

SYMMETRY AND DOUBLE REDUCTION FOR EXACT SOLUTIONS OF
SELECTED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

JOSEPH BOON ZIK HONG

UNIVERSITI TEKNOLOGI MALAYSIA

SYMMETRY AND DOUBLE REDUCTION FOR EXACT SOLUTIONS OF
SELECTED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

JOSEPH BOON ZIK HONG

A thesis submitted in fulfilment of the
requirements for the award of the degree of
Doctor of Philosophy (Mathematics)

Faculty of Science
Universiti Teknologi Malaysia

FEBRUARY 2017

To my beloved father and mother

ACKNOWLEDGEMENTS

First of all, a special thanks to my supervisor, Assoc. Prof. Dr. Shamsuddin Ahmad. He had helped me a lot in my study. He always supports me on what I wish to do. Besides, I also wish to express my deep acknowledgments to my co-supervisor, Dr. Kamran Fakhar. I am given a lot of guidance from him, and he always advise me whenever I met problems. Furthermore, a deep appreciate were given to Prof. Abdul Hamid Kara who giving support in journal publications.

Besides, I also would like to extend my sincere appreciation to my entire friends, who had kindly provided valuable and helpful comments in the preparation of the thesis, and to those who have involve directly or indirectly in the preparation of this thesis, whom I have not mentioned above.

Last but not least, we would like to express my grateful to my parents who always give me assistances in my life.

ABSTRACT

Amongst the several analytic methods available to obtain exact solutions of non-linear differential equations, Lie symmetry reduction and double reduction technique are proven to be most effective and have attracted researcher from different areas to utilize these methods in their research. In this research, Lie symmetry analysis and double reduction are used to find the exact solutions of non-linear differential equations. For Lie symmetry reduction method, symmetries of differential equation will be obtained and hence invariants will be obtained, thus differential equation will be reduced and exact solutions are calculated. For the method of double reduction, we first find Lie symmetry, followed by conservation laws using 'Multiplier' approach. Finally, possibilities of associations between symmetry with conservation law will be used to reduce the differential equation, and thereby solve the differential equation. These methods will be used on some physically very important nonlinear differential equations; such as Kadomtsev-Petviashvili equation, Boyer-Finley equation, Short Pulse Equation, and Korteweg-de Vries-Burgers equations. Furthermore, verification of the solution obtained also will be done by function of PDETest integrated in Maple or comparison to exist literature.

ABSTRAK

Antara beberapa kaedah analitis yang terdapat untuk mendapatkan penyelesaian tepat bagi persamaan terbitan tidak linear, penurunan simetri Lie dan teknik penurunan dua kali ganda telah terbukti merupakan cara yang paling berkesan, dan telah menarik perhatian pengkaji dari berbeza bidang untuk menggunakan kaedah ini dalam pengkajian mereka. Dalam kajian ini, analisis simetri Lie dan penurunan dua kali ganda digunakan untuk mencari penyelesaian tepat bagi persamaan-persamaan terbitan tidak linear. Untuk kaedah penurunan simetri Lie, simetri-simetri persamaan terbitan akan didapatkan dan, maknanya koordinat berkanun akan didapati, oleh itu, persamaan terbitan akan diturunkan dan penyelesaian tepat akan dikira. Untuk kaedah penurunan dua kali ganda, kami cari simetri Lie dulu, diikuti dengan hukum-hukum keabadian dengan menggunakan pendekatan Pendarab. Akhirnya, kemungkinan kesekutuan antara simetri dan hukum keabadian akan digunakan untuk menurunkan persamaan terbitan dan dengan itu menyelesaikan persamaan terbitan tersebut. Kaedah ini akan digunakan pada sesetengah persamaan terbitan tak linear fizikal yang sangat penting; seperti persamaan Kadomtsev-Petviashvili, persamaan Boyer-Finley, Persamaan Nadi Pendek, dan Persamaan Korteweg-de Vries-Burgers. Tambahan pula, pengesahan untuk jawapan yang didapati juga akan ditentusahkan dengan fungsi PDETest yang diintegrasikan dalam Maple atau perbandingan dengan literatur yang wujud.

TABLE OF CONTENTS

CHAPTER	TITLE	PAGE
	DECLARATION	ii
	DEDICATION	iii
	ACKNOWLEDGEMENTS	iv
	ABSTRACT	v
	ABSTRAK	vi
	TABLE OF CONTENTS	vii
	LIST OF TABLES	x
	LIST OF FIGURES	xi
	LIST OF APPENDICES	xii
1	INTRODUCTION	1
	1.1 Introduction	1
	1.2 Background of the Research	1
	1.3 Statement of the Problem	4
	1.4 Objectives of the Study	6
	1.5 Scope of the Study	7
	1.6 Significance of the Study	7
2	LITERATURE REVIEW AND PAST STUDIES	8
	2.1 Introduction	8
	2.2 Lie Symmetry	8
	2.2.1 Lie Group of Transformation	11
	2.2.2 Lie Reduction for ODE	13
	2.2.2.1 Vector Fields of ODE	13
	2.2.2.2 Form Invariant (ODE)	14

