GENERALIZED COMMUTATIVITY DEGREES OF SOME FINITE GROUPS AND THEIR RELATED GRAPHS

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To my beloved mother

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ABSTRACT

The commutativity degree of a finite group G is the probability that two randomly chosen elements of the group G commute and is denoted as $P(G)$. The concept of commutativity degree is then extended to the relative commutativity degree of a group G, denoted as $P(H,G)$, which is defined as the probability that two arbitrary elements, one in the subgroup H and another in the group G, commute. Similarly, the concept of commutativity degree can be extended to two arbitrary elements from two subgroups of the group. In this research, the theory of commutativity degree is extended by defining the probability that the n-th power of a random pair of elements in the group G commute, where it is called the n-th power commutativity degree and is denoted by $P^n(G)$. The computation of n-th power commutativity degree are divided into two cases, namely for $n = 2$ and $n = 3$ where $P^2(G)$ and $P^3(G)$ is called the squared commutativity degree and cubed commutativity degree, respectively. These probabilities have been obtained for dihedral groups. Meanwhile, the productivity degree of two subgroups of a group is also an extension of the commutativity degree and it is defined as the ratio of the order of the intersection of HK and KH with the order of their union, where H and K are two subgroups of a group G, denoted by $P_G(HK)$. The general formula for dihedral groups has been found for this probability. Another extension of the commutativity degree which has been defined in this research is the relative n-th nilpotency degree of two subgroups of a group, denoted as $P_{nil}(n,H,K)$. This probability is defined as the probability that the commutator of two arbitrary elements in H and in K belongs to the n-th central series of the group. Some results that have been found through this probability include its lower and upper bound, its comparison between their factor groups and its relation with extra relative n-isoclinism. All of the results obtained are then applied to graph theory where a graph related to each probability is defined. This includes the graph related to cubed commutativity degree and the product of subgroup graph which is related to the productivity degree of two subgroups. A new graph which is called a complete tripartite graph is introduced. The last graph is the bipartite graph associated to a non-nilpotent group of class $(n-1)$, called as relative non-nil $(n-1)$ bipartite graph. Therefore, some graph properties are found for all mentioned graphs which include the diameter and girth.

ABSTRAK

Darjah kekalisan tukar tertib bagi suatu kumpulan terhingga G adalah kebarangkalian bahawa dua unsur yang dipilih secara rawak dalam kumpulan G adalah kalis tukar tertib dan ditandakan sebagai $P(G)$. Konsep darjah kekalisan tukar tertib seterusnya diperluaskan kepada darjah kekalisan tukar tertib secara relatif bagi suatu kumpulan G, ditandakan sebagai $P(H, G)$, yang mana ditakrifkan sebagai kebarangkalian bahawa dua unsur sebarangan iaitu satu unsur dalam subkumpulan H dan satu unsur dalam kumpulan G berkalis tukar tertib. Begitu juga, konsep darjah kekalisan tukar tertib boleh diperluaskan kepada dua unsur yang sebarangan daripada dua subkumpulan dalam suatu kumpulan. Dalam kajian ini, teori darjah kekalisan tukar tertib diperluaskan dengan mentakrifkan kebarangkalian bahawa kuasa ke-n bagi sepasang unsur yang dipilih secara rawak daripada kumpulan G adalah berkalis tukar tertib, di mana disebut sebagai darjah kekalisan tukar tertib kuasa ke-n dan ditandakan oleh $P^n(G)$. Pengiraan darjah kekalisan tukar tertib kuasa ke-n dibahagikan kepada dua kes iaitu $n = 2$ dan $n = 3$ di mana $P^2(G)$ dan $P^3(G)$, masing-masing disebut sebagai darjah kekalisan tukar tertib kuasa dua dan darjah kekalisan tukar tertib kuasa tiga. Kebarangkalian ini telah diperoleh bagi kumpulan dwihedron. Sementara itu, darjah hasil darab bagi dua subkumpulan bagi suatu kumpulan juga merupakan perluasan daripada darjah kekalisan tukar tertib dan ditakrifkan sebagai nisbah peringkat bagi persilangan HK dan KH dengan peringkat bagi kesatuan mereka, dengan H dan K ialah dua subkumpulan bagi suatu kumpulan G, ditandakan oleh $P_G(HK)$. Rumus umum bagi kumpulan dwihedron telah dijumpai bagi kebarangkalian ini. Perluasan lain bagi darjah kekalisan tukar tertib yang ditakrifkan dalam kajian ini adalah darjah nilpoten kali ke-n secara relatif bagi dua subkumpulan bagi suatu kumpulan, ditandakan sebagai $P_{nil}(n, H, K)$. Kebarangkalian ini ditakrifkan sebagai kebarangkalian bahawa penukar tertib bagi dua unsur berbeza yang berada dalam H dan K adalah kepunyaan siri berpusat kali ke-n bagi suatu kumpulan. Beberapa keputusan yang telah ditemui melalui kebarangkalian ini adalah termasuk batas bawahnya dan batas atasnya, perbandingannya antara kumpulan faktor mereka dan hubungannya dengan isoklinisma-n secara relatif tambahan. Kesemua keputusan yang diperoleh seterusnya diaplikasikan kepada teori graf dengan graf berkait kepada setiap kebarangkalian ditakrifkan. Ini termasuklah graf yang berkait kepada darjah kekalisan tukar tertib kuasa tiga dan graf hasil darab bagi subkumpulan yang berkait dengan darjah hasil darab bagi dua subkumpulan. Graf baru yang dikenali sebagai graf 3-partit lengkap diperkenalkan. Graf terakhir adalah graf bipartit berkaitan dengan kumpulan tak nilpoten bagi kelas $(n-1)$, disebut sebagai graf bipartit $(n-1)$ tak-nil secara relatif. Oleh itu, beberapa sifat graf telah dijumpai bagi kesemua graf yang disebut termasuklah diameter dan ukur lilit.

