THE SCHUR MULTIPLIERS, NONABELIAN TENSOR SQUARES AND CAPABILITY OF SOME FINITE p-GROUPS

ROSITA BINTI ZAINAL

A thesis submitted in fulfilment of the
requirements for the award of the degree of
Doctor of Philosophy (Mathematics)

Faculty of Science
Universiti Teknologi Malaysia

AUGUST 2016

To my beloved parents,

Allahyarhamah Puan Hajjah Rokiah Binti Abdullah,

Tuan Haji Zainal Bin Haji Sulong

and my family

ACKNOWLEDGEMENT

First of all, thanks to Allah S.W.T., who has given me the strength to complete this research. In preparing this thesis, I was in contact with many people. They have contributed towards my understanding and thoughts. In particular, I wish to express my sincere appreciation and deepest gratitude to my supervisor, Dr Nor Muhainiah Mohd Ali for her encouragement, guidance and invaluable suggestions. I am grateful for the help of my co-supervisors, Prof Dr Nor Haniza Sarmin and Assist Prof Dr Samad Rashid in appreciation of the significant contribution, encouragement and guidance which they provided throughout the completion of this thesis. I am thankful for their effort and patience in the preparation and completion of this thesis.

I am also indebted to Ministry of Higher Education (MOHE) Malaysia for the sponsorship of my studies through the MyPhD MyBrain15 Scholarship.

Moreover, I would like to express my special thanks to my beloved parents, Allahyarhamah Puan Hajjah Rokiah Binti Abdullah and Tuan Haji Zainal Bin Haji Sulong for their long lasting love. Lastly, I would like to thank my siblings, family and friends. They really supported me and played an important role in the completion of my thesis. I would like to thank them for their encouragement, love and emotional support.

Al-Fatihah.

ABSTRACT

The homological functors and nonabelian tensor product have its roots in algebraic K-theory as well as in homotopy theory. Two of the homological functors are the Schur multiplier and nonabelian tensor square, where the nonabelian tensor square is a special case of the nonabelian tensor product. A group is said to be capable if it is a central factor group. In this research, the Schur multiplier, nonabelian tensor square and capability for some groups of order p^3, p^4, p^5 and p^6 are determined. An algebraic computation of the center, derived subgroups, abelianization, Schur multipliers, nonabelian tensor squares and capability of the groups are determined with the assistance of Groups, Algorithms and Programming (GAP) software. Using the results of the center, derived subgroups and abelianization, the Schur multiplier, nonabelian tensor square and capability for the groups are determined. The nonabelian tensor squares and capability are also determined using the results of the Schur multipliers. The Schur multiplier of each of the groups considered is found to be trivial or abelian. The results show that the nonabelian tensor square of the groups are always abelian. In addition, a group has been shown to be capable if it has a nontrivial kernel or it is an extra-special p-group with exponent p.

ABSTRAK

Fungtor homologi dan hasil darab tensor tak abelan berasal dari teori K-aljabar serta teori homotopi. Dua fungtor homologi adalah pendarab Schur dan kuasa dua tensor tak abelan, dengan kuasa dua tensor tak abelan adalah kes khas bagi hasil darab tensor tak abelan. Suatu kumpulan dinamai kumpulan berupaya jika ia adalah kumpulan faktor berpusat. Dalam penyelidikan ini, pendarab Schur, kuasa dua tensor tak abelan dan keberupayaan untuk beberapa kumpulan berperingkat p^3, p^4, p^5 dan p^6 telah ditentukan. Pengiraan aljabar bagi pusat kumpulan, subkumpulan terbitan, keabelan, pendarab Schur, kuasa dua tensor tak abelan dan keberupayaan kumpulan tersebut telah ditentukan dengan bantuan perisian *Groups, Algorithms and Programming* (GAP). Dengan menggunakan keputusan pusat kumpulan, subkumpulan terbitan dan keabelan, pendarab Schur, kuasa dua tensor tak abelan dan keberupayaan kumpulan tersebut telah ditentukan. Kuasa dua tensor tak abelan dan keberupayaan juga telah ditentukan dengan menggunakan keputusan pendarab Schur. Pendarab Schur bagi setiap kumpulan yang dipertimbangkan telah didapati remeh atau abelan. Keputusan menunjukkan yang kuasa dua tensor tak abelan kumpulan tersebut adalah sentiasa abelan. Tambahan pula, suatu kumpulan telah ditunjukkan berupaya jika ia mempunyai inti tak remeh atau ia adalah kumpulan-p ekstra-khas dengan eksponen p.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

