FREE VIBRATION OF COMPOSITE LAMINATED RECTANGULAR PLATES UNDER SHEAR DEFORMATION THEORY

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ABSTRACT

In this study, free vibration of anti-symmetric angle-ply, and cross-ply laminated rectangular plates under first order shear deformation theory using clamped boundary condition is analysed. Two different numerical methods called Splines approximation and Radial Basis Functions (RBFs) are considered to approximate the functions. The fundamental frequency for anti-symmetric angle-ply plates are analysed with respect to aspect ratio, length-to-thickness ratio, ply angles and number of layers with different type of material arrangements. The problem of free vibration of cross-ply laminated plates are analysed for frequency parameter with respect to length-to-thickness ratio, aspect ratio, and number of layer with different disposition of materials. For both of the problems, the equations of motion derived using Yang Nooris and Stavsky (YNS) theory and the solution is assumed in separable form to obtain a coupled differential equations in term of displacement and rotational functions. These functions are approximated using cubic Spline function for the first case and the differential equations are then approximated using RBF for the second case. Preliminary studies on anti-symmetric angle-ply laminated plates with higher order shear deformation theory under simply supported boundary condition are studied using Spline method. The free vibration of anti-symmetric angle-ply laminated plates are analysed, under third order shear deformation theory using spline approximation. The equations of motions are derived using Reddy theory with third order shear deformation theory and the solution is assumed in separable form to obtain a coupled differential equations. The displacement and rotational functions are approximated using cubic and quantic Splines. These procedures produce a set of ordinary differential equation, along with boundary condition equations and become a generalized eigenvalue problem. The resulting eigenvalue problem is solved for the frequency parameter. The frequency parameter was analysed with respect to aspect ratio, length-to-thickness ratio, ply angles and number of layers with different type of material arrangements. The aim of this research is to provide the free vibration of layered plates for anti-symmetric angle-ply and cross-ply laminated plate with first order shear deformation theory with clamped boundary condition and identifies the difference between two different methods applied.

ABSTRAK

Dalam kajian ini, getaran bebas bagi plat segi empat tepat berlamina lapis-serong dan lapis silang antisimetri dengan teori ubah bentuk ricih peringkat pertama menggunakan syarat sempadan terkapit dianalisis. Dua kaedah berangka yang berbeza dikenali sebagai, fungsi penghampiran Spline dan Fungsi Asas Jejarian (RBF) dipertimbangkan sebagai penghampiran kepada fungsi. Frekuensi asas bagi plat lapis serong antisimetri, dianalisis terhadap nisbah aspek, nisbah sisi kepada ketebalan, sudut lapis dan bilangan lapisan dengan gabungan jenis bahan yang berbeza. Masalah getaran bebas plat berlamina lapis silang dianalisis untuk parameter frekuensi terhadap nisbah sisi kepada ketebalan, nisbah aspek, dan bilangan lapisan dengan susunan bahan yang berbeza. Bagi kedua-dua masalah, persamaan gerakan telah diterbitkan dengan menggunakan teori Yang Nooris dan Stavsky (YNS) dan penyelesaian diandaikan dalam bentuk bolehpisah untuk mendapatkan persamaan perbezaan berganding dalam sebutan fungsi anjakan dan putaran. Fungsi ini dihampirkan dengan menggunakan fungsi Spline kuasa tiga bagi kes pertama dan persamaan perbezaan telah dihampirkan dengan menggunakan fungsi RBF bagi kes kedua. Kajian permulaan ke atas plat berlamina lapis-serong antisimetri dengan menggunakan teori ubah bentuk ricih aras tinggi dengan syarat sempadan disokong mudah telah dikaji menggunakan kaedah Spline. Getaran bebas bagi plat berlamina lapis-serong antisimetri dianalisis dengan teori ubah bentuk ricih peringkat ketiga, dengan menggunakan penghampiran Spline. Persamaan gerakan diterbitkan dengan menggunakan teori Reddy dengan teori ubah bentuk ricih peringkat ketiga dan penyelesaian diandaikan dalam bentuk bolehpisah untuk mendapatkan persamaan perbezaan berganding. Fungsi anjakan dan putaran dihampirkan menggunakan fungsi Spline kuasa tiga dan kuasa lima. Prosedur ini menghasilkan satu set persamaan pembezaan biasa, berserta dengan persamaan syarat sempadan dan menjadi masalah nilaieigen teritlak. Masalah nilaieigen yang terhasil diselesaikan untuk parameter frekuensi. Parameter frekuensi dianalisis terhadap nisbah aspek, nisbah sisi kepada ketebalan, sudut lapis dan bilangan lapisan dengan gabungan jenis bahan yang berbeza. Tujuan kajian ini adalah untuk menyediakan nilai frekuensi bagi plat berlapis untuk plat berlamina lapis-serong dan lapis silang antisimetri dengan teori ubah bentuk ricih peringkat pertama dengan syarat sempadan terkapit dan mengenalpasti perbezaan diantara dua kaedah berbeza yang digunakan.