2.2.2.3	Prolongation of Vector Field	14
2.2.2.4	Reduction of ODE	17
2.2.3	Reduction of ODE	18
2.2.3.1	Lie Symmetry and Prolongation of Symmetry of PDE	18
2.2.3.2	Reduction of PDE	20
2.3	Conservation Laws	20
2.3.1	Methods to Find Conservation Laws	22
2.3.1.1	Direct Method	23
2.3.1.2	Multiplier Approach	24
2.4	Double Reduction	25
2.4.1	Methods of Double Reduction	26
2.4.1.1	Association	26
2.4.1.2	Double Reduction of PDE	26
2.5	Kadomtsev-Petviashvili (KP) equation	27
2.6	Boyer-Finley Equation	28
2.7	Short Pulse Equation	29
2.8	Wu-Zhang Equation	31
3	KADOMTSEV-PETVIASHVILI EQUATION	33
3.1	Introduction	33
3.2	Symmetry Reduction for Kadomtsev-Petviashvili Equation	34
3.3	Double Reduction for Kadomtsev-Petviashvili Equation	37
3.3.1	First Double Reduction for Kadomtsev- Petviashvili Equation	40
3.3.2	Second Double Reduction for Kadomtsev- Petviashvili Equation	44
3.3.3	Third Double Reduction for Kadomtsev- Petviashvili Equation	50
3.4	Discussions	54

4	BOYER-FINLEY EQUATION	56
	4.1 Introduction	56
	4.2 Symmetry Reduction for Boyer-Finley Equation	56
	4.3 Double Reduction for Boyer-Finley Equation	64
	4.3.1 First Double Reduction for Boyer-Finley Equation	66
	4.3.2 Second Double Reduction for Boyer-Finley Equation	69
	4.4 Discussions	73
5	SHORT PULSE EQUATION	77
	5.1 Introduction	77
	5.2 Double Reduction for Short Pulse Equation	77
	5.2.1 First Double Reduction for Short Pulse Equation	82
	5.2.2 Second Double Reduction for Short Pulse Equation	85
	5.3 Discussions	87
6	KORTEWEG-DE VRIES-BURGERS EQUATIONS	89
	6.1 Introduction	89
	6.2 Symmetry Reduction for Korteweg-de Vries-Burgers Equations	90
	6.3 Discussions	92
7	CONCLUSION	94
	7.1 Introduction	94
	7.2 Summary of the Chapters	94
	7.3 Suggestion for the Future Research	99
	REFERENCES	101
	Appendix A	110-135

LIST OF TABLES

TABLE NO.	TITLE	PAGE
3.1	Association between symmetries and conservation laws of Kadomtsev-Petviashvili Equation	39
3.2	Association between symmetries and conservation laws of Equation (3.10)	43
3.3	Association between symmetries and conservation laws of Equation (3.16)	47
4.1	Association between symmetries and conservation laws of Boyer-Finley Equation	66
4.2	Association between symmetries and conservation laws of Equation (4.19)	71
5.1	Association between symmetries and conservation laws of SPE	82

LIST OF FIGURES

FIGURE NO.	TITLE	PAGE
4.1	PDETest of solution (4.4)	75
4.2	PDETest of solution (4.7) and (4.23)	75
4.3	PDETest of solution (4.15)	75
5.1	PDETest of solution (5.12)	88
6.1	Graph of Equation (6.3)	92

LIST OF APPENDICES

APPENDIX	TITLE	PAGE
A	Examples of the methods	110

CHAPTER 1

INTRODUCTION

1.1 Introduction

In this thesis, Chapter One represents the basic direction of the research. Meanwhile, Chapter Two discusses the previous researches related to the topic focusing on how symmetry and conservation laws are obtained; hence showing the symmetry reduction and double reduction. Chapter Three, Chapter Four, Chapter Five, and Chapter Six presents the methods used to solve Kadomtsev-Petviashvili equation, Boyer-Finley equation, Short Pulse Equation, and Korteweg-De Vries-Burgers equations, respectively. The final chapter, which is Chapter Seven, focuses on addressing the conclusion of this study.

This chapter contains the background of the study, problem statement, objectives, scope as well as the significance of study.

1.2 Background of the Research

Partial differential equations (PDEs) are often used in the modelling of nonlinear physical phenomena. However, nonlinearity is of a great challenge to the researchers. Several authors from different background have responded to this task and consequently, many methods (numerical, analytical) have been developed to determine the possible solutions for nonlinear equations. These types of PDEs normally involve high orders and

there is no universal method to solve all nonlinear PDEs. Apparently, there are some methods that can solve certain complex PDEs under certain restrictions. One of the approaches involves the invariance of PDEs under one parameter known as the Lie group of transformations, or often called the Lie symmetries. After finding Lie symmetry of a PDE, an invariant solution of the PDE via reduction process can be made using a change of variables. Compared to other exact methods that usually require extra conditions, this method is a universal method to solve PDEs and ODEs without many obstacles. Hence, a class of more general solutions can be obtained, which are very helpful in studying the equations under different boundary conditions. Among the many literatures available on the subject are the well known fundamental references including (Ibragimov & Lie, 1994), Application of Lie Groups to Differential Equations by Olver (2000), Elementary Lie Group Analysis and Ordinary Differential Equations by Ibragimov (1999), as well as the Symmetry and Integration Methods for Differential Equations by Bluman & Anco (2010).

