A BOUNDARY INTEGRAL EQUATION FOR THE 2D EXTERNAL POTENTIAL FLOW

M. NASSER*, A. MURID and N. AMIN
Department of Mathematics, University of Technology
81310 UTM Skudai, Johor, MALAYSIA
e-mail: mms_nasser@hotmail.com

Based on the recently discovered second kind Fredholm integral equation for the exterior Riemann problem, a boundary integral equation is developed in this paper for the two-dimensional, irrotational, incompressible fluid flow around an airfoil without a cusped trailing edge. The solution of the integral equation contains one arbitrary real constant, which may be determined by imposing the Kutta-Joukowski condition. Comparisons between numerical and analytical values of the pressure coefficient on the surface of the NACA 0009 and NACA 0012 airfoils with zero angle of attack show a very good agreement.

Key words: boundary integral equation, 2D potential flow, Riemann problem, Kutta-Joukowski condition, Nyström method, least square methods.

1. Introduction

The boundary integral equation method (also called boundary element method, panel method) is a very economical method from the computational point of view for investigating the potential flow past airfoils. According to Carabineanu (1996), the work of Hess and Smith (1967) may be considered as the starting point of this method. The method becomes one of the most frequently used numerical methods for calculating 2D and 3D potential flow (see e.g., Katz and Plotkin (2001)). Various integral equations for studying the external potential flow problem have been discussed in Hess (1975), and recently in Carabineanu (1996) and Hwang (2000). Mokry (1990, 1996) has formulated the external potential flow problem as an exterior Riemann problem. Murid and Nasser (2003) have formulated a new boundary integral equation for the exterior Riemann problem. Based on this formulation, a boundary integral equation for the exterior potential flow problem around an airfoil will be formulated in this paper. The extension of this integral equation to multi-element airfoils is not difficult. In this paper, we shall consider only the two-dimensional, steady-state, irrotantial flow around an airfoil, and we shall assume that the fluid is incompressible and free from viscosity.

Another important method for computing the external potential flow in 2D is the conformal mapping method which consists in transforming the outer-airfoil region (physical domain) onto the exterior of a standard domain (usually the unit circle) for which an analytic form of the solution of the external potential flow problem is known (Abbott and von Doenhoff, 1959; p.47). Since conformal maps cannot be obtained in a closed form, in general, we have to resort to numerical approximations of such maps which is just as difficult as solving the original external potential flow problem.

2. External potential flow as a Riemann problem

Suppose the airfoil is represented by a simple close counterclockwise oriented curve C (see Fig.1). The interior and exterior of C will be denoted by Ω^+ and Ω^-, respectively. If a given function $f(z)$ is

---

* To whom correspondence should be addressed

defined in a domain containing C, then the limits of the function $f(z)$ when the point z moves to a point $t \in C$ from inside or outside of C will be denoted by $f^{+}(t)$ and $f^{-}(t)$, respectively.

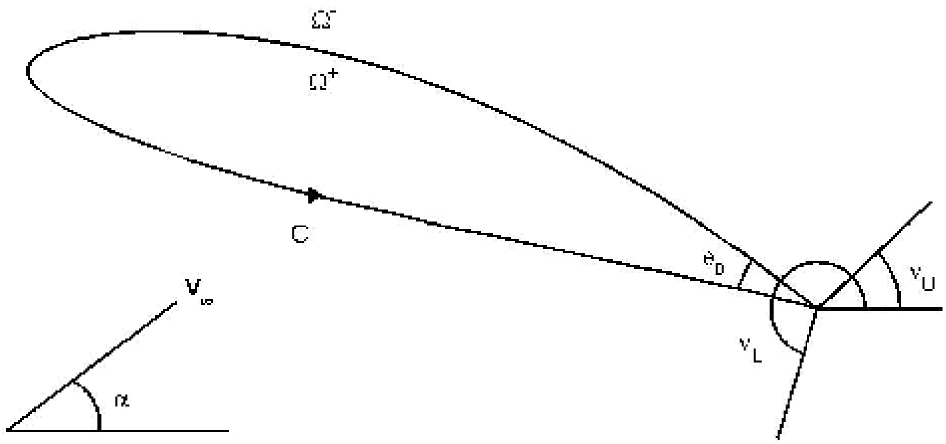


Fig.1. Physical model.

Assume that the curve C has the parametric representation

$$C: \quad t = t(\tau), \quad 0 \le \tau \le \beta, \quad \beta > 0 .$$

The corner point is assumed to be $t_0 := t(0) = t(\beta)$. The points preceding t_0 in describing C in the counterclockwise direction will be denoted by t_L and the points following t_0 will be denoted by t_U, i.e., $t_U := t(0+)$ and $t_L := t(\beta-)$. The function $t(\tau)$ will be assumed to be such that $t''(\tau)$ exists and is continuous for all $\tau \in [0, \beta]$ where the derivatives at the end points are understood to be the one side derivatives, i.e., $t'(0+)$, $t''(0+)$, $t'(\beta-)$, $t''(\beta-)$ and

$$t'(0+) \ne t'(\beta-), \qquad t''(0+) \ne t''(\beta-).$$

Suppose further that the interior angle θ_0 of the corner point t_0 satisfies $0 < \theta_0 < \pi$. Denote the angles between the positive $x-$axis and the normal vectors to C at the points t_U and t_L by v_U and v_L, respectively. Then it is clear from Fig.1 that $v_L = v_U + \pi + \theta_0$.

