THE CONJUGATION DEGREE ON A SET OF METACYCLIC 3-GROUPS AND 5-GROUPS WITH THEIR RELATED GRAPHS

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This thesis is specially dedicated to my dearest family, supervisor, colleagues and friends.

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ABSTRACT

The conjugation degree on a set is the probability that an element of a group fixes a set, whereby the group action considered is conjugation. The conjugation degree on a set is a variation of the commutativity degree of a group, which is the probability that two randomly chosen elements in a group commute. In this research, the presentation of metacyclic p-groups where p is an odd prime is used. Meanwhile, the set considered is a set of an ordered pair of commuting elements in the metacyclic p-groups, where p is equal to three and five, satisfying certain conditions. The conjugation degree on the set is obtained by dividing the number of orbits with the size of the set. Hence, the results are obtained by finding the elements of the group that follow the conditions of the ordered set, followed by the computation of the number of orbits of the set. In the second part of this research, the obtained results of the conjugation degree on a set are then associated with graph theory. The corresponding orbit graph, generalized conjugacy class graph, generalized commuting graph and generalized non-commuting graph are determined where a union of complete and null graphs, one complete and null graphs, one complete and null graphs with one empty and null graphs are found. Accordingly, several properties of these graphs are obtained, which include the degree of the vertices, the clique number, the chromatic number, the independence number, the girth, as well as the diameter of the graph. Furthermore, some new graphs are introduced, namely the orderly set graph, the order class graph, the generalized co-prime order graph, and the generalized non co-prime order graph, which resulted in the finding of one complete or empty graphs, a union of two complete or one complete graphs, a union of complete and empty graphs and a complete or empty graphs. Finally, several algebraic properties of these graphs are determined.

ABSTRAK

Darjah kekonjugatan terhadap suatu set merupakan kebarangkalian bahawa suatu unsur di dalam suatu kumpulan menetapi sesuatu set, dengan tindakan bagi kumpulan yang dipertimbangkan ialah kekonjugatan. Darjah kekonjugatan terhadap suatu set merupakan variasi daripada darjah kekalisan tukar tertib bagi sesuatu kumpulan, iaitu kebarangkalian dua unsur yang dipilih secara rawak di dalam sesuatu kumpulan adalah berkalis tukar tertib. Dalam penyelidikan ini, perwakilan bagi kumpulan-p metakitaran dengan p ialah nombor perdana ganjil digunakan. Sementara itu, set yang dipertimbangkan ialah set pasangan bertertib bagi unsur yang berkalis tukar tertib di dalam kumpulan-p metakitaran, dengan p bersamaan tiga dan lima, yang memenuhi syarat-syarat tertentu. Darjah kekonjugatan terhadap suatu set diperoleh dengan membahagi bilangan orbit dengan saiz set tersebut. Oleh itu, keputusan diperoleh dengan mencari unsur-unsur di dalam kumpulan yang mengikut syarat bertertib set, diikuti oleh pengiraan bilangan orbit di dalam set berkenaan. Dalam bahagian kedua penyelidikan ini, hasil yang telah diperoleh dari darjah kekonjugatan terhadap suatu set tersebut dikaitkan dengan teori graf. Graf orbit, graf kelas kekonjugatan yang teritlak, graf kalis tukar tertib yang teritlak dan graf bukan kalis tukar tertib yang teritlak yang sepadan telah ditentukan, dengan gabungan graf lengkap dan graf nol, satu graf lengkap dan graf nol, satu graf lengkap dan graf nol dengan satu graf kosong dan graf nol telah dijumpai. Berikutnya, beberapa ciri bagi graf yang tersebut telah diperoleh, termasuklah darjah bucu, nombor klik, nombor kromat, nombor tidak bersandar, lilitan serta garis pusat graf. Tambahan lagi, beberapa graf baharu telah diperkenalkan, yang dinamai graf set teratur, graf kelas teratur, graf teratur ko-perdana yang teritlak dan graf bukan teratur ko-perdana yang teritlak, yang menghasilkan penemuan satu graf lengkap atau graf kosong, gabungan dua graf lengkap atau satu graf lengkap, gabungan graf lengkap dan graf kosong, dan satu graf lengkap atau graf kosong. Akhirnya, beberapa ciri aljabar bagi graf-graf ini telah ditentukan.