Another recently developed route for analysing differential equations is by finding the conservation laws. In addition to having physical interpretations like conservations of energy, momentum and volume, these laws provide a mechanism for reducing the differential equations. Many significant methods have been developed to construct conservation laws such as the Noether's theorem for variational problems (Noether, 1971; Wang *et al.*, 2014), multiplier approach (Anco & Bluman, 2002a, 2002b; Bluman & Anco, 2010), symmetry action on a known conservation law (Bluman *et al.*, 2006), partial Noether approach (Kara & Mahomed, 2006) and a new conservation method (Ibragimov, 2007). The classical approach is by Noether's theorem for variational PDEs in which a Lagrangian has to be known. There are other PDEs that belonged to the evolution type equation, which do not admit Lagrangian. To handle these equations, one can use the direct method also known as the 'multiplier' approach (Anco & Bluman, 2002a, 2002b; Bluman & Anco, 2010), which directly utilises the definition. This approach has been actively pursued recently; it involves constructing multipliers for PDEs that are then further analysed to obtain corresponding conserved vectors.

The theory of double reduction of a PDE (or systems of PDEs) is well-known for the association of conservation laws with Noether symmetries (Bluman & Anco, 2010; Cariello & Tabor, 1991). Meanwhile, the association of conservation laws with Lie Backlund symmetries (Kara & Mahomed, 2000) and non-local symmetries (Sjöberg & Mahomed 2004) led to the expansion of the theory of double reduction for PDEs with two independent variables, which do not possess Noether symmetries (Sjöberg 2007). Solving PDEs through double reduction may not be as universal as symmetry method since some PDEs do not possess any unique and non-trivial conservation laws. However, if PDEs do possess non-trivial conservation laws, the double reduction method will be able to find such exact solution that may not be obtained through symmetry method. Besides, this method is straightforward and more effective in reducing the order and variable of an equation in one step. Furthermore, this method provides a mechanism to construct more solutions with less restrictions and limitations compared to other methods. Lastly, a PDE contains more conservation laws, which means that it has high integrability considering that one can perform double reductions through conservation laws.

One way in which PDEs can be used is through models involving more than one independent variable. Plasma is a significant technology recently proposed. It is widely applied in the fields such as Biology, Physics and computing. An important equation in plasma to describe wave surface problem for an incompressible fluid with free surface and rigid horizontal bottom boundary conditions is called the Kadomtsev-Petviashvili (KP) equation (Kadomtsev & Petviashvili, 1970), which is originated from the study on Korteweg–de Vries (KdV) equations.

Data deliver through silica optic are innovated by improving the technology in telecommunication. Huge amount of data can be sent in a short period using this silica optic. The model that describes the propagation of ultra-short light pulse in silica optical fibres is the one preferred for the study. In literature, this method refers to the Short Pulse equation (SPE) (Schäfer & Wayne, 2004).

General relativity theory plays an important role in many fields; for instance, general relativity theory as a backbone of quantum theory. Explanation on general relativity theory exposes many problems in finding all real, Euclidean, self-dual spaces with one Killing vector that was reduced, which was then replaced with another equation namely Boyer-Finley equation named after the Boyer and Finley III (1982) for their contribution in developing this equation. As for the importance of reveal general theory, this equation has been also examined within this study to find the exact solution.

Finally, this study also considered the Korteweg-de Vries-Burgers (KdV-B) equation. This equation yields the famous Korteweg-de Vries (KdV) equation. KdV-B equation exists in various physical situations. Here, KdV-B equation model from the theory of ferroelectricity (Zayko, 1989) were chosen. Ferroelectricity is generally used in choosing suitable material for a capacitor. Basically, capacitors are the main components to construct battery for electronic applicants. Hence, it is worthy to study this equation to help understanding the material for the battery of electronic applicants including smartphone and laptop.

This study was conducted to study the invariance, Lie symmetries and conservation laws of the above equations namely KP, Boyer-Finley, SPE and KdV-B equations, which were mentioned in the few last paragraphs. Meanwhile, the ultimate goal is to obtain the exact solutions that are not yet reported by existing literature.