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LIST OF SYMBOLS

A_{ij}	-	Elastic coefficients representing the extensional rigidity
B_{ij}	-	Elastic coefficients representing the bending-stretching coupling rigidity
D_{ij}	-	Elastic coefficients representing the bending rigidity
E_x	-	Young's modulus along *x* direction
E_y	-	Young's modulus along *y* direction
H	-	Length-to-thickness ratio
$H(X - X_j)$	-	The Heaviside step function in plate
I_0	-	Normal inertia coefficient
I_2	-	
I_4	-	
I_6	-	
K	-	Shear correction factor
$L(x)$	-	Radial basis operator
$\left.\begin{matrix} M_x \\ M_y \\ M_{xy} \end{matrix}\right\}$	-	Moment resultants in respective direction of plates
$\left.\begin{matrix} N_x \\ N_y \\ N_{xy} \end{matrix}\right\}$	-	Stress resultants in respective direction of plates
N	-	Number of intervals of spline interpolation
Q_{ij}	-	Elements of the stiffness matrix for the material

$\bar{Q}_{ij}$	-	Elements of the transformed stiffness matrix for the material
$\left.\begin{matrix} Q_{xz} \\ Q_{yz} \end{matrix}\right\}$	-	Transverse shear resultant in the respective directions on plate
$S(x)$	-	Spline function
U, V, W	-	Displacement function in *x*, *y*, and *z* direction for plate
$\bar{U}, \bar{V}, \bar{W}$	-	Non-dimensionalised displacement function in *x*, *y*, and *z* direction for plate
X	-	Non-dimensionalised distance co-ordinate of plate
X_s	-	the equally spaced knot of spline
$\lVert X - X_j \rVert$	-	Euclidian norm
a	-	Length of the plates
b	-	Width of plates
$\left.\begin{matrix} a_i \\ c_i \\ e_i \\ g_i \\ l_i \end{matrix}\right\}, \left.\begin{matrix} b_j \\ d_j \\ f_j \\ p_j \\ q_j \end{matrix}\right\}$	-	Spline coefficients
$\left.\begin{matrix} a_j \\ b_j \\ c_j \\ d_j \\ e_j \end{matrix}\right\}$	-	Radial basis function coefficients
h	-	Thickness of plate
i, j, k	-	Summation or general indices
u, v, w	-	Displacement of plate in *x*, *y*, and *z* direction
u_0, v_0	-	The displacement reference to mid-plane
x	-	Length coordinate of the plate
y	-	Width coordinate of the plate
$\left.\begin{matrix} \varepsilon_x \\ \varepsilon_y \end{matrix}\right\}$	-	Normal strain in the respective directions

$\left.\begin{matrix}\gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{matrix}\right\}$	-	Shear strain in respective direction
λ	-	Non-dimensional frequency parameter
ϕ	-	Aspect ratio
ψ_x, ψ_y	-	Shear rotational of any point on the middle surface
Ψ_X, Ψ_Y	-	Shear rotational functional
$\bar{\Psi}_X, \bar{\Psi}_Y$	-	Non-dimensionalised shear rotational functional
$\left.\begin{matrix}\sigma_x\\ \sigma_y\end{matrix}\right\}$	-	Normal stress in respective directions
$\left.\begin{matrix}\tau_{xy}\\ \tau_{yz}\\ \tau_{xz}\end{matrix}\right\}$	-	Shear stress at a point on the reference surface of the plate

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LIST OF ABBREVIATIONS

AGE	-	AS4/3501-6 Graphite/epoxy
BC’s	-	Boundary Conditions
C-C	-	Both the *x*-axis ends are fully clamped
CPT	-	Classical Plate Theory
EGE	-	E-glass/epoxy
FSDT	-	First Order Shear Deformation Theory
HSDT	-	Higher Order Shear Deformation Theory
KGE	-	Kevler-49/epoxy
S-S	-	Both the *x*-axis ends are simply supported
YNS	-	Yang Nooris and Stavsky theory

CHAPTER 1

INTRODUCTION

1.1 Introduction

The history of composite material starts three decades ago, where the mixture of straw and mud are used to form bricks to make buildings and shelters. As the time passes the technology has developed and improvisation on mixture of composite plates are increasing, from straw and mud to glued laminated wood, laminated metal and fibre-glass for various purpose. Now the structure of plates are the centre of attraction for most of the contemporary engineers for being stiffer structural element in modern construction. Fields such as aerospace, automobile and shipbuilding are some industries that are widely using this because of composite plates have more desired damping and shock absorbing characteristic.

The natural frequency of the plates play an important role in constructions. The stability of the plates in construction based on the natural frequency of the plates. Free vibration of plates is, the vibration occurs in the absence of loads on the plates but implied some initial boundary conditions on plates. Therefore, natural frequency of the plate must be consider in order to construct any plate structure.

The combination of macroscopic structural unit of two or more separate materials called the composite materials. There are few advantages of composite material design, for instance they are resistant to high temperature, lower in weight and it has better damping and shock absorbing characters. This is because of the composite materials have structural unit, and each of the unit will have specific properties in comparison to their monolithic counterpart.

1.2 Structure of Plate

Nowadays most of the companies and engineers are focusing more on ratio of strength to weight or ratio of stiffness to weight, while designing any structures or object. As result of higher specific modulus and specific strength, the composite materials have higher shear modulus compared to metallic materials. This condition leads to the increased usage of laminated composite materials in engineering sector for the past few decades, and laminated composite materials have been widely used in engineering field due to higher specific modulus and specific strength.

Shear deformation plays an important role during the calculation of materials' frequencies. Shear deformation means the tendency of an element change its shape without change in length of the element. The change in angle at the corner of an original rectangular element called the shear strain while the ratio between shear stress and shear strain define as elasticity modulus of shear.

