NEW METHODS OF COMPUTING THE PROJECTIVE POLYNOMIAL RESULTANT BASED ON DIXON, JOUANOLOU AND JACOBIAN MATRICES

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NEW METHODS OF COMPUTING THE PROJECTIVE POLYNOMIAL RESULTANT BASED ON DIXON, JOUANOLOU AND JACOBIAN MATRICES

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To my beloved mother late Fatima Abdulkarim, my respected father Mallam Sulaiman Abdullahi and my lovely children Abdullahi Surajo, Fatima Surajo and Zainab Surajo.

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ABSTRACT

In elimination theory, particularly when using the matrix method to compute multivariate resultant, the ultimate goal is to derive or construct techniques that give a resultant matrix that is of considerable size with simple entries. At the same time, the method should be able to produce no or less superfluous factors. In this thesis, three different techniques for computing the resultant matrix are presented, namely the Jouanolou-Jacobian method, the Dixon-Jouanolou methods for bivariate polynomials, and their generalizations to the multivariate case. The Dixon-Jouanolou method is proposed based on the existing Jouanolou matrix method which is subjected to bivariate systems. To further extend this method to multivariate systems, the entry formula for computing the Dixon resultant matrix is first generalized. This extended application of the loose entry formula leads to the possibility of generalizing the Dixon-Jouanolou method for the bivariate systems of three polynomials to systems of $n + 1$ polynomials with n variables. In order to implement the Dixon-Jouanolou method on systems of polynomials over the affine and projective space, respectively, the concept of pseudo-homogenization is introduced. Each space is subjected to its respective conditions; thus, pseudo-homogenization serves as a bridge between them by introducing an artificial variable. From the computing time analysis of the generalized loose entry formula used in the computation of the Dixon matrix entries, it is shown that the method of computing the Dixon matrix using this approach is efficient even without the application of parallel computations. These results show that the cost of computing the Dixon matrix can be reduced based on the number of additions and multiplications involved when applying the loose entry formula. These improvements can be more pronounced when parallel computations are applied. Further analyzing the results of the hybrid Dixon-Jouanolou construction and implementation, it is found that the Dixon-Jouanolou method had performed with less computational cost with cubic running time in comparison with the running time of the standard Dixon method which is quartic. Another independent construction produced in this thesis is the Jouanolou-Jacobian method which is an improvement of the existing Jacobian method since it avoids multi-polynomial divisions. The Jouanolou-Jacobian method is also able to produce a considerably smaller resultant matrix compared to the existing Jacobian method and is therefore less computationally expensive. Lastly all the proposed methods have considered a systematic way of detecting and removing extraneous factors during the computation of the resultant matrix whose determinant gives the polynomial resultant.

ABSTRAK

Dalam teori penghapusan, khususnya apabila menggunakan kaedah matriks untuk mengira hasilan multivariat, matlamat utama ialah untuk menerbitkan atau membina teknik yang boleh memberikan matriks hasilan yang mempunyai saiz yang bersesuaian dengan pemasukan unsur yang ringkas. Pada masa yang sama kaedah tersebut berupaya menghasilkan sedikit atau langsung tiada faktor lebihan. Dalam tesis ini tiga kaedah berbeza untuk mengira matriks hasilan dibentangkan iaitu kaedah Jouanolou-Jacobian, kaedah Dixon-Jouanolou bagi polinomial dua pembolehubah dan pengitlakannya kepada kes multivariat. Kaedah Dixon-Jouanolou dicadangkan berasaskan kepada kaedah matriks Jouanolou yang sedia ada tertakluk kepada sistem dua pembolehubah. Untuk melanjutkan kaedah ini kepada sistem multivariat, rumus pemasukan bagi pengiraan matriks hasilan Dixon terlebih dahulu diitlakkan. Dengan menggunakan rumus pemasukan unsur teritlak dapat membawa kepada lanjutan kaedah matriks hasilan Dixon-Jouanolou, daripada tertakluk kepada sistem tiga polinomial dengan dua pembolehubah kepada sistem $n + 1$ polinomial dengan n pembolehubah. Bagi melaksanakan kaedah Dixon-Jouanolou terhadap sistem polinomial masing-masing ke atas ruang afin dan ruang unjuran, konsep penghomogenan pseudo diperkenalkan. Setiap ruang ini tertakluk kepada syarat tertentu; oleh itu, penghomogenan pseudo ini menghubung kaitkan keduanya dengan memperkenalkan satu pembolehubah buatan. Analisis pengiraan masa bagi perlaksanaan rumus pemasukan unsur teritlak dalam pengiraan pemasukan matriks Dixon mendapati bahawa kaedah pengiraan matriks Dixon menggunakan pendekatan berkesan walaupun tidak melibatkan pengiraan selari. Keputusan ini menunjukkan bahawa kos pengiraan matriks Dixon dapat dikurangkan berdasarkan kepada bilangan operasi penambahan dan pendaraban yang dilakukan dalam rumus pemasukan unsur tersebut. Penambahbaikan ini akan lebih ketara apabila menggunakan pengiraan selari. Seterusnya, analisis keputusan perlaksanaan kaedah hibrid Dixon-Jouanolou menunjukkan bahawa kos pengiraan kaedah Dixon-Jouanolou adalah lebih baik, dengan masa perlaksanaan kubik berbanding dengan kos pengiraan matriks Dixon piawai yang mempunyai masa pengiraan kuartik. Satu lagi kaedah yang dihasilkan daripada tesis ini ialah kaedah Jouanolou-Jacobian yang bertujuan mengelakkan pengiraan pembeza melibatkan pembahagian multipolinomial. Kaedah baharu ini dapat menghasilkan saiz matriks hasilan yang jauh lebih kecil berbanding kaedah Jouanolou yang sedia ada dan oleh itu kos pengiraannya dapat dikurangkan. Akhir sekali, semua kaedah yang dicadangkan telah mempertimbangkan satu pendekatan sistematik yang berupaya mengesan dan menghapuskan faktor lebihan dalam pengiraan matriks hasilan di mana penentunya memberikan polinomial hasilan.