1.3 Statement of the Problem

1) To study and tackle the nonlinearity of the following four nonlinear significant equations via Lie symmetry and conservation laws:

i) Kadomtsev-Petviashvili (KP) equation,

$$(u_t + uu_x + u_{xxx})_x + 3s^2u_{yy} = 0,$$

where $u = u(x, y, t)$. This equation is modelled using the Euler Equation describing the wave surface problem for an incompressible fluid with free surface and rigid horizontal bottom boundary condition.

ii) Boyer-Finley equation,

$$u_{xy} = (e^u)_{tt},$$

where $u = u(x, y, t)$. This is an equation of self-dual Einstein spaces of Euclidean signature with one rotational Killing vector.

iii) Short Pulse equation (SPE),

$$u_{xt} = \alpha u + \frac{1}{3}\beta(u^3)_{xx},$$

where $u = u(x, t)$, which the unknown real function and the subscripts denote differentiation with respect to x and t ; α and β are nonzero real parameters. This model describes the propagation of ultra-short light pulses in silica optical fibres.

iv) Korteweg-de Vries-Burgers (KdV-B) equation,

$$u_t + Auu_x + Bu_{xx} + Cu_{xxx} = 0,$$

$$A = \frac{2\alpha}{K'(v)}, \quad B = \frac{v}{K'(v)}, \quad C = -\frac{v^2}{K'(v)}, \quad K'(v) = \frac{dK(v)}{dv},$$

$$K(v) = \frac{\omega_p^2 v^2}{c^2 - v^2} - \omega_0^2 - 2\alpha u_0$$

where u is the first term in expanding the series of polarisation with respect to small attenuation coefficient; $K(v) = 0$ is the dispersive equation for wave velocity; u_0 is the equilibrium value of u ; ω_p, ω_0 are the frequencies of

wave, which vary according to different problems; c represents the velocity of light; α is the coefficient determined by the system. These equations describe a ferroelectric system.

- 2) The method on the four equations, KP, Boyer-Finley equation, SPE and KdV-B equation will involve deriving the symmetry for these four differential equations with the aid of programming.
- 3) Application of symmetry to reduce the KP, Boyer-Finley and KdV-B equation.
- 4) Methods to obtain conservation laws for KP, Boyer-Finley equation, and SPE via Multiplier approach.
- 5) Measuring the association between Lie symmetry and conserved vector.
- 6) Applying the conservation laws combined with underlying associated symmetries to employ the 'double reduction' on KP equation, Boyer-Finley equation and SPE, thus calculating their solutions.

1.4 Objectives of the Study

The objectives of this study are as follow:

1. To calculate the possible Lie symmetries and conservation laws for Short Pulse Equations (SPE), Boyer-Finley equation, Kadomtsev-Petviashvili (KP) equation, and Korteweg-de Vries-Burgers (KdV-B) equation.
2. To determine the possible association between symmetry and conserved vector of SPE, Boyer-Finley equation and KP equation.
3. To utilise symmetry in reducing Boyer-Finley, KP and KdV-B equations.
4. To apply the association of conservation laws and symmetry in double reduction to the equation, and to obtain the exact solution for SPE, Boyer-Finley and KP equations.

1.5 Scope of the Study

This study concentrates on using symmetry approach and/or double reduction approach with the combination of symmetry and conservation laws to reduce non-linear differential equations namely Short Pulse Equation, Boyer-Finley Equation, Kadomtsev-Petviashvili (KP) Equation, and Korteweg-de Vries-Burgers (KdV-B) Equation. Multiplier approach was selected to find the conservation laws of these equations instead of other methods.

1.6 Significance of the Study

The method of Lie symmetry and double reduction provides a mechanism to tackle a considerable amount of nonlinear differential equations that are not easily handled by other integration methods. In some cases, even analytic methods failed to produce any results. In fact, this method is the only universal method that produces analytics solutions (Kara & Mahomed, 2000). For instance, in fluid flow problems, # mathematicians, physicists or engineers often faced many complicated nonlinear high order dimensional differential equations in complex domain with a number of unknown parameters. Nonetheless, the order of equation can be easily reduced using the Lie symmetry and conservation laws technique, thus aiding in solving or investigating the system with much ease. Moreover, as compared to numerical methods, this method is more efficient and cheaper. Besides, the exact solutions obtained by this method can serve as the benchmark for testing the algorithms and accuracy of numerical solutions. Exact solutions of differential equation may also help scientist or physicist to detect or measure the accuracy and sensitivity of various variables in their physical interest that involved in the equation.