Let $T(t)$ denote the tangent function to C at the point $t \in C$ in the direction of C. Under the above assumptions, $T(t)$ is a continuously differentiable function for all $t \in C$ except at the corner point t_0 where the tangent is undefined. However, at the corner point t_0 the one side tangent vector can be defined as

$$T_U = \frac{t'(0+)}{|t'(0+)|} \quad \text{and} \quad T_L = \frac{t'(\beta-)}{|t'(\beta-)|}.$$

It is clear from Fig.1 that

$$T_L = e^{i(\pi+\theta_0)} T_U = -e^{i\theta_0} T_U . \tag{2.1}$$

Hence the tangent function $T(t)$ has a discontinuity of the first type at the corner point t_0. Suppose that the complex analytic function $W(z)$ is the complex velocity. If the free stream velocity is of unit magnitude and angle α to the real axis, then $W(z)$ can be decomposed into the free stream part $e^{-i\alpha}$, and the complex disturbance velocity $w(z)$, respectively, as follow

$$W(z) = e^{-i\alpha} + w(z), \qquad z \in \Omega^- . \tag{2.2}$$

Assuming that one can find a solution for the complex velocity $W(z)$ or the complex disturbance velocity $w(z)$, by the Bernoulli theorem the pressure coefficient $C_p(z)$ is then given by

$$C_p(z) = 1 - W(z)\overline{W(z)}, \qquad z \in \Omega^- \cup \Gamma . \tag{2.3}$$

This pressure coefficient may be then integrated to compute loads on the airfoil or other components such as flaps and slats.

If we assume that the flow does not have suction or blowing, i.e., the normal component of the velocity is zero for all $t \in C \,|\, \{t_0\}$, then from Mokry (1990), the function $w(z)$ satisfies the exterior Riemann problem

$$Re\left[c(t)w^-(t)\right] = \gamma(t) \tag{2.4}$$

where

$$c(t) = -iT(t) \quad \text{and} \quad \gamma(t) = -Im\left[e^{-i\alpha}T(t)\right], \qquad t \in C \,|\, \{t_0\} \tag{2.5}$$

and the circulation Γ of the fluid along the boundary C is given by

$$\Gamma = \int_C w^-(t)dt . \tag{2.6}$$

Since in an unconfined flow, the velocity disturbance is required to vanish far away from the airfoil, the problem (2.4) needs to be supplemented by the far field boundary condition $w(\infty) = 0$. In this way we arrive at the exterior Riemann problem

$$Re\left[c(t)w^-(t)\right] = \gamma(t), \qquad w(\infty) = 0, \qquad t \in C \,|\, \{t_0\}. \tag{2.7}$$

The functions $c(t)$ and $\gamma(t)$ given by Eq.(2.5) are continuously differentiable on C except at the corner point t_0 where they have there a discontinuity of the first type. The Riemann problem with discontinuous coefficients have been treated in the classical references (Gakhov, 1966; pp.449-454) and (Muskhelishvili, 1977; pp.271-275) by reducing it to a Hilbert problem. We shall here extend the integral equation derived in Murid and Nasser (2003) for the exterior Riemann problem in regions with smooth boundaries to solve the Riemann problem (2.7).

The solution of the Riemann problem with discontinuous coefficients at a point t_0 is sought in the class of functions which are integrable on the boundary C. It follows that the solution is Hölder continuous everywhere except possibly at the discontinuous point t_0 (Gakhov, 1966; p.407). At the point t_0, we have two cases, namely,

1. a solution may be required to be bounded at the point t_0, i.e., a solution is sought which is bounded everywhere on C;
2. a solution may be admitted which has integrable singularities at the point t_0.

According to the Kutta-Joukowski condition, the velocity must be bounded at the corner point t_0. Hence the solution of the exterior Riemann problem (2.7) must be sought in the class of bounded functions at the point t_0.

Mathematically, the Kutta-Joukowski condition requires the pressure coefficient $C_p(t)$ to satisfy

$$C_p(t(0+)) = C_p(t(\beta-)). \tag{2.8}$$

This condition implies also that, unless the trailing edge is cusped, the flow has a stagnation point at the trailing edge (Katz and Plotkin, 2001; p.88). Thus by Eq.(2.3), the pressure coefficient $C_p(t)$ at the corner point t_0 equals one, i.e.,

$$C_p(t(0)) = C_p(t(\beta)) = 1. \tag{2.9}$$

3. Solvability of the Riemann problem (2.7)

In this section, we shall study the solvability of the exterior Riemann problem (2.7). Let $\boldsymbol{R_e}$ be the conformal mapping from the exterior region Ω^- onto the interior of the unit circle, $\boldsymbol{D}^+$, such that $R_e(\infty) = 0$, and R_e^{-1} be its inverse. Then by Osgood-Caratheodory theorem (Henrici, 1974; p.383), the mapping function R_e can be extended to a homeomorphism from $\overline{\Omega^-}$ onto $\overline{\boldsymbol{D}^+}$. The exterior Riemann problem (2.7) will then be mapped by the conformal mapping R_e to the interior Riemann problem

$$Re\left[\tilde{c}(\xi)\tilde{w}^+(\xi)\right] = \tilde{\gamma}(t), \qquad \tilde{w}(0) = 0, \qquad \xi \in \boldsymbol{T} \mid \{\xi_0\}, \tag{3.1}$$

in the unit disc $\boldsymbol{D}^+$ where $\boldsymbol{T}$ is the unit circle, $\xi_0 = R_e(t_0)$ and

$$\tilde{c}(\xi) = c\left(R_e^{-1}(\xi)\right) = -iT\left(R_e^{-1}(\xi)\right) \quad \text{and} \quad \tilde{\gamma}(\xi) = \gamma\left(R_e^{-1}(\xi)\right), \qquad \xi \in \boldsymbol{T} \mid \{\xi_0\}. \tag{3.2}$$

The function R_e^{-1} is analytic in $\boldsymbol{T} \mid \{\xi_0\}$, and hence the functions $\tilde{c}(\xi)$ and $\tilde{\gamma}(\xi)$ are continuously differentiable on $\boldsymbol{T}$ except at the point $\xi_0 = R_e(t_0)$ where they have a discontinuous of the first type. Consequently, Eq.(3.1) is a Riemann problem in $\boldsymbol{D}^+$ with discontinuous coefficients.