TABLE OF CONTENTS

CHAPTER	TITLE	PAGE
	DECLARATION	ii
	DEDICATION	iii
	ACKNOWLEDGEMENT	iv
	ABSTRACT	vi
	ABSTRAK	vii
	TABLE OF CONTENTS	viii
	LIST OF TABLES	xi
	LIST OF FIGURES	xii
	LIST OF ABBREVIATIONS	xiii
	LIST OF SYMBOLS	xiv
	LIST OF APPENDIX	xvi
1	**INTRODUCTION**	1
	1.1 Preface	1
	1.2 Research Background	2
	1.3 Statement of the Problem	7
	1.4 Objectives of the Study	8
	1.5 Scope of the Study	9
	1.6 Significance of the Study	9
	1.7 Thesis Organization	10
2	**LITERATURE REVIEW**	13
	2.1 Introduction	13

	2.2	Preliminary Definitions and Theorems	16
		2.2.1 Affine and Projective Space	18
	2.3	Matrix Method for Computing Resultant	26
		2.3.1 Sylvester Resultant	27
		2.3.2 Bezout's Resultant	29
		2.3.3 Multivariate Resultant	32
		2.3.4 Determinantal Formula	35
	2.4	Dixon Resultant	37
		2.4.1 Recursive Algorithm	45
		2.4.2 Transformation from Sylvester to Dixon matrix	46
		2.4.3 Entry Formula	52
	2.5	Macaulay Resultant	57
		2.5.1 Reduced and Non-reduced monomials	60
	2.6	Jouanolou Resultant	65
	2.7	Hybrid Resultant	72
	2.8	Conclusion	75
3	**METHODOLOGY**		76
	3.1	Introduction	76
	3.2	Assumptions	76
	3.3	Research Framework	77
		3.3.1 Stage One: Background of the Research	77
		3.3.2 Stage Two: Generalized Entry Formula	79
		3.3.3 Stage Three: New Hybrid Formulations	79
		3.3.4 Stage Four: Implementation and Complexity Analysis	83
		3.3.5 Stage Five: Summary and General Conclusions	83
	3.4	Computational Tools	83
	3.5	Conclusion	84
4	**JOUANOLOU-JACOBIAN FORMULATION**		85
	4.1	Introduction	85

	4.2	Jacobian Block	86
	4.3	Construction of Jouanolou-Jacobian method	88
	4.4	Complexity Analysis of Jouanolou-Jacobian Method	102
	4.5	Conclusion	108
5	**TWO DIXON-JOUANOLOU RESULTANT FORMULATION**		110
	5.1	Introduction	110
	5.2	Pseudo-Homogenization	111
	5.3	Dixon-Jouanolou Method	113
		5.3.1 Dixon-Jouanolou Method Type 1	113
		5.3.2 Elements of the Entry Formula	115
		5.3.3 Dixon-Jouanolou Method Type 2	120
		5.3.4 Extraneous Factors Reduction	123
		5.3.5 Complexity Analysis of Dixon-Jouanolou Method	124
	5.4	Conclusion	128
6	**GENERALIZED DIXON-JOUANOLOU FORMULATION**		129
	6.1	Introduction	129
	6.2	Generalized Dixon Resultant Matrix	130
	6.3	Generalize Entry Formula	135
	6.4	Generalized Dixon-Jouanolou Method	148
		6.4.1 Computational Cost of the Dixon-Jouanolou	149
	6.5	Conclusion	156
7	**CONCLUSION**		157
	7.1	Introduction	157
	7.2	Summary of the Thesis	158
	7.3	Conclusion	162
	7.4	Future directions	165
REFERENCES			**167**
Appendices A – B			175–176

LIST OF TABLES

TABLE NO.	TITLE	PAGE
2.1	Cost of computing Dixon matrix for bivariate system	51
2.2	Cost of computing the Dixon resultant matrix for system of $n+1$ polynomials in n variables	51
2.3	Summary of multivariate resultant formulations	73
4.1	Estimation of number of monomials in a homogeneous polynomial of degree m with n variables	103
4.2	Cost of computing Bezoutian and Jacobian matrix	107
4.3	Size comparison of some selected resultant formulations	108
5.1	Operational computing time for the standard Dixon method versus the new hybrid Dixon-Jouanolou method	127
6.1	The computational complexity of the generalized standard Dixon method verses the generalized Dixon-Jouanolou method	155

LIST OF FIGURES

FIGURE NO.	TITLE	PAGE
1.1	Usual routines when computing resultant	6
1.2	Thesis organization	12
2.1	Proposed and existing method of computing resultant	15
2.2	Support of the polynomials system	19
2.3	Minkowski Sum of A_0 and A_1	21
2.4	Different points at infinity	24
2.5	Horizon; the vanishing points of different sets of parallel lines at infinity	25
2.6	Projective Plane in 3-Dimensional Plane	25
2.7	Strophoid	33
2.8	Process of computing resultant polynomial	36
2.9	A figure showing the intersection of surfaces	58
2.10	Summary of the Existing Matrix Methods of Computing Resultant	74
3.1	Research Framework	78
3.2	Methodology of Computing Jouanolou-Jacobian Method	80
3.3	Methodology of Computing Dixon-Jouanolou Method type 1 and 2	82
4.1	Procedure for recursive polynomial division of multivariate polynomials using monomial ordering	106
5.1	Method of Computing the Hybrid Resultant Matrix	112

LIST OF ABBREVIATIONS

BKK	-	Bernstein Kouhnirenko Khovanskii
CAS	-	Computer Algebra System
CPU	-	Central Processing Unit
GCD	-	Greatest Common Divisor
GPS	-	Global Positioning System
LCM	-	Lowest Common Multiple
RSC	-	Rank Submatrix Computation
UFD	-	Unique Factorization Domain

LIST OF SYMBOLS

$V(F)$	-	Affine varieties of the system F
$Det(M)$	-	Determinant of the square matrix M
$\Delta_{i,j}$	-	Differentials
$\oplus$	-	Direct sum
$\theta(f_1, f_2, \ldots, f_{n+1})$	-	Dixon polynomial of the system $f_1, f_2, \ldots, f_{n+1}$
Θ	-	Dixon resultant matrix
$R_r(F)$	-	Dixon-Jouanolou resultant matrix
$D(f_1, f_2, \ldots, f_{n+1})$	-	Dixon resultant matrix of the system $f_1, f_2, \ldots, f_{n+1}$
$\xi_{a_1,a_2,\cdots,a_n,b_1,b_2,\cdots,b_n}$	-	Element of the loose entry formula
$D_{a,b,c,d}$	-	Entries of the Dixon matrix for bivariate system
$D_{m_1-1,\ldots,nm_n-1;nm_n-1,\ldots,m_n-1}$	-	Entries of the generalize Dixon matrix of the system F
$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$	-	Exponent vectors of the monomial x^α
$J_r(F)$	-	Jouanolou resultant matrix
$\mu(\mathbb{Q}_1, \mathbb{Q}_2, \ldots, \mathbb{Q}_n)$	-	Mixed volume of the polytopes $\mathbb{Q}_1, \mathbb{Q}_2, \ldots, \mathbb{Q}_n$
x^α	-	Monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ where $\alpha \in \mathbb{N}^n$
$N(F)$	-	Newton polytopes of the system F
$\mathbb{P}^n$	-	Projective space
$P(F)$	-	Projective varieties of the system F
$H_r(f_1, f_2, \ldots, f_n)$	-	Resultant matrix of the Dixon-Jouanolou method
$\mathbb{C}[x_1 x_1^{-1}, \ldots, x_n x_n^{-1}]$	-	Ring of Laurent polynomial over a field $\mathbb{C}$
$K[x_1, x_2, \ldots, x_n]$	-	Ring of polynomial over a field K
$\mathbb{C}$	-	Set of complex numbers