REFERENCES

- Abdur Rab, M., Mias, A. S., & Akter, T., (2012), Some travelling wave solutions of KdV-Burgers Equation. *Int. Journal of Math. Analysis*, 6, 1053-1060.
- Ablowitz, M. J. & Segur, H. (1979). On the evolution of packets of water waves. *Journal of Fluid Mechanics*, 92, pp. 691–715.
- Ablowitz, M. J. & Segur, H. (1981). *Solitons and the Inverse Scattering Transform*, Society for Industrial and Applied Mathematics.
- Ablowitz, M. J. & Clarkson P. A. (1991), *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press.
- Agrawal, G. P. (2013). *Nonlinear Fiber Optics*, Academic Press.
- Alterman, D. & Rauch, J. (2000). Diffractive short pulse asymptotics for nonlinear wave equations. *Physics Letters A*, 264(Dc), pp. 390–395.
- AN, G. & Cole, J. (1969). The general similarity solution of the heat equation. *Journal of Mathematics and Mechanics*, 18(11), pp. 1025–1042.
- Anco, S. C. & Bluman, G. (2002a). Direct construction method for conservation laws of partial differential equations Part I: Examples of conservation law classifications. *European Journal of Applied Mathematics*, 13, pp. 546–566.
- Anco, S. C. & Bluman, G. (2002b). Direct construction method for conservation laws of partial differential equations Part II: General treatment. *European Journal of Applied Mathematics*, 13, pp. 567–585.
- Anco, S. C. & Bluman, G. (1997). Direct Construction of Conservation Laws from Field Equations. *Phys. Rev. Lett.*, 78(15), pp. 2869–2873.
- Ashgar, S., Mahmood, M. & Kara, A. H. (2009). Solutions using symmetry methods and conservation laws for viscous flow through a porous medium inside a deformable channel. *Journal of Porous Media*, 12(8), pp. 811–819.

- Avsarkisov, V., Oberlack, M. & Hoyas, S. (2014). New scaling laws for turbulent Poiseuille flow with wall transpiration. *Journal of Fluid Mechanics*, 746, pp. 99–122.
- Aziz, T., Mahomed, F. M., Shahzad, A. & Ali, R. (2014). Travelling Wave Solutions for the Unsteady Flow of a Third Grade Fluid Induced Due to Impulsive Motion of Flat Porous Plate Embedded in a Porous Medium. *Journal of Mechanics*, 30(05), pp. 527–535.
- Bashan, A., Gazi Karakoc, S. B., & Geyikli, T. (2015). Approximation of the KdVB equation by the quintic B-spline differential quadrature method. *Kuwait J. Sci.* 42(2), pp. 67-92.
- Biswas, A., Milovic, D. & Edwards, M. (2010). Nonlinear Schrödinger's Equation. In *Mathematical Theory of Dispersion-Managed Optical Solitons SE - 2*. Nonlinear Physical Science. Springer Berlin Heidelberg, pp. 5–26.
- Bluman, G. & Anco, S. (2010). *Symmetry and Integration Methods for Differential Equations*, Springer New York.
- Bluman, G., Temuerchaolu & Anco, S. C. (2006). New conservation laws obtained directly from symmetry action on a known conservation law. *Journal of Mathematical Analysis and Applications*, 322(1), pp. 233–250.
- Bokhari, A. H., Al-Dweik, A. Y., Kara, A. H., Mahomed, F. M. & Zaman, F. D. (2011). Double reduction of a nonlinear (2+1) wave equation via conservation laws. *Communications in Nonlinear Science and Numerical Simulation*, 16(3), pp. 1244–1253.
- Bokhari, A. H., Al-Dweik, A. Y., Zaman, F. D., Kara, A. H. & Mahomed, F. M. (2010). Generalization of the double reduction theory. *Nonlinear Analysis: Real World Applications*, 11(5), pp. 3763–3769.
- Boyer, C. & Finley III, J. (1982). Killing vectors in self-dual, Euclidean Einstein spaces. *Journal of Mathematical Physics*, 1126(1982).

- Boyer, C. & Winternitz, P. (1989). Symmetries of the self-dual Einstein equations. I. The infinite-dimensional symmetry group and its low-dimensional subgroups. *Journal of Mathematical Physics*, 30(5), pp. 1081–1094.
- Cariello, F. & Tabor, M., (1991). Similarity reductions from extended Painlevé expansions for nonintegrable evolution equations. *Physica D: Nonlinear Phenomena*, 53(1), pp. 59–70.
- Cheviakov, A. (2007). GeM software package for computation of symmetries and conservation laws of differential equations. *Computer physics communications*, (May 2006).
- Chung, Y., Jones, C. K. R. T., Schafer, T. & Wayne, C. E. (2005). Ultra-short pulses in linear and nonlinear media. *Nonlinearity*, 18(3), pp.1351–1374.
- Chung, Y. & Schafer, T. (2007). Stabilization of ultra-short pulses in cubic nonlinear media. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 361(1-2), pp. 63–69.
- Clarkson, P. A. & Kruskal, M. D. (1989). New similarity reductions of the Boussinesq equation. *Journal of Mathematical Physics*, 30(10), pp. 2201–2213.
- Comejo-Perez O., Negro J., Nieto L. M., Rosu H. C. (2006), Travelling-wave Solutions for Korteweg-de Vries-Burgers equation through factorization, *Foundations of Physics*, 36, 1587 - 99
- David, D., Kamran, N., Levi, D. & Winternitz, P. (1986). Symmetry reduction for the Kadomtsev–Petviashvili equation using a loop algebra. *Journal of Mathematical Physics*, 27(5), pp. 1225–1237.
- Demiray H. (2003), A note on the exact travelling wave solution to the KdV-Burgers equation, *Wave motion*, 11, 367 - 69
- Fakhar, K., Kara, A. H., Khan, I. & Sajid, M. (2011). On the computation of analytical solutions of an unsteady magnetohydrodynamics flow of a third grade fluid with Hall effects. *Computers & Mathematics with Applications*, 61(4), pp. 980–987.
- Fakhar, K., Zu-Chi, C., Ji, X. & Cheng, Y. (2006). Similarity reduction of (3+1) Navier-Stokes system. *Eng. Comp.*, 6, pp. 632–643.