The solution of the Riemann problem (3.1) which satisfies $\tilde{w}(0) = 0$ can be written in the form

$$\tilde{w}(\tilde{z}) = \tilde{z}\, w_1(\tilde{z}), \qquad \tilde{z} \in \boldsymbol{D}^+$$

where the function $w_1(\tilde{z})$ is an analytic function in $\boldsymbol{D}^+$. This implies that the function $w_1(\tilde{z})$ is a solution of the interior Riemann problem

$$Re\left[c_1(\xi)w_1^+(\xi)\right]=\tilde{\gamma}(t), \qquad \xi\in \boldsymbol{T}\,|\,\{\xi_0\}, \tag{3.3}$$

in the unit disc $\boldsymbol{D}^+$ where

$$c_1(\xi)=\xi c\left(R_e^{-1}(\xi)\right)=-i\xi T\left(R_e^{-1}(\xi)\right), \qquad \xi\in \boldsymbol{T}\,|\,\{\xi_0\}. \tag{3.4}$$

Denote the points preceding ξ_0 in describing $\boldsymbol{T}$ in the counterclockwise direction by ξ_L and the points following ξ_0 by ξ_U. Then $\xi_L=e^{(2\pi-0)i}\xi_U$. Since C is counterclockwise oriented, hence $\boldsymbol{T}=-R_e(C)$. Therefore

$$\xi_U=R_e(t_L) \quad \text{and} \quad \xi_L=R_e(t_U).$$

Let us set

$$G_2(\xi):=-\frac{\overline{c_1(\xi)}}{c_1(\xi)}=\overline{\left(\xi T\left(R_e^{-1}(\xi)\right)\right)^2}, \qquad \xi\in \boldsymbol{T}\,|\,\{\xi_0\}, \tag{3.5}$$

then

$$\frac{G_2(\xi_L)}{G_2(\xi_U)}=\frac{\overline{\left(\xi_L T\left(R_e^{-1}(\xi_L)\right)\right)^2}}{\overline{\left(\xi_U T\left(R_e^{-1}(\xi_U)\right)\right)^2}}=\frac{\overline{(\xi_L T_U)^2}}{\overline{(\xi_U T_L)^2}}=\left(\overline{\xi_L}\,\xi_U\,\overline{T_U}\,T_L\right)^2=e^{-2i(\pi-\theta_0-0)}.$$

Thus according to Gakhov (1966); p.413, the index κ of the interior Riemann problem (3.3) in the class of bounded functions at the corner point t_0 is given by

$$\kappa=\left[\frac{-2(\pi-\theta_0-0)}{2\pi}\right]=-1$$

where the symbol $[x]$ denotes the greatest integer not exceeding x.

Consequently, in accordance with Muskhelishvili (1977); pp.273-274, the interior Riemann problem (3.3) is uniquely solvable in the class of bounded functions at the point ξ_0. Hence the exterior Riemann problem (2.7) is uniquely solvable in the class of bounded functions at the corner point t_0.

4. Boundary integral equation for the external flow around airfoils

Two Fredholm integral equations have been recently derived in Murid *et al.* (2002) and Murid and Nasser (2003) for the interior and exterior Riemann problems in domains with C^2 – smooth boundaries. Checking carefully the derivation of the integral equations in those papers, we find that the property of smoothness of the boundary was employed twice, the first time in applying the Sokhotskyi formulas, and again in proving the continuity of the kernel $N(c)(t,w)$. If we assume the boundary C is the airfoil described above (Fig.1), then for the exterior and interior Riemann problems (2.7) and (3.3), the coefficient $c(t)$ is given by $c(t)=-iT(t)$, $t\in C\,|\,\{t_0\}$, and hence $N(c)(t,w)=-N(w,t)$. From Henrici (1986); p.394, the kernel $N(w,t)$ is continuous for all $(w,t)\in C\times C$ such that $t\neq t_0$. Using this fact and the fact that the

Sokhotskyi formulas (Gakhov, 1966; p.32), remain valid for all non-corner points on piecewise smooth Jordan curve, we see that theorems *2.1* and *2.2* in Murid and Nasser (2003) are valid for the above airfoil C except for the corner point t_0. Define the function $L(z)$ by

$$L(z)=\frac{1}{2\pi i}\int_C \frac{2\gamma(t)}{c(t)(t-z)}dt\,, \qquad z\notin C\,, \tag{4.1}$$

then theorems *2.1* and *2.2* in Murid and Nasser (2003) become:

Lemma 1. If $f(z)$ *is a solution of the interior Riemann problem*

$$Re\left[c(t)f^{+}(t)\right]=\gamma(t), \qquad t\in C\,|\,\{t_0\}, \tag{4.2}$$

then the function $\mu_1(t):=Im\left[c(t)f^{+}(t)\right]$, $t\in C\,|\,\{t_0\}$, *satisfies the integral equation*

$$\mu_1(t)+\int_C N(w,t)\mu_1(w)|dw|=Im\left[c(t)L^{-}(t)\right], \quad t\in C\,|\,\{t_0\}. \tag{4.3}$$