$\mathbb{Z}$	-	Set of integers
$\mathbb{N}$	-	Set of natural numbers
$\mathbb{R}$	-	Set of real numbers
$S(H_r(f_1, f_2, \ldots, f_n))$	-	Size of the Generalized Dixon-Jouanolou matrix
$S(R_r)$	-	Size of the Dixon-Jouanolou matrix
$A = \langle A_1, A_2, \ldots, A_{n+1} \rangle$	-	Support of the system F
$Syl(f_1, f_2, \ldots, f_n)$	-	Sylvester resultant matrix of the system $f_1, f_2, \ldots, f_n$
$F = \{f_1, f_2, \ldots, f_{n+1}\}$	-	System of polynomials

LIST OF APPENDIX

APPENDIX	TITLE	PAGE
A	List of Publications and submitted manuscripts	175
B	Polynomial and its coefficients	176

CHAPTER 1

INTRODUCTION

1.1 Preface

The role played by a system of polynomial equations in scientific research has a variety of applications in real life situations. For example, in modelling the components in computer-aided design represented by the Bezier Bernstein splines [1], detecting whether a moving robot will collide with an obstacle or not [2], designing curves and surfaces [3], differential elimination [4] and application of Global Positioning System (GPS) in geodesy and geoinformatics [5]. Another important application is the modelling of geometric and kinematics constraints where a well-constrained system of polynomials equations are used to represent the motion of a camera.

Dealing with the above-mentioned applications requires a technique of variables elimination. There are three powerful elimination techniques; Grobner basis [6, 7], set characteristics or Ritt-Wu method [8] and the resultant matrix method [9, 10]. Some of the disadvantages of both Grobner basis and Ritt-Wu as reported in [11, 12] are:

1. These methods require large storage capacity during the computations.
2. High computational complexity.

The matrix method for computing the resultant is a popular tool used in eliminating variables which reduces the system of polynomial equations into simpler forms. The resultant of a system of polynomial equations can be obtained from the determinant of the resultant matrix. The determinant of the resultant matrix is also referred to as the projection operator. Exact resultant can be achieved if the projection operator exactly equals the resultant. Otherwise, the projection operator consists of a product of polynomials which are multiples of the resultant. These other factors of the projection operator besides the resultant polynomial are called extraneous factors. The presence of extraneous factors in the resultant formulation gives rise to the problem of extracting the resultant polynomial from the determinant.

Much of the concern in researches related to multivariate polynomials resultant is to determine a method that can give exact resultant. In most cases, exact resultant only exist on certain classes of the generic system of polynomial equations and these conditions are determined and proven to give exact resultant. Besides finding methods that can produce a determinantal formula which can give exact resultant, a method that can reduce the presence of extraneous factors in the resultant matrix formulation reduces the complexity of the problem. It becomes the aim of this thesis to find new methods that can reduce the complexity of computing the resultant matrix and resultant polynomial.

The rest of this chapter is as follows. Section 1.2 gives the research background leading to the problem statement in Section 1.3. The objectives of the study are given in Section 1.4 followed by the scope of the study, the significance of the study and thesis organization.

1.2 Research Background

When dealing with systems of polynomials in more than one variable, there are basically two matrix base constructions which depend on the nature of the resultant matrix [13]. If each entry of the matrix is either the coefficient of one of the polynomials or zero, the matrix is regarded as Sylvester type [14]. Sometimes the entries of the resultant matrix are polynomials in terms of the coefficients of the given system of polynomial equations, such type is referred to as Cayley/Dixon type. Methods such as Macaulay, Jouanolou (Generalized Macaulay), Newton sparse, incremental and Salmon Jacobian which is also referred as Sturmfel resultant [5] are considered to be Sylvester type while Dixon is regarded as Cayley/Dixon type [15–17]. All Cayley/Dixon resultant matrix have complicated entries, but with relatively small matrix [18–23]. On the other hand, Sylvester type resultant matrix have simple entries with large size matrix [24, 25].

In a situation where the resultant matrix is constructed based on the two types of the constructions, such formulation is referred to as the hybrid resultant matrix [26]. The foundation work for hybrid resultant was first introduced in [27], derived for certain class of the multivariate polynomials of multi-graded type. Independently, in 1999 Chionh *et al.* in [28] had proposed another hybrid construction which possibly is the first construction that can be applied to a more general class of system of polynomials.

Apart from the classical hybrid resultant matrix, the sparse hybrid formulation was constructed, due to the frequent appearance of such systems in many engineering applications [29]. However, it is not clear whether or not the constructions can generate exact resultant. Another construction was given by [30] and unlike the work of [29], Khetan presents his formulation and computes the hybrid resultant matrix based on certain examples. His construction also only considers systems of polynomials with unmixed support and the size of the matrix can still be very large [30]. A complete implementation of the Sylvester-Bezout construction is given by Ahmad in [13] giving conditions that can give optimal resultant matrix and describes some limitations in the implementations.

Apart from the matrix method for computing resultant, the second most commonly used algorithmic method is the Ritt-Wu's approach introduced by Ritt in [31] and further improved by the Chinese mathematician Wu Wen-Tsün. The method has two important steps namely reduction to triangular form and successive pseudo-division [32, 33].

A triangular set of polynomials with almost the same set of common solutions as the original system of equations is defined as the characteristic set of a set of polynomials [8, 34]. Ritt presents the first algorithm to compute the characteristic set that was resurrected by Wu and Ritt respectively in [8, 31]. Characteristic sets are typically computed by eliminating variables sequentially in some predetermined order using successive pseudo-division of polynomials.

Ritt-Wu's method requires a large storage capacity during the computation. For example, Heymann's question can be resolved using the matrix method within 300 seconds, compared to almost 19 hours using characteristics set method [11, 18]. The implementation by Gao and Wang in [11] is carried out using SUN 4/470.