- Fakhar, K., Hayat, T., Yi, C. & Zhao, K. (2009). Symmetry transformation of solutions for the Navier–Stokes equations. *Applied Mathematics and Computation*, 207(1), pp. 213–224.
- Fakhar, K. & Kara, A. H. (2011). An Analysis of the Invariance and Conservation Laws of Some Classes of Nonlinear Ostrovsky Equations and Related Systems. *Chinese Physics Letters*, 28(1), p. 010201.
- Ferapontov, E., Korotkin, D. & Shramchenko, V. (2004). Boyer-Finley equation and systems of hydrodynamic type. *Classical and Quantum Gravity - CLASS QUANTUM GRAVITY*, 19(24), pp. L205–L210.
- Fu, Z., Chen, Z., Zhang, L., Mao, J. & Liu, S. (2010). Novel exact solutions to the short pulse equation. *Applied Mathematics and Computation*, 215(11), pp. 3899–3905.
- Golin'ko, V. I., Dryuma, V. S. & Yu, A. (1983). Non-linear quasicylindrical waves: exact solutions of the cylindrical Kadomtsev-Petviashvili equation. In *Proc. 2nd Int. Workshop on Nonlinear and Turbulent Processes in Physics, Kiev*. Gordon and Breach: Harwood Acad., pp. 1353–1360.
- Huber, A. (2010). The Short Pulse Equation - a Symmetry Study. *J. Comp. Methods in Sci. and Eng.*, 10(1,2), pp. 79–87.
- Ibragimov, N. H. (2007). A new conservation theorem. *Journal of Mathematical Analysis and Applications*, 333(1), pp. 311–328.
- Ibragimov, N. H., Kara, A. H. & Mahomed, F. M. (1998). Lie-Backlund and Noether Symmetries with Applications. *Nonlinear Dynamics*, 15, pp. 115–136.
- Ibragimov, N. H., Lie, S. (1994). On integration of a Class of Linear Partial Differential Equations by Means of Definite Integrals. *CRC Handbook of Lie Group Analysis of Differential Equations*, 2, pp. 473–508. (English translation of Lie, S. & Arch (1881). Über die Integration durchbestimmte Integrale von einer Klasse linear partieller Differentialgleichungen. *Arch, für Math*, 6(3), pp. 328–368.)
- Ibragimov, N. K. (1999). *Elementary Lie group analysis and ordinary differential equations*, Wiley.
- Jacobi, C. G. J., Borchardt, C. W., Clebsch, A. & Lottner, E. (1884). *Vorlesungen Über Dynamik*, Berlin: G. Reimer.

- Jeffrey A., Xu S. (1989), Exact solutions to the Korteweg-de Vries-Burgers, *Wave motion*, 11, 559 - 64
- Jeffrey A., Mohamad M. N. B. (1991), Exact solutions to the Korteweg-de Vries-Burgers, *Wave motion*, 11, 369 - 75
- Johnpillai, A. G., Kara, A. H. & Biswas, A. (2013). Symmetry reduction, exact group-invariant solutions and conservation laws of the Benjamin–Bona–Mahoney equation. *Applied Mathematics Letters*, 26(3), pp. 376–381.
- Johnson, R. (1980). Water waves and Korteweg–de Vries equations. *Journal of Fluid Mechanics*, 97, pp. 701–719.
- Johnson, R. S. (1997). *A Modern Introduction to the Mathematical Theory of Water Waves*, Cambridge University Press.
- Joseph, L. (1888). The "Mécanique Céleste" of Laplace, and Its Translation, with a Commentary by Bowditch. *Proceedings of the American Academy of Arts and Sciences*, 24, 185-201. (English Translation of Laplace, P. S. (1798). *Traité de Mécanique Céleste*, Paris: Duprat.)
- Kadomtsev, B. & Petviashvili, V. (1970). On the stability of solitary waves in weakly dispersing media. *Sov. Phys. Dokl*, 15, pp. 539–541.
- Kara, A. H. & Khalique, C. M. (2005). Nonlinear evolution-type equations and their exact solutions using inverse variational methods. *Journal of Physics A: Mathematical and General*, 38(21), pp. 4629–4636.
- Kara, A. H. & Mahomed, F. M. (2006). Noether-Type Symmetries and Conservation Laws Via Partial Lagrangians. *Nonlinear Dynamics*, 45(3-4), pp. 367–383.
- Kara, A. H. & Mahomed, F. M. (2000). Relationship between Symmetries and Conservation Laws. *Int. J. Theor. Phys.*, 39(1), pp. 23–40.
- Kara, A. H., Mahomed, F. M. & Unal, G. (1999). Approximate symmetries and conservation laws with applications. *International journal of theoretical physics*, 38(9), pp. 2389–2399.
- Kaur, L. & Gupta, R. K. (2014). Some invariant solutions of field equations with axial symmetry for empty space containing an electrostatic field. *Applied Mathematics and Computation*, 231, pp. 560–565.