G^{ab}	-	Abelianization of G
$G \otimes_{\mathbb{Z}} H$	-	Abelian tensor product of G and H
$Z(G)$	-	Center of G
$[x, y]$	-	Commutator of x and y, $xyx^{-1}y^{-1}$
${}^{y}x$	-	Conjugate of x with respect to y, yxy^{-1}
$\mathbb{Z}_n$	-	Cyclic group of order n
G'	-	Derived subgroup of G
$G \times H$	-	Direct product of G and H
$Z^*(G)$	-	Epicenter of G
$Z^{\wedge}(G)$	-	Exterior center of G
$G \wedge G$	-	Exterior square of G
$G \cong H$	-	G is isomorphic to H
$\langle x \rangle$	-	Group generated by the element x
$H \leq G$	-	H is a subgroup of G
1	-	Identity of the group
$J(G)$	-	Kernel of the homomorphism
$G \otimes G$	-	Nonabelian tensor square of G
$G \otimes H$	-	Nonabelian tensor square of G and H
$\|G\|, \|x\|$	-	Order of G, the order of the element x
G/H	-	Quotient group of G by H
$M(G)$	-	Schur multiplier of G
$\nabla(G)$	-	Subgroup of $J(G)$
$H_3(G)$	-	Third homology group
Γ	-	Whitehead's quadratic functor

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Introduction

This thesis focuses on the Schur multipliers, the nonabelian tensor squares and the capability of some finite groups. A brief introduction on the Schur multiplier, nonabelian tensor square and capability of a group is provided in this chapter.

The Schur multiplier of a group G, denoted by $M(G)$, was first introduced by Issai Schur in 1904 [1]. Issai Schur began the study with fundamental works on multipliers and it lead to the computation of the Schur multiplier for projective of representation groups. Since then, the Schur multiplier has been studied by many researchers.

The nonabelian tensor square of a group G, $G \otimes G$, is a special case of the nonabelian tensor product $G \otimes H$ for two arbitrary groups G and H. The nonabelian tensor square is generated by the symbols $g \otimes h$ and defined by two relations which are

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \text{ and } g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$$

for all g, g', h, $h' \in G$, where the action is taken to be conjugation, i.e. ${}^{h}g = hgh^{-1}$. The determination of the nonabelian tensor square was initiated by Brown and Loday in [2].

The capability of groups was first discussed by Baer [3] in 1938, who determined the capability of finitely generated abelian groups. Following an important remark by Hall in [4], the study on characterizing the capability of groups has become an interest for many researchers over the years. A group G is said to be capable if there exists a group H such that the central factor group $H/_{Z(H)}$ is isomorphic to G, or equivalently if it is an inner automorphism of a group H.

In this research, some properties of the groups including the center, $Z(G)$, derived subgroups, G', and abelianization, G^{ab} are determined. By using these results together with the classifications of the groups, the Schur multipliers, nonabelian tensor squares and capability are then computed for the groups of order p^n, where p is an odd prime and n equals 3, 4, 5 and 6. From here on, p is denoted for an odd prime. A large number of groups is constructed and the classification of the groups is split into a bigger class if $p > 3$ for $n \geq 4$. Furthermore, if $n \geq 7$, the classification is still being improved. In this research, only $n = 3, 4, 5$ and 6 are considered as discussed in the scope of the study.

1.2 Research Background

In 1897, Burnside in [5] constructed the classifications for groups of order p^n. Based on those classifications, this research focuses on case $n = 3, 4, 5$ and 6.

The Schur multiplier of a group is the second cohomology group with coefficients in $\mathbb{Z}$. In 1956, Green in [6] proved that the order of the Schur multiplier of a finite p-group of order p^n is $p^{\frac{n(n-1)}{2}-t(G)}$ for some non-negative integer $t(G)$. Continuing the work in [6], Berkovich [7] characterized the structure of a group when the order of the Schur multiplier is $p^{\frac{n(n-1)}{2}-t(G)}$ for $t(G) = 0$ and 1. These researches have attracted many researchers that lead to the extension until $t(G) = 5$. The nonabelian tensor squares and capability of the groups of order p^3, p^4, p^5 and p^6 are then computed using the Schur multipliers of these groups.

