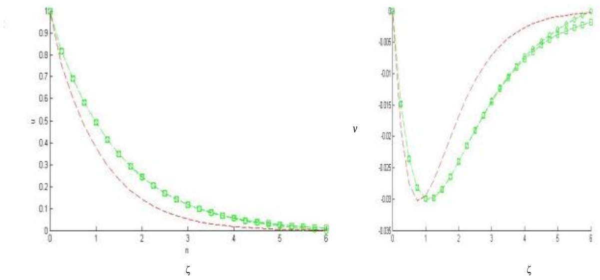

Fig. 2: Effect of difference values of a $(a=0.5(red), a=1(green))$ on the velocity profile with fixed $b=0, \omega=\tau=1$.

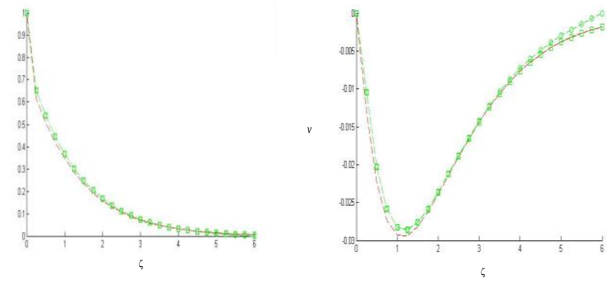

Fig. 3: Effect of difference values of b $(b=1(red), b=3(green))$ on the velocity profile with fixed $a=\omega=\tau=1$

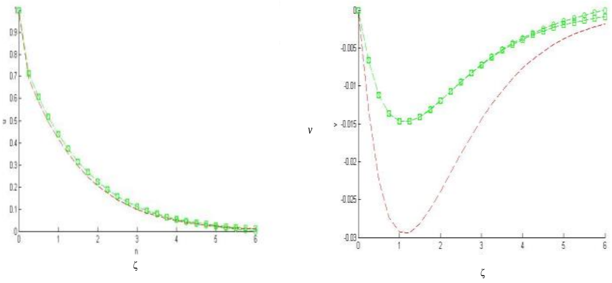

Fig. 4: Effect of difference values of ω $(\omega = 0.5(green), \omega = 1(red))$ on the velocity profile with fixed $a = b = \tau = 1$

Figure 2 until Figure 4 show the real and imaginary parts of velocity profiles with varying the values of material constants of second grade parameter (a), third grade parameter (b) and angular velocity (ω). Figure 2 illustrates that by increasing the second grade parameter, the velocity profile increases in the real part while decreases in the imaginary part. Figure 3 shows the velocity profile increases in both real and imaginary parts by increasing the value of third grade parameter while Figure 4 presents the decrease of velocity profile for both real and imaginary part. The results show that the different values of second grade parameter, third grade parameter and angular rotation will give different effect to the velocity of fluid flow.

6. Conclusion

The governing equation of non-Newtonian third grade fluid in rotating frame with variable acceleration is studied. The hybrid method of FDM and asymptotic interpolation technique is introduced to obtain the approximate numerical solution. The validation of hybrid method is shown in Table 1. Moreover, the different values of material constants are tested and the results are plotted as in Figure 1 until Figure 4. Based on the results, this hybrid method can be applied to solve the problem regarding non-linear equation in fluid flow.

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