- Kaya D. (2004), An application for the decomposition method for the KdVB equation, *Appl Math Comput*, 152, 279 - 88
- Khusnutdinova, K. & Klein, C. (2013). On the integrable elliptic cylindrical Kadomtsev-Petviashvili equation. *Chaos*, 23, p. 013126.
- Khusnutdinova, K. R., Klein, C., Matveev, V. B. & Smirnov, A. O. (2013). On the integrable elliptic cylindrical Kadomtsev-Petviashvili equation. *Chaos (Woodbury, N.Y.)*, 23(1), p. 013126.
- Klein, C., Matveev, V. & Smirnov, A. (2007). Cylindrical Kadomtsev-Petviashvili equation: Old and new results. *Theoretical and Mathematical Physics*, 152(2), pp. 1132–1145.
- Klein, F. (1918). Über die Differentialgesetze für die Erhaltung von Impuls und Energie in die Einsteinschen Gravitationstheories. *Nachr., Göttingen, Math.-Phys. Kl.*, pp. 171–189. (Main point of the paper was taken and english translation courtesy of Basil Gordon)
- Lee, J., Kandaswamy, P., Bhuvaneshwari, M. & Sivasankaran, S. (2009). Lie group analysis of radiation natural convection heat transfer past an inclined porous surface. *Journal of Mechanical Science and Technology*, 22(9), pp. 1779–1784.
- LeVeque, R. & Veque, R. Le (1992). *Numerical methods for conservation laws*, Berlin: Springer.
- Levi, D., Ricca, E., Thomova, Z. & Winternitz, P. (2014). Lie group analysis of a generalized Krichever-Novikov differential-difference equation. *Journal of Mathematical Physics*, 55(10), p. 103503.
- Levi, D., Menyuk, C. R. & Winternitz, P. (1994). Similarity reduction and perturbation solution of the stimulated-Raman-scattering equations in the presence of dissipation. *Phys. Rev. A*, 49(4), pp. 2844–2852.
- Lipovskii, V., Matveev, V. & Smirnov, A. (1989). Connection between the Kadomtsev-Petviashvili and Johnson equations. *Journal of Soviet Mathematics*, 46(1), pp. 1609–1612.
- Liu, H. & Li, J. (2009). Lie symmetry analysis and exact solutions for the short pulse equation. *Nonlinear Analysis: Theory, Methods & Applications*, 71(5-6), pp. 2126–2133.

- Lou, S., Tang, X. & Lin, J. (2000). Similarity and conditional similarity reductions of a (2+1)-dimensional KdV equation via a direct method. *Journal of Mathematical Physics*, 41(12), pp. 8286–8303.
- Lu D., Hong B., & Tian L. (2009), New solitary wave and periodic wave solutions for general types of KdV and KdV-Burgers equations, *Commun. Nonlinear Sci. Numer. Simulat*, 14, 77 - 84
- Ludlow, D. K., Clarkson, P. A. & Bassom, A. P. (1999). Similarity Reductions and Exact Solutions for the Two-Dimensional Incompressible Navier-Stokes Equations. *Studies in Applied Mathematics*, 103(3), pp. 183–240.
- Ma, Y.-L. & Li, B.-Q. (2012). Some new Jacobi elliptic function solutions for the short-pulse equation via a direct symbolic computation method. *Journal of Applied Mathematics and Computing*, 40(1-2), pp. 683–690.
- Martina, L., Sheftel, M. & Winternitz, P. (2001). Group foliation and non-invariant solutions of the heavenly equation. *Journal of Physics A: Mathematical and General*, 34(01), pp. 9243–9263.
- Mindu, N. & Mason, D. P. (2014). Derivation of Conservation Laws for the Magma Equation Using the Multiplier Method: Power Law and Exponential Law for Permeability and Viscosity. *Abstract and Applied Analysis*, 2014(6), pp. 1–13.
- Naz, R. (2012). Conservation laws for some compacton equations using the multiplier approach. *Applied Mathematics Letters*, 25(3), pp. 257–261.
- Naz, R., Ali, Z. & Naeem, I. (2013). Reductions and New Exact Solutions of ZK, Gardner KP, and Modified KP Equations via Generalized Double Reduction Theorem. *Abstract and Applied Analysis*, 2013, pp. 1–11.
- Noether, E. (1971). Invariant variation problems. *Transport Theory and Statistical Physics*, 1, pp. 186–207. (English translation of Noether, E. (1918). Invariante Variations probleme. *Nachr. v. d. Ges. d. Wiss. zu Göttingen*, pp. 235–257.)
- Nucci, M. C. & Clarkson, P. A. (1992). The nonclassical method is more general than the direct method for symmetry reductions. An example of the Fitzhugh-Nagumo equation. *Physics Letters A*, 164(1), pp. 49–56.
- Olver, P. J. (2000). *Applications of Lie Groups to Differential Equations*, Springer New York.