Lemma 2. *If* $w(z)$ *is a solution of the exterior Riemann problem (2.7), then the function* $\mu(t):=Im\left[c(t)w^{-}(t)\right]$, $t\in C\,|\,\{t_0\}$, *satisfies the integral equation*

$$\mu(t)-\int_C N(w,t)\mu(w)|dw|=-Im\left[c(t)L^{+}(t)\right], \qquad t\in C\,|\,\{t_0\}. \tag{4.4}$$

To avoid the difficulties in the calculation of its right hand side, $-Im\left[c(t)L^{+}(t)\right]$, the integral Eq.(4.4) will be modified as follows. It is obvious that the constant function $f(z)=-e^{-i\alpha}$ is an analytic function in the interior simply connected region Ω^{+} and satisfies the interior Riemann problem (16). Thus by Lemma 1, the function

$$\mu_1(t)=Im\left[c(t)f^{+}(t)\right]=Re\left[e^{-i\alpha}T(t)\right], \qquad t\in C\,|\,\{t_0\}, \tag{4.5}$$

is a solution of the integral Eq.(4.3).

By Sokhotskyi formulas, the function $L(z)$ defined by Eq.(4.1) satisfies

$$c(t)L^{+}(t)-c(t)L^{-}(t)=2\gamma(t), \qquad t\in C\,|\,\{t_0\},$$

which implies

$$Im\left[c(t)L^{+}(t)\right]=Im\left[c(t)L^{-}(t)\right], \qquad t\in C\,|\,\{t_0\}. \tag{4.6}$$

Substituting Eq.(4.6) into Eq.(4.3) leads to

$$Im\left[c(t)L^{+}(t)\right]=\mu_{1}(t)+\int_{C}N(w,t)\mu_{1}(w)|dw|\,, \qquad t\in C\,|\,\{t_{0}\} \tag{4.7}$$

where $\mu(t)$ is given by Eq.(4.5). Substituting Eq.(4.7) into Eq.(4.4) implies that the function $\rho(t)$ defined by

$$\rho(t)=\mu(t)-\mu_{1}(t)\,, \qquad t\in C\,|\,\{t_{0}\}, \tag{4.8}$$

is a solution of the integral equation

$$\rho(t)-\int_{C}N(w,t)\rho(w)|dw|=-2\mu_{1}(t)\,, \qquad t\in C\,|\,\{t_{0}\}. \tag{4.9}$$

Thus we have the following corollary:

Corollary 1. *If $w(z)$ is a solution of the exterior Riemann problem (2.7), then the real function*

$$\rho(t)=Im\left[c(t)w^{-}(t)\right]-Re\left[e^{-i\alpha}T(t)\right]\,, \qquad t\in C\,|\,\{t_{0}\}, \tag{4.10}$$

satisfies the integral Eq.(4.9).

If the function $w(z)$ is the unique solution of the exterior Riemann problem (7) in the class of bounded function at the corner point t_0 and the function $\rho(t)$ is given by Eq.(4.10), then

$$w^{-}(t)=\frac{Re\left[c(t)w^{-}(t)\right]+i\,Im\left[c(t)w^{-}(t)\right]}{c(t)}=-e^{-i\alpha}-\frac{\rho(t)}{T(t)}, \qquad t\in C\,|\,\{t_{0}\}. \tag{4.11}$$

In accordance with the Cauchy integral formula this implies that

$$w(z)=\int_{C}\frac{\rho(t)}{T(t)(t-z)}dt\,, \qquad z\in\Omega^{-}\,. \tag{4.12}$$

Since the function $w(z)$ is bounded at the corner point t_0, the function $\rho(t)/T(t)$ must satisfy the condition (Gakhov, 1966; p.55)

$$\frac{\rho(t_{U})}{T(t_{U})}=\frac{\rho(t_{L})}{T(t_{L})}\,. \tag{4.13}$$

According to Eq.(2.1) and since $\rho(t)$ is a real-valued function; the condition (4.13) implies that the function $\rho(t)$ satisfies the conditions

$$\rho(t_{U})=0 \quad \text{and} \quad \rho(t_{L})=0\,. \tag{4.14}$$

Thus we have proved the following lemma:

Lemma 3. *If $w(z)$ is the unique solution of the exterior Riemann problem (2.7) in the class of bounded functions at the corner point t_0, then the real function*

$$\rho(t) = Im\left[c(t)w^{-}(t)\right] - Re\left[e^{-i\alpha}T(t)\right], \qquad t \in C \,|\, \{t_0\}, \tag{4.15}$$

is a solution of the integral Eq.(4.9) with the conditions (4.14).

By Lemma 3 and since the exterior Riemann problem is solvable in the class of bounded functions at the corner point t_0, the integral Eq.(4.9) with the conditions (4.14) is solvable. However, from Henrici (1986; p.398), $\lambda = 1$ is a simple eigenvalue of the kernel $N(w,t)$. This implies in accordance with the Fredholm alternative theorem that the solution of the integral Eq.(4.9) can be written as

$$\rho(t) = \rho_p(t) + c_0\rho_h(t), \qquad t \in C \,|\, \{t_0\} \tag{4.16}$$

where $\rho_p(t)$ is a particular solution of the integral Eq.(4.9), $\rho_h(t)$ is a solution of the homogeneous integral equation corresponding to (4.9) and c_0 is arbitrary real constant. Moreover, from Henrici (1986); pp.380-381; p.397, it follows, that the function $\rho_h(t)$ satisfies

$$\lim_{\tau\to 0+} |t'(\tau)|\rho_h(t(\tau)) \neq 0, \qquad \lim_{\tau\to\beta-} |t'(\tau)|\rho_h(t(\tau)) \neq 0. \tag{4.17}$$