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LIST OF SYMBOLS

Γ_{cent}	-	Centralizers graph
A	-	Commuting elements in the set Ω
$\Pr(G)$	-	The commutativity degree
$[b,a] = bab^{-1}a^{-1}$	-	Commutator of b and a
K_n	-	Complete graph with n vertices
$P_G(\Omega)$	-	Conjugation degree on a set
$\chi(\Gamma)$	-	Chromatic number
$\omega(\Gamma)$	-	Clique number
$d_\Gamma(v)$	-	Degree of a vertex v in Γ
D_3	-	Dihedral group of order six
$\overline{K}_n$	-	Empty graph with n vertices
K_e	-	Empty graph
G/N	-	Factor group
G	-	Finite group
$\alpha(\Gamma)$	-	Independence number
Γ_Ω^{GC}	-	Generalized commuting graph
$\Gamma_G^{\Omega_C}$	-	Generalized conjugacy class graph
Γ^{CO}	-	Generalized co-prime order graph
Γ^{NO}	-	Generalized non co-prime order graph
Γ_Ω^{GN}	-	Generalized non-commuting graph
Γ	-	A graph
$diam(\Gamma)$	-	The largest number of vertices which must be traversed in order to travel from one vertex to another
$girth(\Gamma)$	-	The length of a shortest cycle contained in Γ

$G\backslash Z(G)$	-	Non-central elements in the group G
Υ_G	-	Non-centralizer graph
K_0	-	Null graph
$C(G)\backslash A$	-	Number of proper centralizers in G
$\Omega\backslash A$	-	Number of non-central elements in Ω
$K(\Omega)$	-	Number of orbits in the set Ω
Γ_G^{Ω}	-	Orbit graph
Γ^{OC}	-	Order class graph
$\|G_1\| \cong \|G_2\|$	-	Order of G_1 and G_2 are isomorphic
$o(H)\|o(K)$	-	Order of H divides order of K
$\|\Omega\|$	-	Order of the set Ω
Γ^{OS}	-	Orderly set graph
$P_G(X)$	-	The probability an element chosen at random from G fixes an element chosen at random from X
$P_{A_G}(H, G)$	-	The probability of an automorphism fixes a subgroup element
$P_G(S)$	-	The probability that an element of a group fixes a set S
Ω	-	A set of an ordered pair of elements in the group with certain rules
$V(\Gamma)$	-	A set of vertices
$E(\Gamma)$	-	A set of edges
Γ_{sub}	-	A subgraph
S_3	-	Symmetric group of order six

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CHAPTER 1

INTRODUCTION

1.1 Introduction

The probability that two random elements from a group G commute is called the commutativity degree. The research on this topic has gathered various interests among researchers in the study of group theory and algebra. Hence, several extensions and generalizations of the commutativity degree have been introduced. One of the extensions is called the probability that an element of a group fixes a set, which was first introduced by Omer $et\ al.$ [1] in 2013. In this research, the probability mentioned is focused on only the conjugation action, and it is defined as the conjugation degree on a set. The conjugation degree on a set is computed for metacyclic p-groups, where p is equal to three and five.

In mathematics, specifically in graph theory, a graph can be presented whenever there exist points and lines. To put it simply, a graph consists of a set of objects or vertices which are connected by links or edges. Precisely, a graph, which is denoted as Γ, is a mathematical structure containing two sets, which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. This concept has been first introduced by Leonard Euler, a Swiss mathematicians on his attempt in solving the popular puzzle about bridges by drawing points and lines [2]. Since then, vast application of graph theory with mathematical problems, especially group theory has been conducted by various researchers. In this research, the obtained results from the conjugation degree on a set are applied into several graphs.