Most of the composite plates consist of multiple laminae or plies oriented in the desired direction. Hosokawa *et al.*, (2001), Hua *et al*.,(2001), mentioned that the ply orientation, geometric parameters and boundary conditions affect the natural frequency of the plate. Apart from that, Abrate (1994) mention that, by having the angle-ply laminations for the composite plates, the design can reach its optimization and it contribute to adjust the natural frequency of the structure.

There are two methods are adopted in this study, Spline and Radial Basis Function (RBF). Spline functions are fast convergence function and it successfully applied to boundary value problems. The existing spline method already analysed by Viswanathan and Kim (2008), Viswanathan and Lee (2007), Javed *et al* (2016), Viswanathan *et al* (2016) and Viswanathan (2016). The RBF is a new method, which is widely used by Ferreira *et al* (2003), Ferreira *et al* (2004), and Ferreira *et al* (2010) to solve the plate problems. In this study, the work is based on comparing the two numerical approximations. The problems on anti-symmetric angle-ply and cross-ply plates under simply supported (S-S) boundary conditions already done by Viswanathan and Kim (2008) and Viswanathan and Lee (2007). In the present work anti-symmetric angle-ply and cross-ply plates under clamped-clamped (C-C) boundary conditions are analysed using Spline approximations and also motivated to compare the results to one of the approximation called Radial Basis Function (RBF) approximations.

In this research, angle-ply and cross-ply laminated plate are used. Angle-ply laminates are laminates, which have ply angles other than 90˚, and cross-ply laminated which contain ply angles 0˚ and 90˚ only. There two types of boundary condition used in this research, one clamped and another boundary condition is simply supported. For all the problem both end the of *y*-axis are considered as simply-supported, and the *x*-axis are considered as clamped-clamped (C-C) and simply supported (S-S) boundary conditions, and the equation of motion are derived under first order shear deformation theory (FSDT) and higher order shear deformation theory (HSDT). The frequency parameters are analysed using two different numerical methods, Spline and Radial Basis Function (RBF) with respect to various parameters. The frequency parameters obtained by using Spline functions subtracted with respective to the frequency parameter obtained by RBF method. The differences in these two values are recorded and analysed the maximum difference between frequency parameter obtained using these two methods.

1.3 Problem Statement

There are numerous studies that have been carried out on composite structure in line with increase of its applications in various industries. A study is needed to produce better approximation method to compute frequency parameter for laminated plates. It is needed to fulfil the results of the current study, there are some essential results to be carried out for further analysis in composite structure. Relative layers, aspect ratio, side-to-thickness ratio, ply angles and material properties with boundary conditions affect the strength and stiffness of the structural composite materials. Therefore, this work shows the latest result on further analysis on anti-symmetric angle-ply, cross-ply laminated rectangular plates under first and higher order shear deformation. The free vibration of anti-symmetric angle-ply, cross-ply laminated plates under first order shear deformation and free vibration of anti-symmetric angles-ply laminated plate under third order shear deformation theory problems are studied. The frequency parameter need to be identify by using two different methods with clamped-clamped and simply supported boundary conditions. The differences between frequency parameters obtained using these two methods are calculated, and identify the maximum differences between frequency parameter obtained using Spline approximation and RBF approximation.

1.4 Objectives

1. To compare the differences between frequency parameters obtained using spline approximation and radial basis function method.
2. To study the frequency parameter for various fixed parameters for angle-ply under first order shear deformation theory.
3. To study the frequency parameter for various fixed parameters for cross-ply under first order shear deformation theory.
4. To extend the angle-ply laminated plate problem for higher order shear deformation theory.

1.5 Scope of the Study

This study focused on anti-symmetric laminated angle-ply and cross-ply plates under first order shear deformation theory. Two different numerical methods, Spline approximation and Radial Basis Function (RBF) approximation are adopted and obtain the frequency parameters. The displacement and rotation functions are obtained from governing equation is approximated using spline functions for first case and the differential functions are approximated using RBF for the second case, and reduce to a system of homogeneous simultaneous algebraic equations by imposed the boundary conditions. Now this becomes as a generalized eigenvalue problem, which can be solved to obtain eigenfrequencies as well as corresponding eigenvectors. In our problem, we analyse the fundamental frequencies of the composite rectangular plates. Preliminary study for a new problem also focused in this study. Here the angle-ply plate problem extended under third order shear deformation theory and preliminary studies conducted for anti-symmetry angle-ply laminated plates under simply supported boundary conditions.

1.6 Significance of the Study

In this research, the vibration of cross-ply and angle-ply plates are analysed under the first order shear deformation theory using Spline method and Radial Basis Function method. The main purpose of this study is to compare the frequency parameters obtained using two different numerical methods, Spline and Radial Basis Functions with respect to the various parameters. Then the problem is extended for higher order shear deformation theory and analysed using Spline approximation under simply supported (S-S) boundary conditions. The fundamental frequency parameter is analysed with respect to the aspect-ratio, side-to-thickness ratio, ply angels, number of layers and material properties with clamped-clamped and simply supported boundary conditions. Hence it can be helpful for designers to analyse the problem for various parameters and boundary conditions.

1.7 Definitions and Preliminaries

1.7.1 The Classical Plate Theory (CPT)

Ventsel and Krauthammer (2001) mention that Euler proposed membrane theory for thin plate. Furthermore, he solved some problems, which includes rectangular, triangular and circular elastic membrane. Bernoulli, was the student of

Euler, who successfully expanded Euler's analogy by replacing the string net with a grid work of beam and focused mainly on bending rigidity, he found the results for both the experimental, and theoretical values were not close to each other.