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$a\|b$	-	a divides b
$A(n)$	-	Alternating group of degree n
$Z(G)$	-	Center of the group G
$C_G(a)$	-	Centralizer of a in G
$P(G)$	-	Commutativity degree of G
$P_g(G)$	-	Commutator degree of G
$[g, h]$	-	The commutator of g and h
K_n	-	Complete graph of n vertices
$K_{m,n}$	-	Complete bipartite graph
$d(\Gamma)$	-	The diameter of Γ
D_m	-	Dihedral groups of order $2m$
$G \times H$	-	Direct product of G and H
${}^{G}/_{H}$	-	The factor group of H in G
$\forall$	-	For all
$g_2 = hg_1h^{-1}$	-	g_2 is a conjugate of g_1 by h
Γ	-	A graph
G	-	A group
$\langle x \rangle$	-	A group generated by the element x
$H \leq G$	-	H is a subgroup of G
$H \trianglelefteq G$	-	H is a normal subgroup of G
1, e	-	Identity element in a group
$\|G : H\|$	-	Index of the subgroup H in the group G
$\cap$	-	Intersection

$\cong$	-	Isomorphic
ϕ	-	Isomorphism mapping
S	-	A non-empty set
$K(G)$	-	The number of conjugacy classes in G
$Z^n(G)$	-	n-th center of the group G
$Z_n(G)$	-	n-th central series of G
$C_G^n(a)$	-	n-th centralizer of a in G
$P_n(G)$	-	n-th commutativity degree of G
$\|a\|$	-	Order of the element a
$\|G\|$	-	Order of the group G
$P_n(H,G)$	-	Relative n-th commutativity degree of G
$E(\Gamma)$	-	The set of edges Γ
$V(\Gamma)$	-	The set of vertices Γ
$\sum$	-	Summation
$\cup$	-	Union

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CHAPTER 1

INTRODUCTION

1.1 Introduction

The theory of commutativity degree in group theory plays a major role in determining the abelianness of a group and has attracted many researchers in various directions. This chapter starts with the history of the commutativity degree of a group. If G is a finite group, then the commutativity degree of G, denoted by $P(G)$, is the probability that two elements of a group G commute. The first appearance of this concept was in 1944 by Miller in [1]. In more than two decades later, the idea of commutativity degree was then investigated for symmetric groups by Erdos and Turan [2]. The results obtained encourage many researchers to work on this topic thus various generalizations have been done.

Studies on the algebraic structures of a finite group using the properties of their associated graphs have become an exciting research area in the recent years. Graph theory is a mathematical branch that concerns with connections of points and lines. The beginning of graph theory is in 1736 when the Swiss mathematician called Leonard Euler considered the Königsberg bridge problem by drawing points and lines. Gerhard and Betsy [3] have mentioned Euler's work in their book. Many years later, graph theory has developed into an extensive and popular branch of mathematics, which has been applied to many problems in mathematics, computer science and other scientific areas. As a consequence, the algebraic part of this research is then applied to some graphs.