Thus one of the conditions (4.14) is enough to determine the arbitrary constant c_0 in (4.16). Consequently, we have the following lemma:

Lemma 4. *The integral Eq.(4.9) with the conditions (4.14) is uniquely solvable.*

As a result of the Corollary 1, and Lemmas 3, 4, we have the following corollary:

Corollary 2. *If* $\mu(t)$ *is the unique solution of the integral Eq.(4.9) with the conditions (4.14), then the function* $w(z)$ *given by (4.12) is the unique solution of the exterior Riemann problem (2.7) in the class of bounded functions at the corner point* t_0 *and its boundary values are given by*

$$w^{-}(t) = -e^{-i\alpha} - \frac{\rho(t)}{T(t)}, \qquad t \in C \,|\, \{t_0\}. \tag{4.18}$$

From Eqs.(2.2) and (4.18), the complex velocity on the boundary C is given by

$$W^{-}(t) = -\frac{\rho(t)}{T(t)}, \qquad t \in C \,|\, \{t_0\}. \tag{4.19}$$

On substituting Eq.(4.19) into Eq.(2.3) we conclude that the pressure coefficient on the surface of the airfoil is given by

$$C_p(t) = 1 - \rho(t)^2, \qquad t \in C \,|\, \{t_0\}. \tag{4.20}$$

Since the function $\rho(t)$ satisfies the conditions (4.14), it is clear from (4.20) that the Kutta-Joukowski condition (2.8) is satisfied. Moreover, it is clear that the function $\rho(t(\tau))$ is continuous for all

$\tau \in (0, \beta)$. Since the limits of the function $\rho(t)$ when the point t tends to t_0 from left and right exist and are equal, i.e.

$$\lim_{\tau \to 0+} \rho(t(\tau)) = 0, \qquad \lim_{\tau \to \beta-} \rho(t(\tau)) = 0,$$

the function $\rho(t)$ can be defined at t_0 by $\rho(t_0) = 0$. Thus the function $\rho(t)$ is continuous for all $t \in C$. The function $C_p(t)$ will then be given by (4.20) for all $t \in C$ and it will satisfy also the relation (2.9). Thus, the condition (4.14) is equivalent to the Kutta-Joukowski condition, and the unique solution of the Riemann problem (2.7) in the class of bounded functions at the corner point t_0 satisfies automatically the Kutta-Joukowski condition.

5. Numerical results

In this section, we shall give some numerical results of the integral Eq.(4.9) on the airfoil C (Fig.1). There are significant differences from the smooth boundary case, both in the behaviour of the solutions and in the properties of the integral operator. If C has corners, the integral operator is no longer compact (Atkinson and Kendall, 1997; p.389). But it can be expressed as the sum of a compact operator and a non-compact operator with norm less than one in suitable function spaces. Then we can use an error analysis based on collectively the compact operator theory with some modification.

For a boundary integral equation on a smooth boundary, i.e., boundaries of class C^{2+q}, $q \geq 2$, and smooth boundary values, i.e., of class C^2, the Nyström method with the trapezoidal rule converges with order $O(1/n^q)$ where n is the number of mesh points (Kress, 1990). When the boundary has corners, the situation is much less convenient. However, it has been shown in Graham and Chandler (1988) that a slightly modified Nyström method may still be applied to solve the integral equation and that optimal uniform convergence is obtained when the mesh is graded near each corner. But in Graham and Chandler (1988), the Nyström method is used with a locally approximating quadrature method on each graded subinterval. However, because the boundary integral Eq.(4.9) has an integrand that is smooth exception of the corners points, it is appropriate to use a quadrature based on a smooth global approximation over each smooth section of the boundary. Such method has been proposed by Kress (1990).

If an integrand g is smooth in $(0,1)$ but has singularities at the end-points $\tau = 0$ and $\tau = 1$, Kress (1990) proposes the following quadrature formula

$$\int_0^1 g(\tau) d\tau \approx \sum_{i=1}^{2n-1} \omega_i g(\tau_i) \tag{5.1}$$

where the weights and mesh points are given by

$$\omega_i = \frac{1}{2n} w'\left(\frac{i\pi}{n}\right), \qquad \tau_i = \frac{1}{2\pi} w\left(\frac{i\pi}{n}\right), \qquad i = 1, 2, \ldots, 2n-1, \tag{5.2}$$

and the function $w : [0, 2\pi] \to [0, 2\pi]$ is a bijective, strictly monotonically increasing, infinity differentiable function and is given by

$$w(s) = 2\pi \frac{v(s)^p}{v(s)^p + v(2\pi - s)^p}, \quad 0 \le s \le 2\pi, \quad p \ge 2,$$

$$v(s) = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{\pi - s}{\pi}\right)^3 + \frac{1}{p}\frac{s-\pi}{\pi} + \frac{1}{2}. \tag{5.3}$$