Ventsel and Krauthammer (2001) said that Kirchhoff derived the same differential equation as Navier, yet Kirchhoff used a different technique, where he successfully introduced some boundary forces to the plates. He also derived the equation based on Bernoulli's beam hypothesis. He proposed two basic assumptions, which known as Kirchhoff's hypothesis and it is useful to solve 3-D into bending plate theory. The Kirchhoff's assumptions are as follows;

a) The material obey Hooke's law.
b) The plate is initially flat.
c) When bending, the middle surface remains unstrained.
d) The thickness of the plate is small compare to the other dimensions.

Ventsel and Krauthammer (2001) mention that, Love extended the Kirchhoff plate theory related to 2-D problems of plates. Kirchhoff ignored the shear deformation, but Reissner and Mindlin had come out with a new theory for moderate thick plate that includes shear deformation as well.

1.7.2 Shear Deformation Theory

1.7.2.1 First Order Shear Deformation Theory (FSDT)

In the year 1951, Mindlin proposed that the middle surface of the plate will remain straight and not necessarily be perpendicular to the middle surface. Moreover, the Mindlin mentioned that, there is a displacement across the thickness yet there is no change in the thickness during the shear deformation; of the plate and the normal stress through the thickness is not considered. Therefore, Reissner assumed that the bending stress is linear and shear stress is quadratic through the thickness. "Reissner

– Mindlin" theory allows to calculate the deformation and stress of the thick plate moderately. This theory known as the first order shear deformation theory (FSDT).

1.7.2.2 Higher Order Shear Deformation Theory (HSDT)

The FSDT is limited for linear distribution of transverse shear strain along the thickness of plate. Hence, researchers develop higher order shear deformation theory. To illustrate the nonlinear deformation, Levinson (1980) developed a model with the third order polynomial, which excluded the shear correction factor. In 1984, Reddy comes out with higher order in plane displacement with constant deflection and known as simple higher order shear deformation theory. Reddy (1984) introduced nonlinear (cubic) shear deformation theory for flat plates.

1.7.3 Laminated Theories

Lamina is a basic building block of a composite structure that usually consists of one of the fibre configurations. However, the unidirectional reinforced or unidirectional lamina with an arrangement of parallel, continuous fibres is the most appropriate starting point for the purpose of mechanics analysis.

Orientations of lamina determined the type of material, which differentiated between isotropic and anisotropic. Figure 2.1 depicts the lamina of four plies in x, y, and z directions.

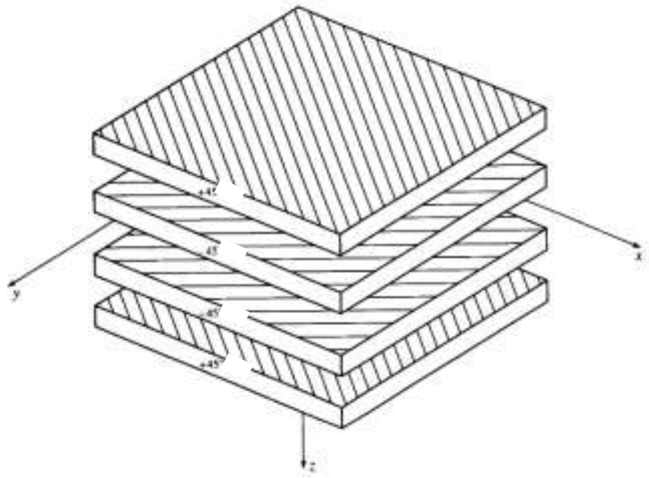

Figure 1.1: Lamina of four plies (Ronald,1994)

Lamination is combination the best aspects of the constituent layers in order to achieve a more useful material. Two or more laminae will form laminated which bonded together as an integral structural element (Ye, 2003). The capability to the structure and orient material layers in a prescribed sequence leads to several particularly significant advantages for composite materials compared with conventional monolithic materials. There are few type of laminates such as angle-ply, cross-ply and unidirectional. Angle-ply laminates have lamina orientations of either +θ or –θ with $0^{\circ} \leq \theta \leq 90^{\circ}$, whereas the cross-ply laminates are oriented at either $\theta = 0^{\circ}$ and $\theta = 90^{\circ}$. In Figure 2.2, the laminates for angle-ply and cross-ply were explained.

(a)	(b)
+45°	90°
-45°	0°
+45°	90°
-45°	0°
+45°	90°
-45°	0°

Figure 1.2: Examples of (a) angle-ply and (b) cross-ply anti-symmetric laminates

The initiation of laminated structures have a close connection with the shear deformable theories. Since composite materials have high ratio of in-plane Young's moduli to transverse shear moduli, shear deformation effects are included as well.

This theory originally proposed for the first time by Stavsky (1965), for laminated isotropic plates. He applied Reissner-Mindlin theory for layered plates, which was later generalized by Yang *et al.* (1966), to laminate anisotropic plates also known as YNS (Yang Nooris and Stavsky) theory. Pagano (1970), evaluated laminated plates theories. At the same time, Whitney and Pagano (1970) presented the first application of YNS theory for symmetric and anti-symmetric rectangular plates.