In this research, the concept of commutativity degree is further extended by defining a concept called the probability that the n-th power of a random pair of elements in the group commute, denoted as $P^n(G)$. This new probability is then determined for dihedral groups. The second part of this research deals with graph theory. New graph which is related to the n-th power commutativity degree is introduced. Moreover, some characterizations of all finite groups in term of this graph are found.

The probability that two arbitrary elements, one in H and another in G commute, is called the relative commutativity degree of a subgroup H of a group G, denoted by $P(H, G)$. This concept was generalized by Erfanian $et\ al.$ [4] in 2007. Similarly, the concept of this probability can be extended to the probability of two arbitrary elements, one in H and another in K, commute, where H and K are two subgroups of G. In this research, the concept of relative commutativity degree is further extended to the productivity degree of two subgroups of dihedral groups and also to the relative n-th nilpotency degree of two subgroups of a group G. Both new probabilities are then applied to graph theory where the graph related to those probability is found. Furthermore, some graph properties which include the diameter and girth are also found.

1.2 Research Background

A group is called an abelian group if every pair of its elements commutes. It means that for a group G, $ab = ba$, for all $a, b \in G$. However, not all groups are abelian, thus are called non-abelian groups. The commutativity degree that is defined on finite groups is a simple tool that measures how much a group is close or far from being abelian.

The commutativity degree, which is denoted by $P(G)$ is the probability that two elements of a group G, chosen randomly with replacement, commute. The first appearance of this concept was in 1944 by Miller [1]. Then the idea to compute $P(G)$ for symmetric groups has been introduced by Erdos and Turan [2] in 1968. Few years later, Gustafson [5] and MacHale [6] used the techniques by Erdos and Turan and showed that the probability of a random pair of elements commute in any finite non-abelian group is less than or equal to $\frac{5}{8}$. In 1979, Rusin [7] computed $P(G)$ for a finite group G meanwhile Sherman [8] used the concept of $P(G)$ and proved that $P(G)$ cannot be arbitrarily close to one if G is a finite non-abelian group.

The concept of commutativity degree was then generalized to the relative commutativity degree which was introduced by Erfanian $et\ al.$ [4] in 2007. Following that year, many studies have been done on this topic. Mohd Ali and Sarmin [9] in 2010 introduced the n-th commutativity degree of G as the probability that the n-th power of a random element commutes with another random element from the same group. A year later, Erfanian $et\ al.$ [10] gave the relative case of n-th commutativity degree. They identified the probability that the n-th power of a random element of a subgroup, H commutes with another random element of a group G, denoted as $P_n(H, G)$.

The second part of this research deals with graph theory. A graph can be represented diagrammatically by means of points and lines. The points are called vertices, the lines are called edges, and the whole diagram is called a graph. The graph is the fundamental structure in the field of graph theory. Gerhard and Betsy [3] have mentioned that Leonard Euler who considered Königsberg bridge problem by using graph was firstly introduced and created a new branch of mathematics called graph theory in the year 1736. Years later, the usefulness of graph theory has been proven to a large number of various fields.

The topic of non-commuting graph, Γ_G has been studied by many authors. For instance Abdollahi $et\ al.$ [11] in 2006 studied the relation between some graph theoretical properties of Γ_G and the group theory properties of the group G. Few years later, Erfanian and Tolue [12] introduced a new graph, namely conjugate graph with vertices are non-central elements of a group. In 2011, Mashkouri and Taeri [13] defined a graph associated to groups where they find a characterization of some dihedral groups in terms of this graph.

1.3 Problem Statements

There is an extensive literature in the use of probability in finite group theory. One of the most important aspects is the probability that two elements of a finite group commute which is called as the commutativity degree. As a result of several attempts to develop the concept of the commutativity degree, this concept has been generalized in a number of ways. Moreover, the study of algebraic graph theory becomes an exciting research topics nowadays dealing with the study of algebraic objects applied to graphs and then to derive properties of graphs by using the tools from algebra. Hence, in this research, the following question are addressed and answered.

1. What is the probability that the n-th power of a random pair of elements of dihedral groups commute for $n = 2$ and $n = 3$?
2. What is the productivity degree of two subgroups of dihedral groups?
3. What is the relative n-th nilpotency degree of two subgroups of a finite group?
4. What is the comparison between relative n-th nilpotency degree of two subgroups of a group and their factor group?
5. What is the relation between extra relative n-isoclinism and relative n-th nilpotency degree of two subgroups of a group?
6. What is the graph related to probability in (1), (2) and (3)?