The Grobner basis of a polynomial ideal is a basis with many useful properties and provides answers to most of the theoretic questions about the ideals, such as ideal description and membership problem. The notion generalizes three well-known algorithms namely; Gaussian elimination algorithm, particularly reduced row echelon form for linear systems, the Euclidean algorithm for computing the greatest common divisor of both univariate and multivariate polynomials and lastly, the simplex algorithm for minimizing or maximizing linear and non-linear functions.

Buchberger's algorithms resolved the issue of the ideal membership using S$-$polynomial of $f_1, f_2 \cdots, f_n \in k[x_1, \ldots, x_n]$ which is defined to be $S(f_1, f_2) = \dfrac{\mathrm{x}^\alpha}{\mathrm{LT}(f_1)} f_1 - \dfrac{\mathrm{x}^\alpha}{\mathrm{LT}(f_2)} f_2$, where LT is the leading term of f_i and x^α is the least common multiple (LCM) of the leading monomials LM of f_1 and f_2 ($\mathrm{x}^\alpha = \mathrm{lcm}(\mathrm{LM}(f_1), \mathrm{LM}(f_2))$) [32, 35].

The first algorithm to compute the Grobner basis of an ideal is given by [7, 36] and since then, many efficient variations have been proposed. Along with other resultant methods, Grobner basis can be considered as an effective tool for solving a polynomial equation which also include finding the solutions of the system of polynomial equations, variables elimination and ideal membership problem. The approach of Grobner basis provides a criterion for which a polynomial must satisfy in order to be a member of a certain ideal.

The Grobner basis method can also be used in a variety of applications such as solving polynomials systems and implicitization of curves and surfaces. This method computes the exact resultant [18, 37]. However, the Grobner basis approach is not as simple as the matrix method and run out of time when the total degree is very large since it requires large storage capacity during the computations.

The Grobner basis method also is less effective, when computing the resultant of a polynomial system, for example, deriving the implicit equation of a bi cubic surface takes only 50 seconds using the matrix method, compared to almost 10,000 seconds using Grobner basis. In an implementation using SUN 4/470, sometimes the system runs out of memory before the computation ends [11, 18].

Another setback of the Grobner basis method reported by Zheng *et al.* [12] is that the approach fails to generate the implicit equation of some parametric equations with base points as given in Equation (1.1). On the other hand, the matrix method of computing the resultant is able to compute the implicit equation despite having these base points. For rational parametric equations defined as

$$x = \frac{x(s,t)}{w(s,t)}, \quad y = \frac{y(s,t)}{w(s,t)} \quad \text{and} \quad z = \frac{z(s,t)}{w(s,t)},$$

a base point is a value (s, t) for which $x(s, t) = y(s, t) = z(s, t) = w(s, t) = 0$. At this point the values x, y and z are not defined. Another implication of the base point is that, no matter what values the coefficients of the rational curves or surfaces will be, there is always a common solution at infinity.

$$F = \begin{cases} x(s,t) = 2t^3 + 4t^2 + 2t + 4st + s^2t + 2 + 3s + s^2 \\ y(s,t) = -2st^2 - 2t - st + 2 + s - 2s^2 - s^3 \\ z(s,t) = 2t^2 - 3st^2 - 2t - 3st - 2s^2t - 2s - 3s^2 - s^3 \\ w(s,t) = t^3 + t^2 - t + s^t - 1 - s + s^2 + s^3 \end{cases} \tag{1.1}$$

In the implementation of the Grobner basis method, some of the reasons for large storage requirement and the CPU time is the swell of intermediate system of equations encountered during the computation of the basis. These intermediate polynomials do not satisfy the requirement of the basis, thus, are not included in the resulting Grobner basis [38, 39].

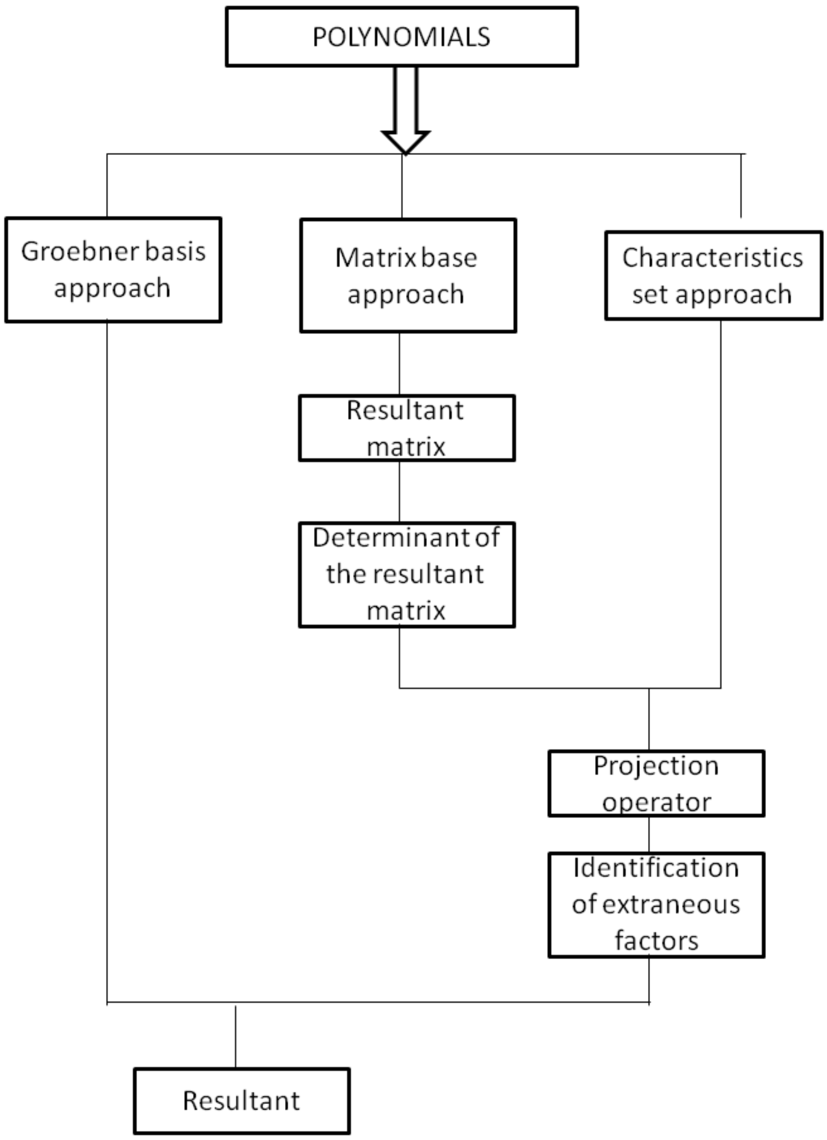

Figure 1.1 Usual routines when computing resultant

In an attempt to improve the effectiveness of the Grobner basis, several algorithms were introduced by different scholars such as signature base algorithm [40, 41], F4, F5, F5C algorithms among others [42–44]. Until today reducing the cost of computing Grobner bases remain an open area of research. Figure 1.1 shows the different techniques of computing resultant and how they are related.