1.2 Background of the Research

As mentioned earlier, the probability that an element of a group fixes a set, from now on will be written as the conjugation degree on a set, is an extension of the concept of the commutativity degree of a group. The concept of the commutativity degree was first explained by Erdös and Turán [3] in one of their series on the statistical group theory. In their study, Erdös and Turán [3] also discussed on the commutativity degree for some symmetric groups. Later in 1973, Gustafson [4] gave the exact definition of the commutativity degree, named as the probability that two random elements commute. Gustafson [4] also showed that the commutativity degree can be computed by dividing the number of conjugacy classes with the size of the group and proved that $\text{Pr}(G) \leq \frac{5}{8}$.

In the early of 21st century, Puornaki and Sobhani [5] investigated the probability that the commutator of two randomly chosen elements in a group is equal to a given element in the same group. Nevertheless, this probability was also generalized by Alghamdi and Russo [6] in 2012 where the upper and lower bound for finite groups were determined. In addition to that, another study was done by Castelaz [7] on solvable and non-solvable groups where two upper bounds on the commutativity degree of non-solvable groups were considered. Later on, Barzgar [8] studied the probability that two subsets of a group commute where several results including lower and upper bounds were obtained. In conclusion, broad extensions on the concept of the commutativity degree which are related to different types of finite groups have been explored.

Throughout this research, another extension of the commutativity degree namely the probability that an element of a group fixes a set, denoted by $P_G(\Omega)$, which was first introduced by Omer $et\ al.$ [1] is considered. In their research, Omer $et\ al.$ [1] focused on the dihedral groups of order $2n$, where the elements of order two are considered. In this research, this probability is extended by finding the probability that an element of metacyclic p-groups fixes a set, where p is equal to 3 and 5, by using conjugation action on a set. Besides that, this probability is defined as the conjugation degree on a set, which focuses only on conjugation action of the group. Furthermore, the results on the conjugation degree are further investigated by connecting them with several graphs, where some algebraic properties of the graphs are determined.

1.3 Motivation of the Research

The study on the commutativity degree and its extensions have been of interest by many authors. In 1975, Sherman [9] introduced the probability of an automorphism of a finite group fixes an arbitary element in the group. In 2011, Moghaddam $et\ al.$ [10] extended the definition given by Sherman [9] and introduced the probability of an automorphism of a finite group fixes a subgroup element. Later in 2013, Omer $et\ al.$ [1] introduced another probability which is the probability that an element of a group fixes a set.

Previously, the probability that an element of a group fixes a set for some finite non-abelian groups which include metacyclic 2-groups, dihedral groups and quaternion groups has been determined. However, the probability that an element of a group fixes a set for metacyclic p-groups where p is an odd prime has not been found. A metacyclic group is a group where both its commutator subgroup and quotient group are cyclic. Since early of 70's, the classification of metacyclic group has been done by many authors. In 1973, King [11] gives a classification of finite metacyclic p-groups by using group-theoretic argument. In 2005, Beuerle [12] follows the classification given by King [11] and introduced a complete list of metacyclic p-groups of nilpotency class at least three which consists of exactly one representative for each isomorphism class. Recently in 2014, Basri [13] added extra conditions on the classification of metacyclic p-group where p is an odd prime given in Beuerle [12] . Therefore, in this research, the revised classification introduced by Basri has been used to determine the probability that an element of a group fixes a set for metacyclic p-groups where p is an odd prime, using conjugation action on a set. This probability has been re-defined as the conjugation on a set.

On the other hand, various studies have been done in relating the elements of groups with graph, namely the algebraic graph theory which can be found in [14–40]. The results on the conjugation degree on the set defined from the metacyclic p-groups can further be applied by connecting them with several graphs where some of the algebraic properties of the graphs can be determined.

1.4 Problem Statements

Previously, the probability that an element of a group fixes a set for metacyclic p-groups where p is an odd prime has not been found. Thus, this motivates the research in finding the probability that an element of a group fixes a set for metacyclic p-groups, where p is an odd prime by using the conjugation action on a set. In this research, this probability is defined as the conjugation degree on a set. Throughout this research, the set of the metacyclic p-groups, where p are the odd primes 3 and 5, denoted by Ω is restricted in the form of $(x, y) \in G \times G$ where $\text{lcm}(|x|, |y|) = p$, $xy = yx$, $x \neq y$, and $(y, x) \in G \times G$ is not included. In order to determine the probability, the size of the set Ω and the number of orbits of Ω are also determined.