A detail overview on composite laminates presented by George (1999), and Soedel (1993, 2004) explained the vibration of plates comprehensively. Recently, composite laminated structures focused more in detail by Autar (2006), and Reddy (2004, 2007). Laminated structures mostly used in various types of engineering industries.

1.7.4 Splines

Schoenberg and Whitney (1953) introduce spline functions. The attention of researchers increased by the year 1960. Spline composed of weight attached to a flat surface at the points to be connected and a flexible strip then bent across each of these weights, drawing in a smooth curve. Homogeneously, mathematically the spline could be taken into account the points in this case are numerical data and the weights are the coefficients on the cubic polynomials used to insert the data. The data with uneven response, the coefficients bend the line so that it passes through each point.

A spline function is a complex curve; it consists of a number of polynomial arcs of a given degree pieced. Greville (1967) mention that, a function of S(*x*) with degree *m* has points of knots under a sequence of real numbers, $x_0, x_1, \ldots\ldots, x_N$, satisfies the properties:

(i) Each sub-interval $\left(x_i + x_{i+1}\right)$, i = 0, 1, 2,…., N, S(x) is given by some polynomial of degree m or less.

(ii) S(x) and its derivatives of all order up to (m-1) are continuous, for m > 0.

Holladay theorem mention that, cubic spline fulfils the minimal curvature properties, where all functions of $f\left(x_i = y_i\right), i = 0,1,....,N$, the spline function $S_{\Delta}^{''}\left(f;a\right) = S_{\Delta}^{''}\left(f;b\right) = 0$ minimizes the integral;

$$\int_a^b \left| f^{''}\left(x\right)\right|^2 dx$$

Spline function is the truncated power function x_+^m , defined by;

$$x_+^m = \begin{cases} x^m, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

For m = 0 the function defines as Heaviside Unit function.

Greville (1967), represent spline function S(x) of degree m and knot sequence: $x_0, x_1, ..., x_N$ as follow,

$$S\left(x\right) = P_m\left(x\right) + \sum_{j=0}^{N} C_j \left(x - x_j\right)_+^m$$

An even degree spline or a parabolic spline, is very limited in its scientific applications. Hence, Ahlberg *et al*. (1967) used the point's collocation between the knot points. This is always a disadvantage because polynomial splines of even degree, interposing to prescribed function at mesh points may not exist as shown by Ahlberg *et al.* (1967).

1.7.4.1 Splines of Odd Degree

A piecewise polynomial of odd degree, (*2K+1*) which is continuously differentiable up to the even order, *2K*, with mesh points;

$$\Delta : a = x_0 < ... < x_N = b$$

and a corresponding set of ordinates;

$$Y : y_0, y_1, ..., y_N$$

A function, $S^{2k+1}\Delta(Y;x)$, can be represented as $S(x)$ when there is no ambiguity constructed satisfying the following properties. An odd order spline function with respect to the mesh Δ, satisfies the following properties;

i) $S(x_i) = y_1, i = 0,1,2,..., N$

ii) $S(x)$ is a polynomial of odd degree in each sub-interval $(x_i + x_{i+1}), i$ = 0, 1, 2,…., (*N*-*1*), and

iii) $S(x) \in C^{2k}[x_0, x_N]$

The spline function is cubic when *k* = 1, and quantic when *k* =2.

A cubic spline of the above interpretation can be form starting from the fact that, since S(*x*) is a cubic $S''(x)$ is linear. $S'''(x)$ correlate to the moment fact $M(x)$ in the draftsman's spline, which can be held by simple supports at the knots, therefore making the variation of the moment linear, afterward one appeared at the recurrence relation.

$$h_{i-1}M_{i-1}+2\left(h_{i-1}+h_i\right)M_i+h_iM_{i+1}=6\frac{y_{i+1}-y_i}{h_i}-\frac{y_i-y_{i-1}}{h_i}\qquad \left(i=1,2,...,N-1\right) \quad (1.1)$$

where $h_i = x_{i+1} - x_i$

The phrase $S'\left(x\right)$, which corresponds to the slope m of the draftman's spline, one can get the recurrence relation

$$\frac{m_{i-1}}{h_{i-1}}+2\frac{1}{h_i}+\frac{1}{h_{i-1}}m_i+\frac{m_{i+1}}{h_i}=3\frac{y_i-y_{i-1}}{h_{i-1}^2}+\frac{y_{i+1}-y_i}{h_i^2}\qquad \left(i=1,2,....,N-1\right) \quad (1.2)$$

Whenever the knots are equally separated, given by $x_i = x_0 + ih,\ \ i = 0,1,...,\mathrm{N}$ equation (1.1) and equation (1.2) respectively became

$$M_{i-1}+4M_i+M_{i+1}=\frac{6}{h^2}\left(\mathrm{y}_{i+1}-\mathrm{y}_{i-1}\right) \quad (1.3)$$

and

$$m_{i-1}+4m_i+m_{i+1}=\frac{3}{h}\left(\mathrm{y}_{i+1}-\mathrm{y}_{i-1}\right)\qquad \left(i=1,2,....,N-1\right) \quad (1.4)$$

Equation (1.3) or equation (1.4) consists of set of (*N-1*) equations in (*N+1*) in knowns. Meanwhile, in order to determine the interpolating spline uniquely supplied by the end, two additional conditions are required.