Since both Ritt-Wu and Grobner basis techniques require large storage capacity and huge CPU time while computing the resultant polynomial, this work focuses on the matrix approach of computing the resultant. Existing methods are revisited giving emphasis on the method of construction, complexity, size of the matrix, nature of the entries, size of the unwanted factors and space requirement in the implementation of these methods.

1.3 Statement of the Problem

The resultant techniques for solving multivariate polynomial equations have received lots of attention with emphasis on eliminating or at least reducing the terms of the extraneous factors in the projection operator. This is because the presence of extraneous factors constitute to one of the biggest problem common to all matrix methods. These factors do not provide any information on the solutions of the polynomials; thus can be misleading and the process of identifying them is time-consuming.

Recent research on the resultant matrix methods focus on the hybrid resultant formulations. However, the existing hybrid resultant matrix methods either produce a large resultant matrix size or extraneous factors embedded in the projection operator [13, 29]. On the other hand, there exist hybrid resultant matrix that gives exact resultant [30], but the method is confined to certain class of polynomials. The Sylvester Bezout type resultant matrix is implemented by [13] and proven to produce an exact result, but under certain conditions, the method had failed to generate the desired Bezout block of the matrix.

Generally, for any given system of multivariate polynomials, none of the existing resultant matrix methods can give exact resultant. However, in some special cases, almost all existing method can produce exact resultant [45] which is due to the special structure of the Newton polytopes corresponding to the system. Among the factors that contribute to the effectiveness and the efficiency of the resultant matrix method is the nature of the matrix elements and the large matrix size. If the entries of the matrix are polynomials, the symbolic computation of the determinant will be more complex then if the entries are numerical values [46]. Therefore the nature of the matrix entries as well as the size of the matrix determine the efficiency of computing the resultant polynomial.

Several formulations have been given with notable improvements. Yet the problem of reducing the size of the resultant matrix and reducing or eliminating extraneous factors is still an open problem in the study of resultant. Thus, deriving or constructing a new hybrid resultant matrix with considerable size, that can eliminate, or at least reduce, the number of extraneous factors remains an important problem of research, which when solved adequately will produce positive dividends.

1.4 Objectives of the Study

Based on the formulated problem, the following research objectives are outlined:

1. To derive, construct and implement the Dixon-Jouanolou methods for bivariate systems of polynomial equations and Jouanolou-Jacobian method for n polynomial equations based on the Dixon, Jouanolou and Jacobian matrices.
2. To generalize the loose entry formula for computing the entries of the Dixon matrix and generalize the construction of Dixon-Jouanolou method to multivariate systems of $n+1$ polynomials with n variables, applying the generalize loose entry formula to compute the entries of the Dixon-Jouanolou matrix.
3. To determine the computational complexity of computing the Dixon-Jouanolou and Jouanolou-Jacobian matrices and compare with the complexity of computing the

Dixon and Jouanolou matrices respectively.

4. To determine the possible causes for the existence of extraneous factors and provide a suitable approach of eliminating them.

1.5 Scope of the Study

The research focuses on the construction of the hybrid resultant matrix methods for computing the resultant of a system of multivariate polynomial equation. The methods involved elimination theory, an area under algebraic geometry. The polynomials under consideration are assumed to be unmixed, generic and symbolic. Although, the new hybrid methods can handle n system of polynomials with n or $n-1$ variables, depending on the requirements of the method, the examples given in this thesis only include system of polynomials with at most four variables. Basic tools of algebraic geometry are applied in solving some problems encountered throughout this research. The computer algebra system Maple version 2015 is used to evaluate the resultant matrices.

1.6 Significance of the Study

So far most of the matrix-based elimination techniques fail to produce an exact resultant. Instead, these methods generate a polynomial called a projection operator which is a multiple of the resultant containing some unwanted factors which looks like an integral part of the resultant. For lower dimensional cases the approach of computing and extracting the resultant is well understood [47], but for higher dimensional cases the problem is still subjected to further research. The contribution of this work is to be able to produce new resultant matrix method that can eliminate or minimize the difficulties faced when extracting the resultant from the projection operator. This study will be beneficial to many industrial applications, in areas like computer-aided design, robot design and control, modeling of geometric object and many other applications within the scope of algebraic geometry.

1.7 Thesis Organization

Chapter 1 introduces the concept of polynomial resultant which begins with preface, research background, statement of the problem, objectives of the study, scope of the study and finally the significance of the study. This chapter provides the introduction to the research area and highlights some of the existing problems. This chapter served as introductory part of this research work.

Chapter 2 serves as the review of the existing literatures. Referring to Figure 1.2, this chapter contains eight sections which include introduction, preliminary definitions and theorems and the matrix methods for computing resultant. Others are Dixon resultant, Macaulay resultant, Jouanolou resultant and the hybrid resultants. This chapter highlights major setbacks of the existing classical and hybrid techniques of computing resultant. Based on these limitations, the research problem have been identified. Hence the new constructions presented in Chapter 4, 5 and 6 are designed to reduce the size of extraneous factors, space requirements and cost of computations. The eighth section concludes the chapter.

Chapter 3 presents the methodology of this research work. As described in Figure 1.2, this chapter contains five sections which include introduction, research assumptions, research framework and computational tools. Details of the three constructions are provided with explanation. The chapter describes how these methods are designed to produce relatively smaller resultant matrix. Finally, the fifth section concludes the methodology.

Chapter 4 presents the Jouanolou-Jacobian constructions, To provide a clear presentation, this chapter contains four sections. The first three sections are introduction, Jacobian block and construction of Jouanolou-Jacobian method. The fourth section presents the complexity analysis of the Jouanolou-Jacobian method. This complexity analysis provides a yardstick for comparison with the existing Jouanolou method to determine whether the Jouanolou-Jacobian technique is computationally expensive or not. Referring to Figure 1.2, the fifth section concludes the chapter.

Chapter 5 presents the Dixon-Jouanolou constructions of type 1 and 2. This chapter contains four sections which include introduction, pseudo-homogenization and Dixon-Jouanolou formulations for bivariate systems. The fourth section concludes the chapter. The concept of pseudo-homogenization allows the constructions to switch from a projective space to affine space using an artificial variable.