In this research, the results on the conjugation degree on the set Ω, specifically the elements of Ω and their orbits are applied into four graphs, namely the orbit graph, the generalized conjugacy class graph, the generalized commuting graph and the generalized non-commuting graph, where some of their properties are also found. Moreover, four new graphs related to the conjugation degree on the set Ω are also introduced, namely the orderly set graph, the order class graph, the generalized co-prime order graph and the generalized non co-prime order graph.

1.5 Objectives of the Research

Let G be a metacyclic p-group, where p is an odd prime and $\Omega \subseteq G \times G$ such that $\Omega = \{(x,y) \in G \times G : \mathrm{lcm}(|x|,|y|) = p, xy = yx, x \neq y\} \setminus \{(x,y)\}$ The objectives of this research are:

(i) To determine the elements and the size of the set Ω.

(ii) To find the conjugation degree on the set Ω for metacyclic p-groups, where p is equal to 3 and 5, by following the restriction of the set Ω.

(iii) To apply the results in (i) and (ii) to graph theory, namely the orbit graph, the generalized conjugacy class graph, the generalized commuting graph and the generalized non-commuting graph.

(iv) To find several algebraic properties of the graphs in (iii) such as the degree of the vertices, the clique number, the chromatic number, the independent number, the girth and the diameter of the graph.

(v) To introduce four new graphs related to the set and conjugacy classes of metacyclic 3-groups and metacyclic 5-groups, as well as finding some of their algebraic properties.

1.6 Scope of the Research

This research has two parts which are group theory and graph theory. In the first part, namely group theory, an extension of the commutativity degree, which is the probability that an element of a group fixes a set, which is defined as the conjugation degree on a set, are determined. the group under consideration is a metacyclic p-group, G and the set $\Omega \subseteq G \times G$. This research is conducted by restricting the set Ω to be an ordered pair of elements in $G \times G$ of the form (x, y), with the condition of $\text{lcm}(|x|, |y|) = p$, $xy = yx$, $x \neq y$, and $(y, x) \in G \times G$ is excluded. Throughout this research, p is equal to 3 and 5, where the presentation of metacyclic p-groups, where p an odd prime is considered. In addition, this research is focusing only on the conjugation action on the set.

In the second part of this research, the results on the conjugation degree on the set Ω are applied to graph theory, namely the orbit graph, the generalized conjugacy class graph, the generalized commuting graph and the generalized non-commuting graph. Furthermore, the algebraic properties of these graphs namely the degree of the vertices, the clique number, the chromatic number, the independent number, the girth and the diameter of the graph, are also considered. In addition, four new graphs namely the orderly set graph, the order class graph, the generalized co-prime order graph and generalized non co-prime order graph are introduced, together with their algebraic properties mentioned earlier.

1.7 Significance of the Research

In other area, such as in telecommunications, specifically in the multi-antenna setting, some researchers found that metacyclic group is one the finite groups that is applicable in the analysis of the setting. Therefore, the study on finite groups, specifically metacyclic group has become of interest for group theorists. In this research, new theoretical results on metacyclic p-groups, focusing on the commutativity degree of the groups are provided. Eventually, this research provides theoretical results in terms of lemmas and theorems which can be applied in other related areas as well.

Extremal graph theory is a branch which connects mathematical field with graph theory. This branch studies the maximal and minimal graphs which satisfy certain properties, including the size, order and girth of the graph. Eventually, extremal graph theory is significance in other field such as coding theory and cryptography. In this research, the results on the conjugation degree on a set of metacyclic p-groups are applied into graph theory, where the degree of the vertex, the clique, chromatic and independent numbers, as well as girth and diameter of the graph are also determined. Consequently, this research is closely related to extremal graph theory, which can also be applied in coding theory and cryptography as well. Moreover, four new graphs related to the conjugation degree on a set are also introduced, including some of their algebraic properties.