In case the end moments are set equal to zero, i.e. if $M_0 = M_n = 0$, the resulting spline is called a natural spline. If the end slopes m_0 and m_n are authorised, the resulting spline is a D_1 spline. A spline with authorised end moments is called a D_2 spline. From the recurrence relations of equation (2.3) and equation (2.4), the orders of error occur in the cubic spline derivatives can be formally shown to be given by, (Spath, 1969);

$$y_i' = S_i' + O\left(h^4\right) \tag{1.5}$$

$$y_i'' = S_i'' + \frac{h^2}{12} y_i^{iv} + O\left(h^4\right) \tag{1.6}$$

$$y_i = \frac{1}{2}\left(S_i + S_{i-1}\right) + O\left(h^2\right) \tag{1.7}$$

and

$$y_i^{iv} = \frac{1}{h}\left(S_i + S_{i-1}\right) + O\left(h^4\right) \tag{1.8}$$

Approximations for y_i', y_i'', and y_i^{iv} can be done more precisely than for y_i.

Spath (1969) proved that a quantic spline could be construct over the mesh points Δ with the piecewise quantic polynomials:

$$S(x) = A_j\left(x - x_j\right)^5 + B_j\left(x - x_j\right)^4 + C_j\left(x - x_j\right)^3 + D_j\left(x - x_j\right)^2 + E_j\left(x - x_j\right) + F_j \tag{1.9}$$

where $A_j, B_j, C_j, D_j, E_j,$ and F_j are spline coefficients.

A recurrence relation that results is

$$D_{i-1} + 4D_i + D_{i+1} = \frac{10}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right) + \frac{4}{h}\left(y_{j+1}' - y_{j-1}'\right) \qquad \left(i = 1, 2, ..., N-1\right)$$
(1.10)

The recurrence relation it can be show that,

$$y_j'' = S_j'' + O\left(h^4\right) \tag{1.11}$$

$$y_j = S_j + O\left(h^4\right) \tag{1.12}$$

$$y_j^{iv} = \frac{1}{h}\left(S_{j^+}^{iv} + S_{j^-}^{iv}\right) + \frac{h^2}{12} y_j^{iv} + O\left(h^4\right) \tag{1.13}$$

These show that the quantic spline approximations to the second and third derivatives are more efficient than that of the fourth derivatives.

The analysis of errors shows that the quantic spline method provides quick convergences – a condition that contributes to its success. Another plus point of this technique, that is in starting with $S\left(x\right)$ is that it provides the values of y_j as well as those of its derivatives, which are often required in physical problems.

The convergences of spline approximations $S_\Delta^{(\alpha)}\left(f,x\right)$ to the approximated function $f^{(\alpha)}\left(x\right)$ as the mesh norm $\Delta = \max\left|x_i - x_{i-1}\right|$ approaches zero, has been examined by many authors. Walsh, Ahlberg and Nilson (1963) showed that if $f\left(x\right) \in C^2\left[a,b\right]$, then, for cubic splines of interpolation to $f\left(x\right)$ at the mesh points, $S_\Delta^{(\alpha)}\left(f,x\right)$ converges uniformly to $f^{(\alpha)}\left(x\right)$ for $\alpha = 0,1$ Ahlberg and Nilson (1963) investigated the Converges of polynomial splines of odd degree.

1.7.4.2 Bickley Spline

Bickley spline used to analyse linear differential equations very precisely. Although the method presented by Schoenberg (1946) and Bickley (1968) were similar but Bickley presented a practical and efficient technique to study two-point boundary value problems for the first time. Bickley's technique yields quick convergence and better accuracy for lower- order approximation as compared to a global higher order approximation. Bickley formulate his cubic spline over the mesh $\Delta : x_0 < x_1 < ... < x_N$ as follows,

$$y^*(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \sum_{j=0}^{N-1} b_j (x - x_0)^3 H(x - x_j) \qquad (1.14)$$

where H is the Heaviside step function,

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0 \end{cases} \qquad (1.15)$$

and for third derivation of $y^*(x) \in C^2[x_0, x_N]$ it's discontinuous.

The number of unknown in $y^*(x)$ is (N+3) and the number of knots in $[x_0, x_N]$ is (N+1), including the end points.

If $y^*(x)$ approximates the solution of the boundary value problem

$$p(x)y'(x) + q(x)t' + r(x)y = s(x) \qquad (1.16)$$

$$\alpha_0 y + \beta_0 y' = y_0 \quad \text{at } x = x_0 \qquad (1.17)$$

$$\alpha_N y + \beta_N y' = y_N \quad \text{at } x = x_N \qquad (1.18)$$

Moreover, we require that $y^*(x) = y(x)$ at the knots x_i, for $i = 0, 1, ..., N.$ The compensation of the differential equation by collection at these points givens (N+1) equation in the spline coefficients equation (1.17), the total number of these resulting equations is (N+3), equal to the number of spline coefficients. Where this condition is against with the fact that in conventional cubic spline interpolation, there would yet remain two degrees of freedom.