Chapter 6 presents the generalization of the Dixon-Jouanolou method, from the bivariate system to the system of $n + 1$ equations with n variables. The loose entry formula for computing the Dixon resultant matrix is generalized to the system of $n + 1$ equations with n variables. This allows the generalization of the Dixon-Jouanolou method. Figure 1.2 shows that this chapter contained five sections which include introduction, generalized Dixon resultant matrix, generalized entry formula and the generalized Dixon-Jouanolou method is presented followed by conclusion.

Chapter 7 presents the summary of the thesis and highlights how each of the objectives are achieved. This chapter also provide the direction for further research. These suggestions are derived from the conclusions of this chapter.

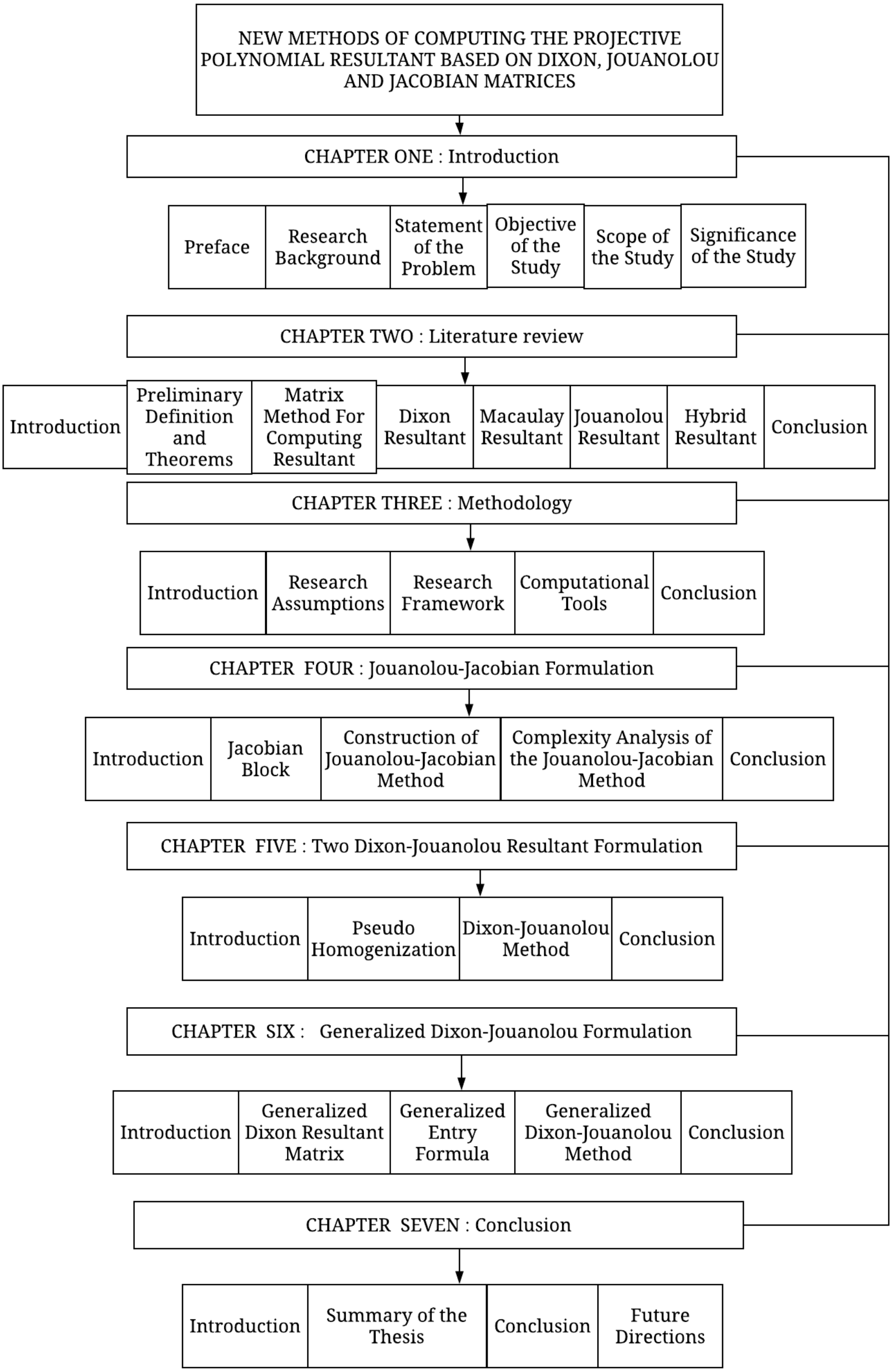

Figure 1.2 Thesis organization

REFERENCES

1. Lewis, R. H. and Stiller, P. F. Solving the recognition problem for six lines using the Dixon resultant. *Mathematics and Computers in Simulation*. 1999. 49(3): 205–219.

2. Yang, L., Fu, H. and Zeng, Z. A practical symbolic algorithm for the inverse kinematics of 6r manipulators with simple geometry. *Automated DeductionCADE-14*. 1997: 73–86.

3. Wang, X. and Chen, F. Implicitization, parameterization and singularity computation of steiner surfaces using moving surfaces. *Journal of Symbolic Computation*. 2012. 47(6): 733–750.

4. Yang, L., Zeng, Z. and Zhang, W. Differential elimination with Dixon resultants. *Applied Mathematics and Computation*. 2012. 218(21): 10679–10690.

5. Awange, J. L., Grafarend, E. W., Paláncz, B. and Zaletnyik, P. *Algebraic geodesy and geoinformatics*. Springer Science & Business Media. 2010.

6. Ahmad, S. N. and Aris, N. The Grobner package in maple and computer algebra system for solving multivariate polynomial equations. *Academic Journal UiTM Johor*. 2011.

7. Buchberger, B. Bruno Buchbergers Ph.D Thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal. *Journal of symbolic computation*. 2006. 41(3): 475–511.

8. Wu, W. Basic principles of mechanical theorem proving in elementary geometries. *Journal of Systems Sciences & Mathematical Sciences*. 1984. 4(3): 207–235.

9. Sylvester, J. J. On a theory of the syzygetic relations of two rational integral functions, comprising an application to the theory of sturm's functions, and that of the greatest

algebraical common measure. *Philosophical transactions of the Royal Society of London*. 1853. 143: 407–548.

10. Bézout, E. *Théorie générale des équations algébriques; par m. Bézout...* de l'imprimerie de Ph.-D. Pierres, rue S. Jacques. 1779.