1.8 Research Methodology

In this research, an extension of the commutativity degree of a group, defined as the conjugation degree on a set is explored. Throughout this research, the group G stands for the metacyclic p-group, where p are the odd primes 3 and 5. Meanwhile, the set considered is Ω, a set of ordered pair of elements in G of the form (x, y), such that the lcm$(|x|, |y|) = p, xy = yx, x \neq y$ and if $(x, y) \in \Omega$, then $(y, x) \notin \Omega$. In addition, the group action employed in this research is conjugation action. The computation of the conjugation degree on the set Ω is started by determining the order of each element in the group, whereby the elements that follow the restriction of the set Ω are considered. Throughout this research, the presentation of metacyclic p-groups, where p is an odd prime given by Basri [13] is referred. The presentation is categorized as Type 1 and Type 2. Since the value p is either 3 or 5, there are a total of four groups that are considered in this research, namely metacyclic 3-group of Type 1, metacyclic 3-group of Type 2, metacyclic 5-group of Type 1, and metacyclic 5-group of Type 2. Thereafter, the elements in the group are gathered together by following the restrictions in the set Ω. Once the set Ω has been determined, the computation of the orbits in the set Ω is conducted. Next, the number of orbits in the set Ω is determined, by following the size of each orbit. Subsequently, the conjugation degree on the set Ω is computed by dividing the number of orbits with the size of the set Ω.

In the second phase of the research, the results found based on the conjugation degree on the set Ω are applied into several graphs, namely the orbit graph, the generalized conjugacy class graph, the generalized commuting graph and the generalized non-commuting graph. By following the definition of each graph, the elements in the set Ω as well as the the orbits of Ω are considered to be the vertices, and they are connected by an edge based on certain rules. Furthermore, some algebraic properties of these graphs are determined including the degree of the vertices, the clique number, the chromatic number, the independent and dominating sets, the girth and the diameter of the graph. Lastly, four new graphs related to the set and conjugacy classes of metacyclic 3-groups and 5-groups are also introduced, including some of their algebraic properties. These graphs are named as the orderly set graph, the order class graph, the generalized co-prime order graph and the generalized non co-prime order graph. The overall research methodology is illustrated in Figure 1.1.

1.9 Thesis Organization

This thesis consists of six chapters. The first chapter, which is the Introduction consists of a brief overview of the thesis, which includes the introduction, the background of the research, research objectives, scope of the research, problem statements, significance of the research, research methodology as well as thesis organization.

Next, Chapter 2 covers on the basic concepts in group theory and graph theory which are related to this research. The research background and the literature review on the concept of the commutativity degree are also discussed. In addition, previous