The equations will be obtained would be

$$a_0 r_k + a_1\left[r_k\left(x_k - x_0\right) + q_k\right] + a_2\left[r_k\left(x_k - x_0\right)^2 + 2q_k\left(x_k - x_0\right) + 2p_k\right] +$$
$$\sum_{j=0}^{k-1} b_j\left[r_k\left(x_k - x_j\right)^3 + 3q_k\left(x_k - x_j\right)^2 + 6p_k\left(x_k - x_j\right)\right] = s_k \qquad \left(k = 1, 2, ..., N\right) \qquad (1.19)$$

and the pair

$$\alpha_0 a_0 + \beta_0 a_1 = \gamma_0 \qquad (1.20)$$

$$\alpha_0 a_0 + \alpha_1\left[a_N\left(x_N - x_0\right) + \beta_N\right] + \alpha_2\left[a_N\left(x_N - x_0\right)^2 + 2\beta_N\left(x_N - x_0\right)\right] +$$
$$\sum_{j=0}^{k-1} a_N\left(x_N - x_j\right)^3 + 3\beta_N\left(x_N - x_j\right)^2 b_j = \gamma_N \qquad (1.21)$$

This case appears in Hessenberg form where the ordering the equations and the coefficient matrix are properly arranged. The system can be solved by forward elimination and back substitution. The form may be different for different sort of problems, but a suitable method can always be used.

Few researchers had work on the computational efficiency and preciseness yield by the spline. This spline technique has been used effectively by Viswanathan *et al.* (2007, 2008) to solve a few problems.

1.7.5 Radial Basis Function(RBF)

Radial Basis Function (RBF), is one of the grid-free numerical method where it can give out an accurate boundary representations and it is a simpler and easier way to use in calculation (Baxter, 1992). It is classified as infinitely smooth piecewise radial functions with the shape parameter, ε. This parameter varies the radial function from sharp peaked until flatten gradient peak. In this function, it is approximated as linear combination with its radial function $\phi(\|.\|)$, where it is radially symmetry about the centre. The RBF approximation, $s(\bar{x})$ can be represented in the form of (Ferreira *et al*., 2004),

$$u(\bar{x}) \approx s(\bar{x}) = \sum_{j=1}^{n} \lambda_j \phi\left(\left\|\bar{x} - \bar{x}_j\right\|\right) + \sum_{j=1}^{m} \gamma_j p_j(\bar{x}) \tag{1.22}$$

with the constraint condition λ_j as

$$\sum_{j=1}^{m} \lambda_j p_j(\bar{x}) = 0 \qquad j = 1, 2, \ldots, m \tag{1.23}$$

The equation above is for given $\bar{x}_i \in \mathbb{R}^d$ with $u(\bar{x}_i), i = 1, 2, \ldots, n$, where $\phi\left(\left\|\bar{x} - \bar{x}_j\right\|\right)$ is radial basis function and $\|.\|$ is Euclidean norm. λ_j will be unknown RBF coefficient with *n* number of interpolation, p_j is a polynomial with *m* terms, and $\bar{x} - \bar{x}_j$ will be the distance between point *x* and a node x_j.

The most interesting characteristic of radial basis function methods is the feature that has a unique interpolant which is always guaranteed under relatively conditions on the centres. But, the only restriction in few important cases, are that there are at least two centres and they are all distinct, which are as simple as one could wish (Ferreira, 2003).

Multiquadratic (MQ), Inverse Multiquadratic (IMQ), Gaussian (GA), and Polyharmonic Splines are examples of radial basis function. Here, MQ radial function will be used, which can be defined as,

$$\phi(r) = \left[r^2 + (C.d_c)^2\right]^q \quad (1.24)$$

where there r is the distance between point x and a node, x_j, q is a constant which dependent on domain, the average nodal spacing between two nodal defined as d_c and C is the dimensionless shape parameter, in our study we take $C = \varepsilon$.

The functions that have an arbitrary order of accuracy are considered as a complete approximation. This is mainly because an approximation which can certainly regenerate linear polynomials can reproduce any smooth function and its derivative with high accuracy as the approximation, and this approximation has linear consistency and completeness (Hubbert, 2002). Wang and Liu (2002) stated that RBF without a polynomial could not produce the exact linear polynomial and would not able to pass the standard patch test even when the nodes are refined it and can approach accuracy. Powell (1992) proposed that by adding the polynomial term in RBF, it could reproduce any function in the basis and the consistency can be assured.

Pascal's triangle can be chosen as polynomial basis, and the polynomial term, m must be less than the number of nodes. The linear and quadratic polynomial for 2-D problem can be representing as follow;

$$p^T(x) = [1, x, y] \quad (1.25)$$

$$p^T(x) = \left[1, x, y, x^2, xy, y^2\right] \quad (1.26)$$

The compute coefficient λ_j and γ_j in equation 1.21, this equation must satisfies n nodes surrounding point x. Equation (1.21) and (1.22) can represent in matrix form as

$$\begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} \phi_0 & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} \quad (1.27)$$

where,

$$P^T = \begin{bmatrix} p_1(x_1, y_1) & p_1(x_2, y_2) & \cdots & p_1(x_n, y_n) \\ p_2(x_1, y_1) & p_2(x_2, y_2) & \cdots & p_2(x_n, y_n) \\ \vdots & \vdots & \vdots & \vdots \\ p_m(x_1, y_1) & p_m(x_2, y_2) & \cdots & p_m(x_n, y_n) \end{bmatrix} \quad (1.28)$$

$$\phi_0 = \begin{bmatrix} \phi_1(r_1) & \phi_2(r_1) & \cdots & \phi_n(r_1) \\ \phi_1(r_2) & \phi_2(r_2) & \cdots & \phi_n(r_2) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_1(r_n) & \phi_2(r_n) & \cdots & \phi_n(r_n) \end{bmatrix} \quad (1.29)$$