Suppose that C is a NACA airfoil with the counterclockwise parametric representation

$$C: \quad t = t(\tau), \quad 0 \le \tau \le 2$$

where $t(0) = t(2)$ is the trailing edge and $t(1)$ is the leading edge. The integral Eq.(4.9) is then reduced to

$$\tilde{\rho}(\tau) - \int_0^1 K(\sigma,\tau)\tilde{\rho}(\sigma)d\sigma - \int_1^2 K(\sigma,\tau)\tilde{\rho}(\sigma)d\sigma = \psi(\tau), \quad 0 < \tau < 2 \tag{5.4}$$

where

$$\tilde{\rho}(\tau) = \rho(t(\tau)), \quad \psi(\tau) = -2\,Re\left[e^{-i\alpha}T(t(\tau))\right], \quad K(\sigma,\tau) = N(t(\sigma),t(\tau))|t'(\sigma)|,$$

for $0 < \sigma, \tau < 2$. Using the quadrature formula (5.1) with $p = 2$ to approximate the integrals in Eq.(5.4), we get for $0 < \tau < 2$

$$\tilde{\rho}_n(\tau) - \sum_{j=1}^{2n-1}\omega_j K(\tau_j,\tau)\tilde{\rho}_n(\tau_j) - \sum_{j=1}^{2n-1}\omega_j K(1+\tau_j,\tau)\tilde{\rho}_n(1+\tau_j) = \psi(\tau), \quad 0 < \tau < 2 \tag{5.5}$$

where ω_j and τ_j are given by Eq.(5.2). Define s_j and α_j, $j = 1,2,\ldots,4n-2$, by

$$s_j = \begin{cases} \tau_j, & 1 \le j \le 2n-1 \\ 1+\tau_j, & 2n \le j \le 4n-2 \end{cases} \quad \text{and} \quad \alpha_j = \begin{cases} \omega_j, & 1 \le j \le 2n-1 \\ \omega_{j-2n+1}, & 2n \le j \le 4n-2, \end{cases} \tag{5.6}$$

then Eq.(5.5) becomes

$$\tilde{\rho}_n(\tau) - \sum_{j=1}^{4n-2}\alpha_j K(s_j,\tau)\tilde{\rho}_n(s_j) = \psi(\tau). \tag{5.7}$$

Collocating at the node points s_i, $i = 1,2,\ldots,4n-2$, one obtains the equivalent linear system

$$\tilde{\rho}_n(s_i) - \sum_{j=1}^{4n-2}\alpha_j K(s_j,s_i)\tilde{\rho}_n(s_j) = \psi(s_i). \tag{5.8}$$

Define the matrix $\boldsymbol{K}_n = (k_{ij})_{(4n-2)\times(4n-2)}$, the vectors $\boldsymbol{x}_n = (x_i)_{(4n-2)\times 1}$ and $\boldsymbol{y}_n = (y_i)_{(4n-2)\times 1}$ by

$$k_{ij} = \alpha_j K(s_j, s_i), \qquad x_i = \tilde{\rho}_n(s_i), \qquad y_i = \psi(s_i),$$

we obtain the $(4n-2)\times(4n-2)$ linear system

$$(\boldsymbol{I} - \boldsymbol{K}_n)\boldsymbol{x}_n = \boldsymbol{y}_n. \tag{5.9}$$

By the Kutta-Joukowski condition (4.14), we have

$$\tilde{\rho}_n(0+) = 0, \qquad \tilde{\rho}_n(2-) = 0.$$

This will be imposed into the system (5.9) by approximate $\tilde{\rho}_n$ at the nodes s_1 and s_{4n-2} as follows

$$\tilde{\rho}_n(s_1) \approx \frac{\tilde{\rho}_n(0+) + \tilde{\rho}_n(s_2)}{2} = \frac{\tilde{\rho}_n(s_2)}{2},$$

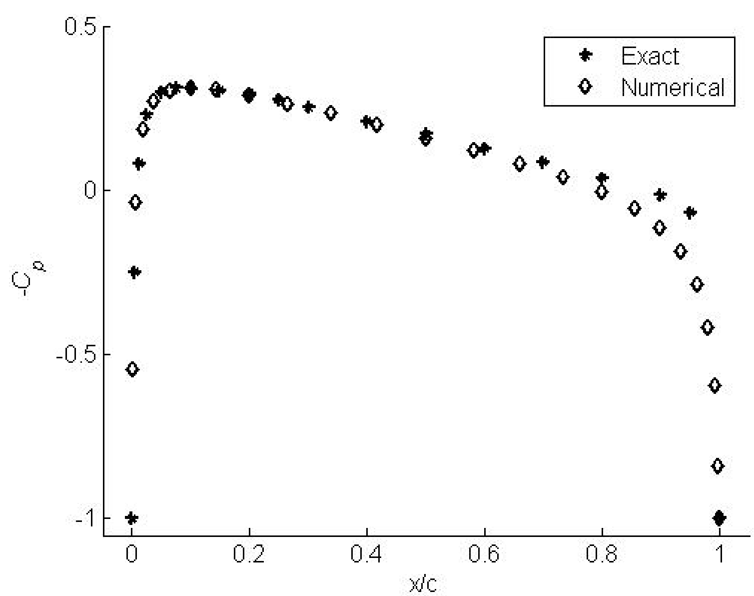
$$\tilde{\rho}_n(s_{4n-2}) \approx \frac{\tilde{\rho}_n(2-) + \tilde{\rho}_n(s_{4n-3})}{2} = \frac{\tilde{\rho}_n(s_{4n-3})}{2}.$$

Consequently, we have two additional equations

$$x_1 - \frac{1}{2}x_2 = 0, \qquad x_{4n-2} - \frac{1}{2}x_{4n-3} = 0. \tag{5.10}$$

By adding Eqs (5.10) to the linear system (5.9), we end up with a $4n\times(4n-2)$ over-determined system whose solution $\boldsymbol{x}_n$ can be computed using the MATLAB's operator \ that makes use of QR factorization with column pivoting (Trefethen and Bau, 1997; p.139). Once the solution has been computed, the Nyström interpolation formula (5.5) can be used to obtain the approximate solution $\rho_n(t)$. The pressure coefficient $C_p(t)$ on the boundary C can then be calculated from (4.20).