11. Gao, X.-S. and Wang, D.-K. On the automatic derivation of a set of geometric formulae. *Journal of Geometry*. 1995. 53(1): 79–88.

12. Zheng, J., Sederberg, T. W., Chionh, E.-W. and Cox, D. A. Implicitizing rational surfaces with base points using the method of moving surfaces. *Contemporary Mathematics*. 2003. 334: 151–168.

13. Ahmad, S. N. *A Hybrid Resultant Matrix Algorithm Based on the Sylvester-Bezout Formulation*. Universiti Teknologi Malaysia: Ph.D. Thesis. 2016.

14. Ahmad, S. N. *et al.* Sylvester-type matrices for sparse resultants. *Malaysian Journal of Fundamental and Applied Sciences*. 2014. 6(1).

15. Karimisangdehi, S., Akbarpoor, S., Valipour, A., Khabiri, B. and Moosaviankhatir, S. Dixon resultant without extraneous factors. *International Journal of Mechatronics, Electrical and Computer Technology*. 2014: 2305–0543.

16. Mantzaflaris, A. and Tsigaridas, E. Resultants and discriminants for bivariate tensor-product polynomials. In *ISSAC 2017-International Symposium on Symbolic and Algebraic Computation*. 2017. 8.

17. Zippel, R. *Effective polynomial computation*. vol. 241. Springer Science & Business Media. 2012.

18. Karimisangdehi, S. *New Algorithms for Optimizing the Sizes of Dixon and Dixon Dialytic Matrices*. Universiti Teknologi Malaysia: Ph.D. Thesis. 2012.

19. Chtcherba, A. and Kapur, D. A complete analysis of resultants and extraneous factors for unmixed bivariate polynomial systems using the Dixon formulation. In *Proc. of 8th Rhine Workshop (RCWA02)*. 2002. 137–165.

20. Chtcherba, A. D. and Kapur, D. On the efficiency and optimality of Dixon-based resultant methods. In *Proceedings of the 2002 international symposium on Symbolic and algebraic computation*. ACM. 2002. 29–36.

21. Chtcherba, A. D. and Kapur, D. Resultants for unmixed bivariate polynomial systems produced using the Dixon formulation. *Journal of Symbolic Computation*. 2004. 38(2): 915–958.

22. Woody, H. Polynomial resultants. *GNU operating system*. 2016.

23. Qin, X., Wu, D., Tang, L. and Ji, Z. Complexity of constructing Dixon resultant matrix. *International Journal of Computer Mathematics*. 2017: 1–15.

24. Kalorkoti, K. On Macaulay form of the resultant. *School of Informatics, University of Edinburgh. Web*. 2016. 25.

25. Jeronimo, G. and Sabia, J. Sparse resultants and straight-line programs. *Journal of Symbolic Computation*. 2017.

26. Zhang, M., Chionh, E. and Goldman, R. Hybrid Dixon resultants. In *Proceedings of the Eighth IMA Conference on the Mathematics of Surfaces*. Citeseer. 1998. 193–212.

27. Weyman, J. and Zelevinsky, A. Determinantal formulas for multigraded resultants. *Journal of Algebraic Geometry*. 1994. 3(4): 569–598.

28. Chionh, E.-W., Zhang, M. and Goldman, R. *Transformation and transitions from the Sylvester to the Bézout resultant*. Technical Report. Citeseer. 1999.

29. DAndrea, C. and Emiris, I. Z. Hybrid sparse resultant matrices for bivariate polynomials. *Journal of Symbolic Computation*. 2002. 33(5): 587–608.

30. Khetan, A. The resultant of an unmixed bivariate system. *Journal of Symbolic Computation*. 2003. 36(3): 425–442.

31. Ritt, J. F. *Differential algebra*. vol. 33. American Mathematical Soc. 1950.

32. Cox, D., Little, J. and O'shea, D. *Ideals, varieties, and algorithms*. vol. 3. Springer. 1992.

33. Jin, M., Li, X. and Wang, D. A new algorithmic scheme for computing characteristic sets. *Journal of Symbolic Computation*. 2013. 50: 431–449.

34. Wu, W. and Gao, X. Mathematics mechanization and applications after thirty years. *Frontiers of Computer Science in China*. 2007. 1(1): 1–8.

35. Cox, D. A. Introduction to Gröbner bases. In *Proceedings of Symposia in Applied Mathematics*. American Mathematical Society. 1998. vol. 53. 1–24.

36. Buchberger, B. and Winkler, F. *Gröbner bases and applications*. vol. 251. Cambridge University Press. 1998.

37. Elkadi, M., Mourrain, B. and Piene, R. *Algebraic Geometry and Geometric Modeling*. Springer. 2006.

38. Neun, W. and Melenk, H. Very large Gröbner basis calculations. In *Computer Algebra and Parallelism*. Springer. 89–99. 1992.

39. Gao, S., Volny, F. and Wang, M. A new algorithm for computing Grobner bases. *IACR Cryptology ePrint Archive*. 2010. 2010: 641.

40. Eder, C. and Perry, J. E. Signature-based algorithms to compute Gröbner bases. In *Proceedings of the 36th international symposium on Symbolic and algebraic computation*. ACM. 2011. 99–106.

41. Sun, Y., Wang, D., Ma, X. and Zhang, Y. A signature-based algorithm for computing Gröbner bases in solvable polynomial algebras. In *Proceedings of the 37th International Symposium on Symbolic and Algebraic Computation*. ACM. 2012. 351–358.

42. Faugere, J.-C.A new efficient algorithm for computing Gröbner bases (f 4). *Journal of pure and applied algebra*. 1999. 139(1): 61–88.

43. Eder, C. and Perry, J. F5c: a variant of faugeres f5 algorithm with reduced Gröbner bases. *Journal of Symbolic Computation*. 2010. 45(12): 1442–1458.

44. Bardet, M., Faugere, J.-C. and Salvy, B. On the complexity of the f5 Gröbner basis algorithm. *Journal of Symbolic Computation*. 2015. 70: 49–70.

45. Kapur, D. and Saxena, T. Comparison of various multivariate resultant formulations. In *Proceedings of the 1995 international symposium on Symbolic and algebraic computation*. ACM. 1995. 187–194.

46. Chen, L. and Zeng, Z. Parallel computation of determinants of matrices with multivariate polynomial entries. *Science China Information Sciences*. 2013: 1–16.

47. Chtcherba, A. D. *A new Sylvester-type resultant method based on the Dixon-Bezout formulation*. University of New Mexico: Ph.D. Thesis. 2003.