7. What are some graph properties that can be obtained in (6)?

1.4 Research Objectives

The objectives of this research are:

1. To determine the n-th power commutativity degree of dihedral groups for the case $n = 2$ and $n = 3$.
2. To compute the productivity degree of two subgroups of dihedral groups and give the generalized formula.
3. To construct the lower and upper bounds for relative n-th nilpotency degree of two subgroups of a group.
4. To give a comparison between relative n-th nilpotency degree of two subgroups of a group and their factor group.
5. To find a relation between extra relative n-isoclinism and relative n-th nilpotency degree of two subgroups of a group.
6. To define a new graph related to the commutativity degree found in (1), (2) and (3).
7. To obtain some graph properties namely the diameter and girth for objective (6).

1.5 Scope of the Study

This research consists of two parts. The first part focuses on the commutativity degree and its generalizations. This study focuses on dihedral groups, both for determining the n-th power commutativity degree and the productivity degree of two subgroups. Meanwhile, the relative n-th nilpotency degree of two subgroups are introduced and computed for all finite non-abelian groups.

The second part of this research focuses on graph theory. The probability under this study is then applied to graph theory. Some graph properties are obtained which include the diameter and girth.

1.6 Significance of Findings

The major contribution of this thesis is to give new theoretical results on the generalization of the commutativity degree and graphs of the groups in the scope which have not been provided in existing literatures. The results obtained by the extensions of commutativity degree which include the n-th power commutativity degree, the productivity degree of two subgroups and the relative n-th nilpotency degree of two subgroups contributed to new findings in the field of group theory. In addition, three new graphs related to each probability are introduced which help to contribute new findings in the field of graph theory.

1.7 Research Methodology

This research starts by examining the commutativity degree and its generalizations, namely the relative commutativity degree and the n-th commutativity degree. The concept of commutativity degree has been extended by many authors and various results have been achieved. For example, Mohd Ali and Sarmin in [9] generalized the concept of commutativity degree by defining the n-th commutativity degree for two-generator two-groups of nilpotency class two. Then the concept of n-th commutativity degree was later extended to the relative n-th commutativity degree by Erfanian $et\ al.$ [10]. In this research, there are some extensions of the concept of commutativity degree. Firstly, the n-th power commutativity degree, secondly the productivity degree of two subgroups and thirdly is the relative n-th nilpotency degree of two subgroups. Dihedral groups are used to compute the first and second probabilities. Meanwhile, in computing the third probability, all finite non-abelian groups are considered. Lastly, by using the definition of n-th power commutativity degree, the productivity degree of two subgroups and also relative n-th nilpotency degree of two subgroups, new graphs have been defined. Some graph properties are also found which include the diameter and girth.

1.8 Thesis Organization

This thesis is divided into six chapters which includes the introduction, literature review, the n-th power commutativity degree and the productivity degree of two subgroups, the relative n-th nilpotency degree of two subgroups, on the graphs associated to various commutativity degrees and the conclusion.

In the first chapter, the introduction to the whole thesis is given including the research background, problem statements, research objectives, scope of the study, significance of findings, research methodology and thesis organization.

In Chapter 2, the literature review of this research is presented. In this chapter, a brief history about commutativity degree is stated in the background of the study, where some earlier and recent works related to commutativity degree and its generalizations namely the relative commutativity degree, n-th commutativity degree and the relative n-th comutativity degree are provided. This chapter also gives some fundamental concepts of group and graph theory. Historical briefing on graph theory is also reviewed, where different type of graphs and their algebraic properties are stated.

In Chapter 3, the n-th power commutativity degree is computed for $n = 2$ and $n = 3$. If $n = 2$ then the probability is called as squared commutativity degree meanwhile if $n = 3$, it is called as cubed commutativity degree. The productivity degree of two subgroups is also computed in this chapter. General formula for each probability is found for the groups in the scope of this research, namely for dihedral groups.

Chapter 4 presents the relative n-th nilpotency degree of two subgroups of a finite group. The results including lower and upper bounds of relative n-th nilpotency degree of two subgroups are established for this probability. In addition, a comparison between relative n-th nilpotency degree of two subgroups of a group and their factor group is found. Chapter 4 also includes the relation between extra relative n-isoclinism and relative n-th nilpotency degree of two subgroups of a group.

Next, in Chapter 5, new graphs related to each probability have been introduced. First of all, a graph related to the n-th power commutativity degree is defined. This graph is found only for cubed commutativity degree of a group. Furthermore, some properties of those graphs have been found for all finite groups. The second graph which is introduced is a graph related to the productivity degree of two subgroups of a group, which focuses only for dihedral groups. Another graph which is newly defined is relative non-nil $(n-1)$ bipartite graph where the vertices of this graph are all elements in the group except the elements in the centralizer. This new graph is related to the relative n-th nilpotency degree of two subgroups of a finite group which is a generalization of the non-commuting graph. Some related properties of those graphs are investigated and determined for a finite group.