a)



b)

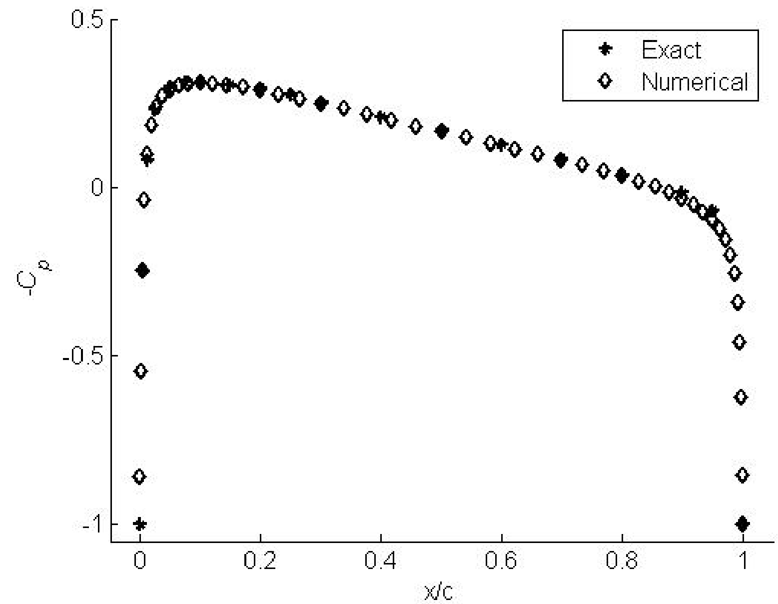
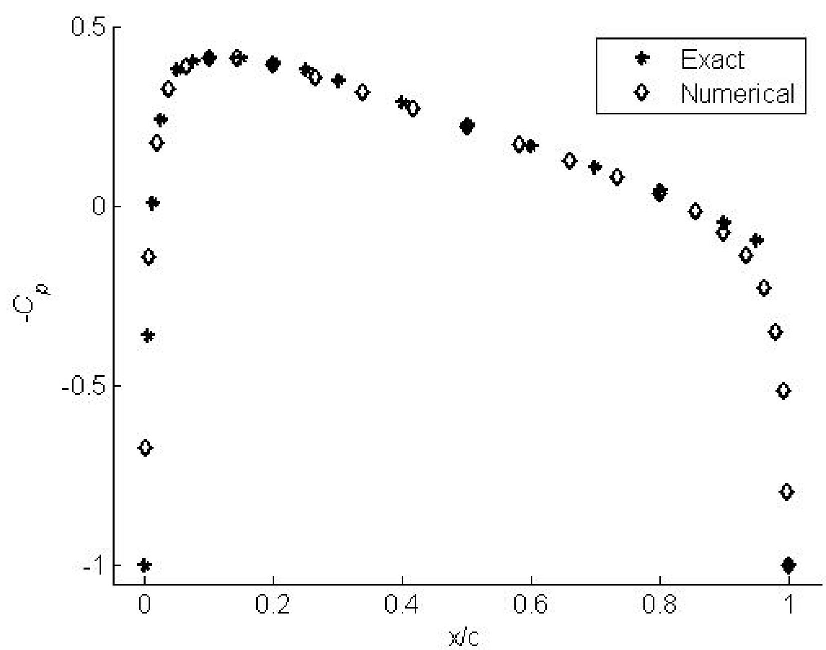


Fig.2. Comparison of numerical and experimental surface pressure distribution on NACA 0009 airfoil with zero angle of attack. (a) $n = 12$ (46 node points), (b) $n = 24$ (94 node points).

a)



b)

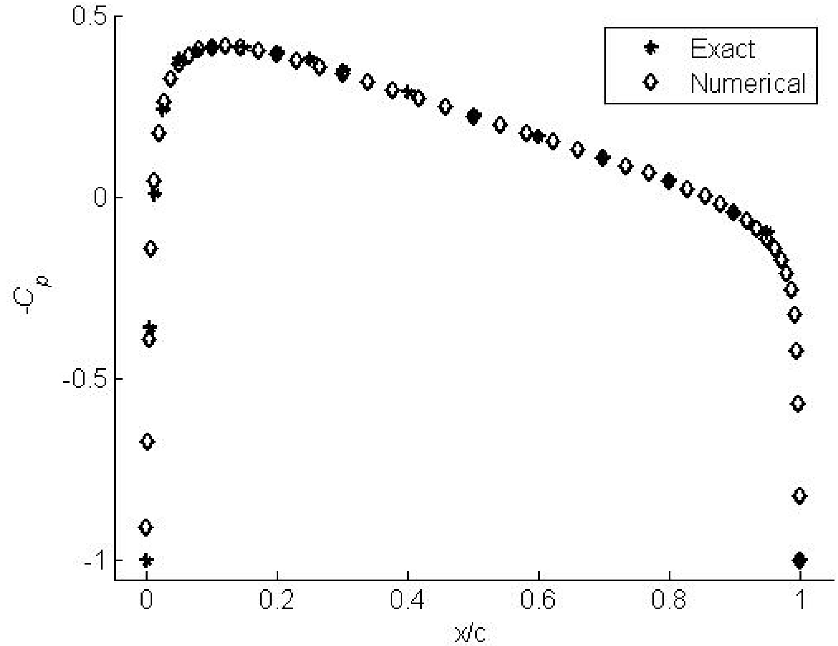


Fig.3. Comparison of numerical and experimental surface pressure distribution on NACA 0012 airfoil with zero angle of attack. (a) $n=12$ (46 node points), (b) $n=24$ (94 node points).