Since there is no direction in the distance of $\bar{x} - \bar{x}_j$ and matrix $\phi_0(r)$ is symmetry, and the interpolation express as follows,

$$u(x) = \Phi(x) u_e \quad (1.30)$$

In which $\Phi(x)$ will be the shape function with *n* number of shape function,

$$\Phi(x) = \left[\varphi_1(x), \varphi_2(x), \ldots \varphi_j(x), \ldots, \varphi_n(x)\right] \quad (1.31)$$

and

$$u_e^T = [u_1, u_2, \ldots, u_n] \quad (1.32)$$

In order to obtain a unique solution, an inverse matrix must exist. Since Powell (1992) and Wendland (1998) had proved the existence of ϕ_0 inverse for arbitrary scattered nodes, this became as an advantage for polynomial basis function, and the order of polynomial in equation (1.20) is low, therefore in general the inverse matrix do exist. The homogenous property of equation (1.21) and non-singular property of matrix ϕ_0^{-1} leads to obtain the shape function $\Phi(x)$,

$$\varphi(x) = \phi^T(r)S_\lambda + P^T(x)S_\gamma = \sum_{j=1}^{n}\phi(r)S_\lambda + \sum_{j=1}^{m}p(r)S_\gamma \tag{1.32}$$

where,

$$S_\gamma = \left[P^T \phi_0^{-1} P\right]^{-1} P^T \phi_0^{-1} \tag{1.33}$$

$$S_\lambda = \phi_0^{-1}\left(I - PS_\gamma\right) \tag{1.34}$$

The interpolation can be written as

$$u(x) = \left[\phi^T(r)S_\lambda + P^T(x)S_\gamma\right]u \tag{1.35}$$

For given point *L*, the function $\phi(x)$ can be written as

$$\varphi(x_L) = \sum_{k=1}^{n}\phi_k(r_L)S_{kj}^{\lambda} + \sum_{j=1}^{m}p_j(r_L)S_{kj}^{\gamma} \tag{1.36}$$

where S_{kj}^{λ} is the (*k*, *j*), S_λ matrix elements, and S_{kj}^{γ} is the (*m*, *j*) S_γ matrix element. These two matrices are constant matrices for the given location of the *n* nodes in the support domain, and the matrix represent as

$$\Phi = \left[\varphi(x_L)\right] = \phi_0 S_\lambda + P^T S_\gamma \tag{1.37}$$

Where P^T is given in equation. (1.26) and ϕ_0 is given in equation. (1.27). Substituting equation. (1.33) and (1.34) into the equation. (1.37) to produce an outcome as

$$\varphi(x_L) = \phi_0 S_\lambda + P^T S_\gamma = \phi_0 \phi_0^{-1}\left(I - PS_\gamma\right) + P^T S_\gamma = I \quad (1.38)$$

Therefore, the shape function $\Phi(x)$ obtained through the above procedure possesses Kronecker delta function properties

$$\varphi_j\left(x_L\right) = \delta_{jL} = \begin{cases} 1 & j = L \\ 0 & j \neq L \end{cases} \quad (1.39)$$

The term interpolation and approximation is referring to the data that fitting the curve and do not pass through nodal data.

1.8 Thesis Outline

This thesis consists of seven chapters, where Chapter 1 is an introduction to the research. Chapter 2 contains literature review, Chapter 3 is the methodology used to solve the problems. Result are discussed in Chapter 4, Chapter 5 and Chapter 6, and Chapter 7 is the closure of the thesis.

Chapter 1 includes a brief introduction and background study about the, problem statement, research objectives, scope and significant of study and chapter organization. Chapter 2 gives an overview of previous work done by various researchers on free vibration of plate and literature review on methods, Spline and Radial Basis approximations, and Chapter 3 discusses the methodology for all the problems.

In Chapter 4, the first problem, anti-symmetric angle-ply laminated plates including first order shear deformation studied. The frequency parameter are analysed with respect to aspect ratio (a/b), length-to-thickness ratio (a/h), and ply angle for two- and four-layered plates consisting of two different materials, AS4/350-6 Graphite/epoxy (AGE) and Kevlar-49/epoxy (KGE) under clamped-clamped boundary conditions using Spline approximation and Radial Basis Function method. The model validated with available literature and the new results are depicted in term of tables and graphs.

Chapter 5 presents the cross-ply laminated plates with odd and even number of layered plates. Here the governing equations developed using first order shear deformation theory. The frequency parameter are analysed with respect to aspect ratio (a/b), and length-to-thickness ratio (a/h) with odd and even number of layered plates using material combinations of AS4/350-6 Graphite/epoxy (AGE) Kevlar-49/epoxy (KGE) and E-glass/Epoxy (EGE). The problem is analysed for three-, four-, five-, six-, seven-, and eight- layered plates. The results compared with the available data and the new results presented.

In Chapter 6, the problem of free vibration of anti-symmetry angle-ply laminated plates under higher order shear deformation theory has been analysed. The problem was analysed with simply supported boundary conditions using the materials AS4/350-6 Graphite/epoxy (AGE) Kevlar-49/epoxy (KGE) and E-glass/Epoxy (EGE). The frequency parameter is studied with respect to aspect ratio (a/b), length-to-thickness ratio (a/h), and ply angle for two- four- and six-layered plates. The results compared with the existing results and the new results shown in tables and graphs. General conclusions are be given based on this study and the possible extension studies proposed.

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