Finally, Chapter 6 serves as the conclusion chapter which is the summary to the whole thesis. Some useful suggestions and ideas for future research on the n-th power commutativity degree, productivity degree of two subgroups, relative n-th nilpotency degree of two subgroups, and graphs associated to various commutativity degrees are also given in this chapter.

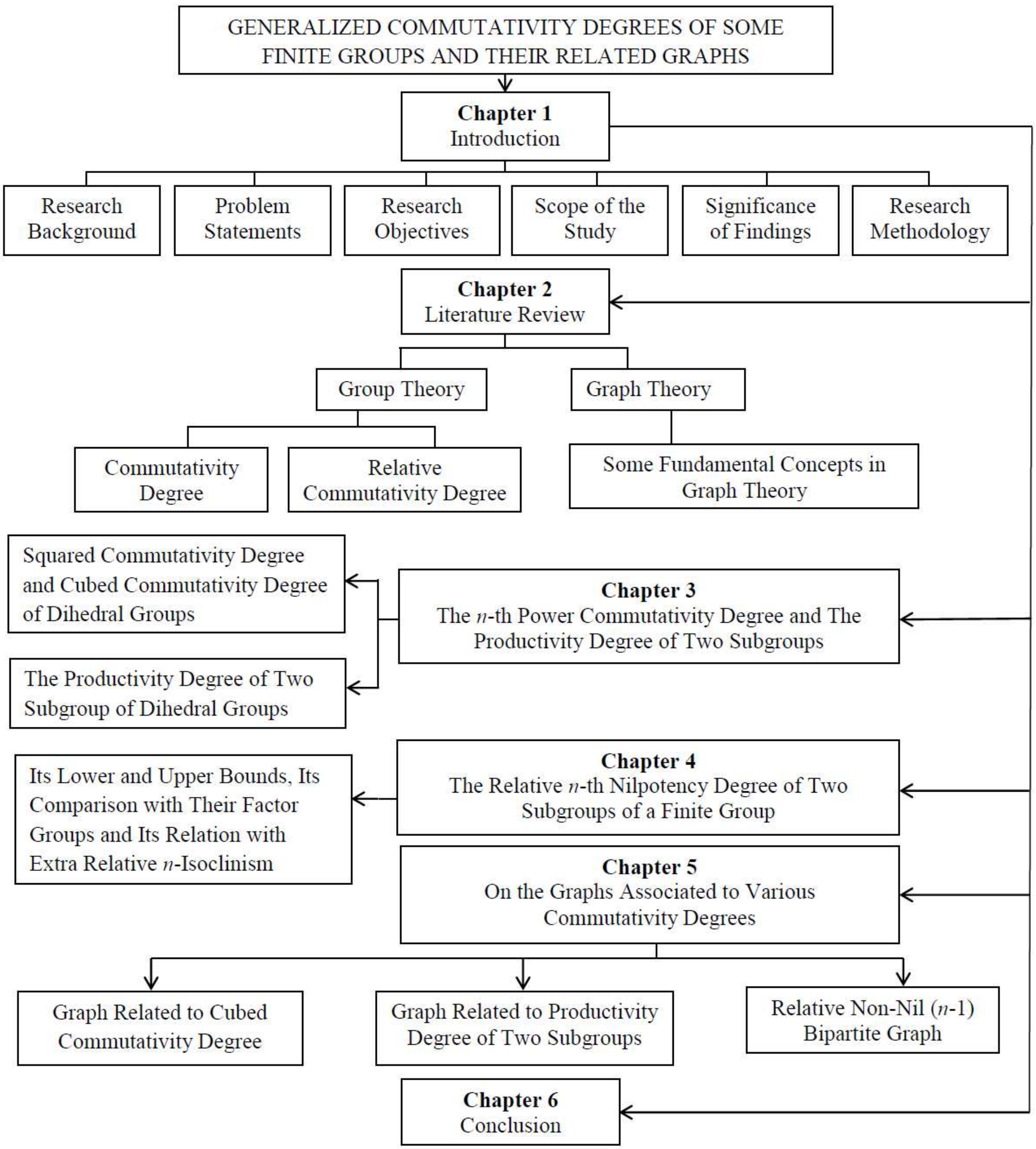

Figure 1.1 Thesis organization

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