48. Sederberg, T. W. *Computer-aided geometric design*. 2012.

49. Jia, X., Shi, X. and Chen, F. Survey on the theory and applications of μ-bases for rational curves and surfaces. *Journal of Computational and Applied Mathematics*. 2018. 329: 2–23.

50. Shen, L.-Y. and Goldman, R. Algorithms for computing strong μ-bases for rational tensor product surfaces. *Computer Aided Geometric Design*. 2017. 52: 48–62.

51. Li, Y. An effective hybrid algorithm for computing symbolic determinants. *Applied Mathematics and Computation*. 2009. 215(7): 2495–2501.

52. Emiris, I. Z. and Canny, J. F. Efficient incremental algorithms for the sparse resultant and the mixed volume. *Journal of Symbolic Computation*. 1995. 20(2): 117–149.

53. Wang, W. and Lian, X. Computations of multi-resultant with mechanization. *Applied mathematics and computation*. 2005. 170(1): 237–257.

54. Gelfand, I. M., Kapranov, M. and Zelevinsky, A. *Discriminants, resultants, and multidimensional determinants*. Springer Science & Business Media. 2008.

55. Sturmfels, B. *Solving systems of polynomial equations*. 97. American Mathematical Soc. 2002.

56. Cox, D. A., Little, J. and O'shea, D. *Using algebraic geometry*. vol. 185. Springer Science & Business Media. 2006.

57. Wildberger, N. J. *Divine Proportions: Rational trigonometry to universal geometry*. Wild Egg. 2005.

58. Aris, N. and Ahmad, S. N. The Sylvester methods of constructing resultant matrices. *J. Sains Malaysiana*. 2012.

59. Cayley, A. On the theory of elimination. *Cambridge and Dublin Math. J*. 1848. 3: 116–120.

60. Macaulay, F. Some formulae in elimination. *Proceedings of the London Mathematical Society*. 1902. 1(1): 3–27.

61. Kapur, D. and Lakshman, Y. N. Elimination methods: an introduction. symbolic and numerical computation for artificial intelligence b. donald et. al. 1992.

62. Ahmad, S. N. and Aris, N. Optimal Sylvester-type matrices for sparse resultants. *The 2nd International Conference on Applied Physics and Mathematics (ICAPM 2010)*. 2010.

63. Emiris, I. Z. and Mourrain, B. Matrices in elimination theory. *Journal of Symbolic Computation*. 1999. 28(1): 3–44.

64. Dixon, A. L. The eliminant of three quantics in two independent variables. *Proceedings of the London Mathematical Society*. 1909. 2(1): 49–69.

65. Kapur, D., Saxena, T. and Yang, L. Algebraic and geometric reasoning using Dixon resultants. In *Proceedings of the international symposium on Symbolic and algebraic computation*. ACM. 1994. 99–107.

66. Kapur, D. and Saxena, T. Extraneous factors in the Dixon resultant formulation. In *Proceedings of the 1997 international symposium on Symbolic and algebraic computation*. ACM. 1997. 141–148.

67. Ahmad, S. N. and Aris, N. The implementation of a hybrid resultant matrix formulation. In *AIP Conference Proceedings*. AIP Publishing. 2015. vol. 1682. 020015.

68. Chionh, E.-W., Zhang, M. and Goldman, R. N. The block structure of three Dixon resultants and their accompanying transformation matrices. In *J. of SymbolicComputation*. 1999.

69. D'Andrea, C. and Emiris, I. Z. Hybrid sparse resultant matrices for bivariate systems. In *Proceedings of the 2001 international symposium on Symbolic and algebraic computation*. ACM. 2001. 24–31.

70. Sturmfels, B. and Zelevinsky, A. Multigraded resultants of Sylvester type. *Journal of Algebra*. 1994. 163(1): 115–127.

71. Chionh, E.-W., Zhang, M. and Goldman, R. N. Implicitization by Dixon a-resultants. In *Geometric Modeling and Processing 2000. Theory and Applications. Proceedings*. IEEE. 2000. 310–318.

72. Mourrain, B. Computing the isolated roots by matrix methods. *Journal of Symbolic Computation*. 1998. 26(6): 715–738.

73. Fu, H., Wang, Y., Zhao, S. and Wang, Q. A recursive algorithm for constructing complicated Dixon matrices. *Applied Mathematics and Computation*. 2010. 217(6): 2595–2601.

74. Chionh, E.-W., Zhang, M. and Goldman, R. N. Fast computation of the Bezout and Dixon resultant matrices. *Journal of Symbolic Computation*. 2002. 33(1): 13–29.

75. Zhao, S. and Fu, H. An extended fast algorithm for constructing the Dixon resultant matrix. *Science in China Series A: Mathematics*. 2005. 48(1): 131–143.

76. Chionh, E.-W. Concise parallel Dixon determinant. *Computer Aided Geometric Design*. 1997. 14(6): 561–570.

77. Jónsson, G. and Vavasis, S. Accurate solution of polynomial equations using Macaulay resultant matrices. *Mathematics of computation*. 2005. 74(249): 221–262.

78. LeGrand, K. A., DeMars, K. J. and Darling, J. E. Solutions of multivariate polynomial system using Macaulay resultant expressions. In *paper AAS 2014–229*. 2014.

79. Li, Y. New method to extend Macaulay resultant. In *Intelligent Computation Technology and Automation, 2009. ICICTA'09. Second International Conference on*. IEEE. 2009. vol. 4. 562–565.

80. Stiller, P. An introduction to the theory of resultants. *Mathematics and Computer Science, T&M University, Texas, College Station, TX*. 1996.

81. D'Andrea, C. and Dickenstein, A. Explicit formulas for the multivariate resultant. *Journal of Pure and Applied Algebra*. 2001. 164(1): 59–86.

82. Jouanolou, J.-P. Formes d'inertie et résultant: un formulaire. *Advances in Mathematics*. 1997. 126(2): 119–250.

83. Szanto, A. Multivariate subresultants using jouanolous resultant matrices. *Preprint*. 2001.

84. Canny, J. and Emiris, I. An efficient algorithm for the sparse mixed resultant. In *International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes*. Springer. 1993. 89–104.

85. Canny, J. F. and Emiris, I. Z. A subdivision-based algorithm for the sparse resultant. *Journal of the ACM (JACM)*. 2000. 47(3): 417–451.

86. Salmon, G. *Lessons introductory to the modern higher algebra*. Hodges, Figgis, and Company. 1885.

87. Cohen, J. S. *Computer algebra and symbolic computation: Mathematical methods*. Universities Press. 2003.