- Parkes, E. J. (2008). Some periodic and solitary travelling-wave solutions of the short-pulse equation. *Chaos, Solitons and Fractals*, 38(1), pp. 154–159.
- Qian, S. & Tian, L. (2008). Group-invariant solutions of a integrable coupled system. *Nonlinear Analysis: Real World Applications*, 9(4), pp. 1756–1767.
- Sahadevan, R. & Khousalya, S. (2000). Similarity reduction of a (2+ 1) Volterra system. *Journal of Physics A: Mathematical and General*, 171, pp. L171–L176.
- Sahin, D., Antar, N. & Ozer, T. (2010). Lie group analysis of gravity currents. *Nonlinear Analysis: Real World Applications*, 11(2), pp. 978–994.
- Sakovich, A. & Sakovich, S. (2006). Solitary wave solutions of the short pulse equation. *Journal of Physics A: Mathematical and General*, 39(22), pp. L361–L367.
- Sakovich, A. & Sakovich, S. (2005). The short pulse equation is integrable. *Journal of the Physical Society of Japan*, 74, pp. 239–245.
- Saleem, U. & Hassan, M. (2012). Darboux Transformation and Multisoliton Solutions of the Short Pulse Equation. *Journal of the Physical Society of Japan*, 81(9), p. 94008.
- San, S. & Yaşar, E. (2014). On the Conservation Laws and Exact Solutions of a Modified Hunter-Saxton Equation. *Advances in Mathematical Physics*, 2014, pp. 1–6.
- Schäfer, T. & Wayne, C. (2004). Propagation of ultra-short optical pulses in cubic nonlinear media. *Physica D: Nonlinear Phenomena*, 196, pp. 90–105.
- Shen, S. (2007). Lie symmetry reductions and exact solutions of some differential–difference equations. *Journal of Physics A: Mathematical and Theoretical*, 40(8), pp. 1775–1783.
- Shu, J. J. (1987) The proper analytical solution of the Korteweg-de Vries-Burgers equation. *J. Phys. A.: Math. Gen.* 20, L49-L56.
- Singh, K. & Gupta, R. K. (2005). On symmetries and invariant solutions of a coupled KdV system with variable coefficients. *International Journal of Mathematics and Mathematical Sciences*, 23, pp. 3711–3725.
- Sjöberg, A. (2007). Double reduction of PDEs from the association of symmetries with conservation laws with applications. *Applied Mathematics and Computation*, 184(2), pp. 608–616.

- Sjöberg, A. (2009). Nonlinear Analysis : Real World Applications On double reductions from symmetries and conservation laws. *Nonlinear Analysis: Real World Applications*, 10(6), pp. 3472–3477.
- Sjöberg, A. & Mahomed, F. M. (2004). Non-local symmetries and conservation laws for one-dimensional gas dynamics equations. *Applied Mathematics and Computation*, 150(2), pp. 379–397.
- Soliman A. A. (2008), Exact solutions of the KdV - Burgers equation by Exp-function method, *Chaos, Solitons and Fractals*, 04, 038.
- Tufail, M. N., Butt, A. S. & Ali, A. (2013). Heat source/sink effects on non-Newtonian MHD fluid flow and heat transfer over a permeable stretching surface: Lie group analysis. *Indian Journal of Physics*, 88(1), pp. 75–82.
- Victor, K. K., Thomas, B. B. & Kofane, T. C. (2007). On exact solutions of the Schäfer–Wayne short pulse equation: WKI eigenvalue problem. *Journal of Physics A: Mathematical and Theoretical*, 40(21), p. 5585.
- Wang, G. W., Xu, T. Z. & Biswas, A. (2014). Topological solutions and conservation laws of the coupled Burgers equations. *Rom. Rep. Phys.*, 66(c), pp. 274–285.
- Wang, M. L. (1996). Exact solution for a compound KdV-burgers equation, *Physics Letter A*, 213, 279-287.
- Wolf, T. (2002). A comparison of four approaches to the calculation of conservation laws. *European Journal of Applied Mathematics*, 13(02), pp. 129–152.
- Xie Y. X. & Tang J. S. (2005), New solitary wave solutions to the KdV-Burgers equation, *Int J Theor Phys*, 44(3), 293 - 301.
- Zayko, Y. N. (1989). Polarization waves in nonlinear dielectric. *Zhurnal Technicheskoy Fiziki*, 59(9), pp. 172-173.
- Zhang, L. H. (2014), Conservation laws, symmetry reductions, and new exact solutions of the (2 + 1)-dimensional Kadomtsev-Petviashvili equation with time-dependent coefficients, *Abs. & App. Ana.*, 2014, ID 853578.
- Zhi, H. (2009). Symmetry reductions of the Lax pair for the 2+1-dimensional Konopelchenko–Dubrovsky equation. *Applied Mathematics and Computation*, 210(2), pp. 530–535.