In this paper, the integral Eq.(4.9) is tested to calculate numerically the pressure coefficient C_p on the boundary C for the two airfoils, namely, NACA 0009 and NACA 0012, with zero angles of attack. After solving the integral Eq.(4.9) with the conditions (4.14), the pressure coefficient $C_p(t)$ on the boundary C will be calculated from (4.20). The numerical results obtained are shown in Figs 2 and 3 and compared to the analytic solutions from Abbott and von Doenhoff (1959). It can be seen from these figures that the agreement between the analytical and the numerical solutions is rather good.

6. Conclusions

A boundary integral equation has been given in this paper for the two-dimensional external potential flow around airfoils. Its kernel is the transposed of the Neumann kernel and its right hand side has a discontinuity of the first type. Once the solution of the integral Eq.(4.9) with the conditions (4.14) has been determined, the velocity at any point z in Ω^- can be calculated from the integral (4.12). The integral equation can be generalized straightforward to multi-element airfoils.

The external potential flow that we have considered in this paper can be formulated as an exterior Neumann problem (Hess, 1975) which can be solved using the well known second kind Fredholm integral equation for the Neumann problem (Henrici, 1986; p.281). The integral equation for the Neumann problem has the same kernel of the integral Eq.(4.9) and its right hand side has a discontinuity of the first type. Thus solving the integral Eq.(4.9) is equivalent to solving the integral equation for the Neumann problem. The advantage of the present method over the method of reducing the external potential flow to the Neumann problem is that for the present method the solution of the Riemann problem is given by the Cauchy integral (4.12) which can be calculate easily and sufficiently compared with the solution of the Neumann problem which is given by a potential of a single layer (Henrici, 1986; p.281).

Acknowledgments

This work was supported in part by MOSTE, Project Vote: 74049. This support is gratefully acknowledged. Words of thanks are also due to Prof. Ioan Pop and to an unknown referee for suggesting a number of improvements.

Nomenclature

C – boundary of the airfoil
C_p – pressure coefficient
Im – imaginary part
Re – real part
$T(t)$ – unit tangent vector to C at $t \in C$
$w(z)$ – disturbance velocity
$w^{-}(t)$ – $w^{-}(t) = \lim_{\substack{z \to t \in C, \\ z \in \Omega^{-}}} w(z)$
$W(z)$ – complex velocity
Γ – circulation
θ_0 – interior angle of the corner point
Ω^{+} – interior domain of the airfoil
Ω^{-} – exterior domain of the airfoil

References

Abbott I.H. and von Doenhoff A.E. (1959): *Theory of Wing Sections.* – New York: Dover Publications, Inc.

Atkinson K.E. (1997): *The numerical solution of integral equation of the second kind.* – Cambridge University Press.

Carabineanu A. (1996): *A boundary element approach to the 2D potential flow problem around airfoils with cusped trailing edge.* – Comput. Methods Appl. Mech. Eng., vol.129, pp.213-219.

Churchill R.V., Brown J.W. and Verhey R.F. (1984): *Complex variables and applications.* 3rd ed. – McGraw-Hill.

Gakhov F.D. (1966): *Boundary Value Problem.* – English translation of Russian edition 1963. –Oxford: Pergamon Press.

Graham I. and Chandler G. (1988): *High order methods for linear functionals of solutions of second kind integral equations.* – SIAM J. Numer. Anal., vol.25, pp.1118-1137.

Henrici P. (1974): *Applied and Computational Complex Analysis. vol.1.* – New York: John Wiley.

Henrici, P. (1986): *Applied and Computational Complex Analysis. vol.3.* –New York: John Wiley.

Hess J.L. (1975): *Review of integral-equation techniques for solving potential-flow problems with emphasis on the surface-source method.* – Comput. Methods Appl. Mech. Eng., vol.5, pp.145-196.

Hess J.I. and Smith A.M.O. (1967): *Calculation of potential flow about arbitrary bodies.* – In: Kuchemann, D. ed. *Progress in Aeronautical Sciences.* vol.8. London: Pergamon. pp.1-138.

Hwang W.S. (2000): *A boundary node method for airfoils based on the Dirichlet condition.* – Comput. Methods Appl. Mech. Eng., vol.190, No.13, pp.1679-1688.

Katz J. and Plotkin A. (2001): *Low-Speed Aerodynamics.* 2nd ed. – Cambridge: Cambridge University Press.

Kress R. (1990): *A Nyström method for boundary integral equations in domains with corners.* – Numer. Math., vol.58, pp.145-161.

Mokry M. (1990): *Complex Variable Boundary Element Method for External Potential Flows.* – AIAA 90-0127.

Mokry M. (1996): *Potential Flow Past Airfoils as a Riemann-Hilbert Problem.* – AIAA 96-2161.

Murid A.H.M. and Nasser M.M.S. (2003): *Eigenproblem of the generalized Neumann kernel.* – Bull. Malaysian Math. Sc. Soc. (Second Series). vol.26, pp.13-33.

Murid A.H.M., Razali M.R.M. and Nasser M.M.S. (2002): *Solving Riemann problem using Fredholm integral equation of the second kind.* –In: Proceeding of Simposium Kebangsaan Sains Mathematik ke-10, UTM, pp.172-179.

Muskhelishvili N.I. (1977): *Singular Integral Equations*. English translation of Russian edition 1953. – Leyden: Noordhoff.

Trefethen L. N. and Bau III, David. (1997): *Numerical Linear Algebra*. –SIAM Philadelphia.

Received: April 28, 2004

Revised: December 3, 2004