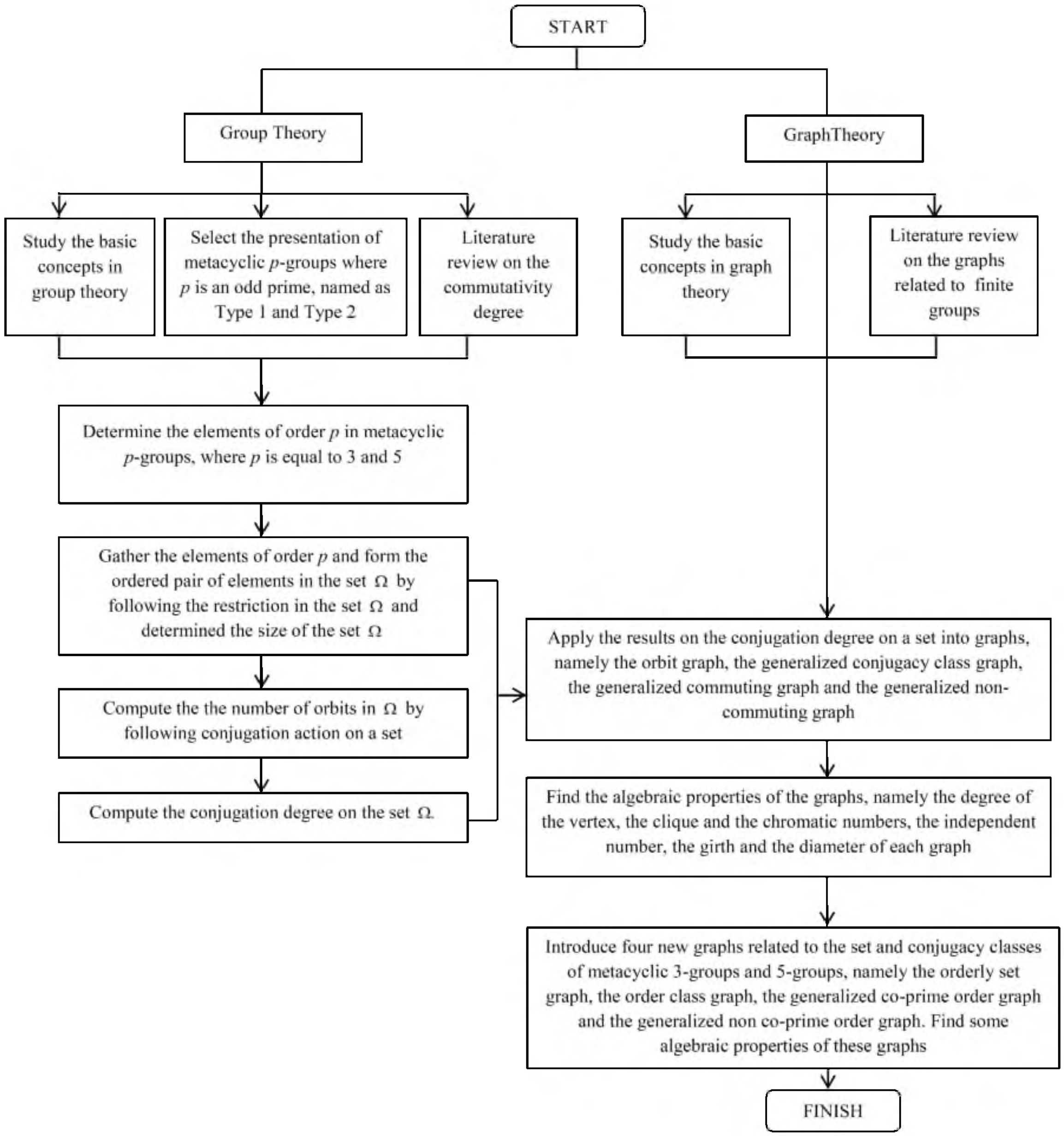

Figure 1.1 Research methodology

studies relating graphs with finite groups are also presented.

Chapter 3 highlights the results of this research on the conjugation degree on the set Ω for the metacyclic 3-groups and metacyclic 5-groups of Type 1 and Type 2. The computation of the conjugation degree is conducted by determining the order of each element in the group, which follows the order restriction of the set Ω, as well as the computation of the orbits of the set Ω. The results computed are then presented in the form of lemmas and theorems.

In Chapter 4, the results on the conjugation degree on the set Ω are applied into graph theory. The elements of the set Ω are presented as the vertices, in which they are joined by an edge according to certain rules applied in different type of graphs. These graphs include the orbit graph, the generalized conjugacy class graph, the generalized commuting graph and the generalized non-commuting graph. The results are discussed in the form of theorems, where several algebraic properties of these graphs are also determined.

In Chapter 5, four new graphs of metacyclic 3-groups and metacyclic 5-groups are introduced, named as the orderly set graph, the order class graph, the generalized co-prime order graph and the generalized non co-prime order graph. These four graphs are computed based on the order of the elements in the set Ω, as well as the conjugacy classes of the same set. In addition, some algebraic properties of these graphs are also presented.

Chapter 6, which is the last chapter of the thesis concludes the overall content of the whole thesis. In this chapter, the summary of the research is given, where several suggestions for future research on the conjugation degree on a set are also presented. Figure 1.2 illustrates the content of the whole thesis.

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