In 1987, Brown and Loday [2] introduced the nonabelian tensor square as a special case of the nonabelian tensor product. Later, the open problems on the nonabelian tensor squares of some finite groups had been posed by Brown *et al.* in [8]. The use of the nonabelian tensor squares for computing the Schur multipliers and capability of the groups were shown in this research.

Extending the research on the capability of a group by Baer in [3], many researches on the capability for various groups have been conducted over the years. Hall *et al.* [9] stated that a group is said to be capable if it is a central factor group. In 1979, the conditions for a group to be capable had been established by Beyl *et al.* [10]. They found that a group is capable if and only if the epicenter of the group is trivial. Later in 1995, Ellis in [11] characterized that a group is capable if and only if its exterior center is trivial.

In this research, by using the classifications of the groups of order p^3, p^4, p^5 and p^6, some of the properties of the groups, the Schur multipliers and nonabelian tensor squares are computed. After that, the Schur multipliers and nonabelian tensor squares of the groups are used in computing which groups are capable. Some results from previous researchers are used in determining the Schur multipliers, nonabelian tensor squares and capability of the groups.

1.3 Problem Statements

Given the groups of order p^3, p^4, p^5 and p^6, the following questions are addressed and answered:

(i) What are the center, derived subgroups and abelianization of these groups?

(ii) What are the Schur multipliers of these groups?

(iii) What are the nonabelian tensor squares of these groups?

(iv) Are these groups capable?

1.4 Research Objectives

The objectives of this research are:

(i) to compute the center, derived subgroups and abelianization for all groups of order p^4 together with abelian groups of order p^5 and p^6.

(ii) to determine the Schur multipliers, $M(G)$, where G is the group of order p^3, p^4 and abelian groups of order p^5 and p^6,

(iii) to characterize the nonabelian tensor squares, $G \otimes G$,

(iv) to classify the capability of G.

1.5 Scope of the Study

This research focuses on the computations of the Schur multipliers, nonabelian tensor squares and capability of groups of order p^3, p^4, p^5 and p^6. For the groups of order p^3, there are five groups in the classifications which consists of three abelian groups and two nonabelian groups. For the nonabelian groups, these groups are also known as extra-special p-groups. All groups in the classifications for groups of order p^3 are considered in this research. For the groups of order p^4, 15 groups were classified and they consist of four abelian groups and 11 nonabelian groups. However, only 11 groups of groups of order p^4 are determined, which are four abelian groups and seven nonabelian groups. The remaining four nonabelian groups are not determined since the classification is split into a bigger class if $p > 3$. Meanwhile, for groups of order p^5 and p^6, the computation focuses only on the abelian groups. There are seven abelian groups and 11 abelian groups for groups of order p^5 and p^6, respectively. For both of the groups, a large number of nonabelian groups are constructed. Note that there exists 60 nonabelian groups of order p^5 and 493 nonabelian groups of order p^6 for $p = 3$ and there are supplementary if $p > 3$ are found. Therefore, it is too complex to classify the groups into one big family. Thus, these groups are not considered in this research.

1.6 Significance of Findings

The results obtained from this research contribute to new findings in Groups Theory. The results obtained can be used for further determination of the Schur multipliers, nonabelian tensor squares and capability of a family of groups and related research area. Besides, this research is to provide new theoretical results on the Schur multipliers, nonabelian tensor squares and capability of the groups in the scope which has not been stated in existing literatures. In addition, with the help of Groups, Algorithms and Programming (GAP) software, new algorithms are also produced. Thus, the findings of this research provide new results in Computational Group Theory of finite p-groups.

1.7 Research Methodology

The research starts from examining the classifications of groups of order p^3, p^4, p^5 and p^6. As mentioned earlier, classification of groups or order p^n are constructed by Burnside in [5]. Rashid in [12] had discussed the determinations of the Schur multipliers, nonabelian tensor squares and capability of groups of order p^2q and p^3q. In the first step, Groups, Algorithms and Programming (GAP) software is used in identifying the properties of the groups, the Schur multipliers, nonabelian tensor squares and capability of the groups. Next, the properties of the groups such as the center, derived subgroups and abelianization are computed. All of these properties are used in determining the Schur multipliers, nonabelian tensor squares and capability of the groups. Then, by using the same method as in [12], the Schur multipliers, nonabelian tensor squares and capability of groups of order p^3, p^4, p^5 and p^6 are determined. The Schur multipliers of the groups is also used in characterizing the capable groups. The computations are conducted by using some of the definitions and established results from previous researches. The research methodology is illustrated in Figure 1.1.

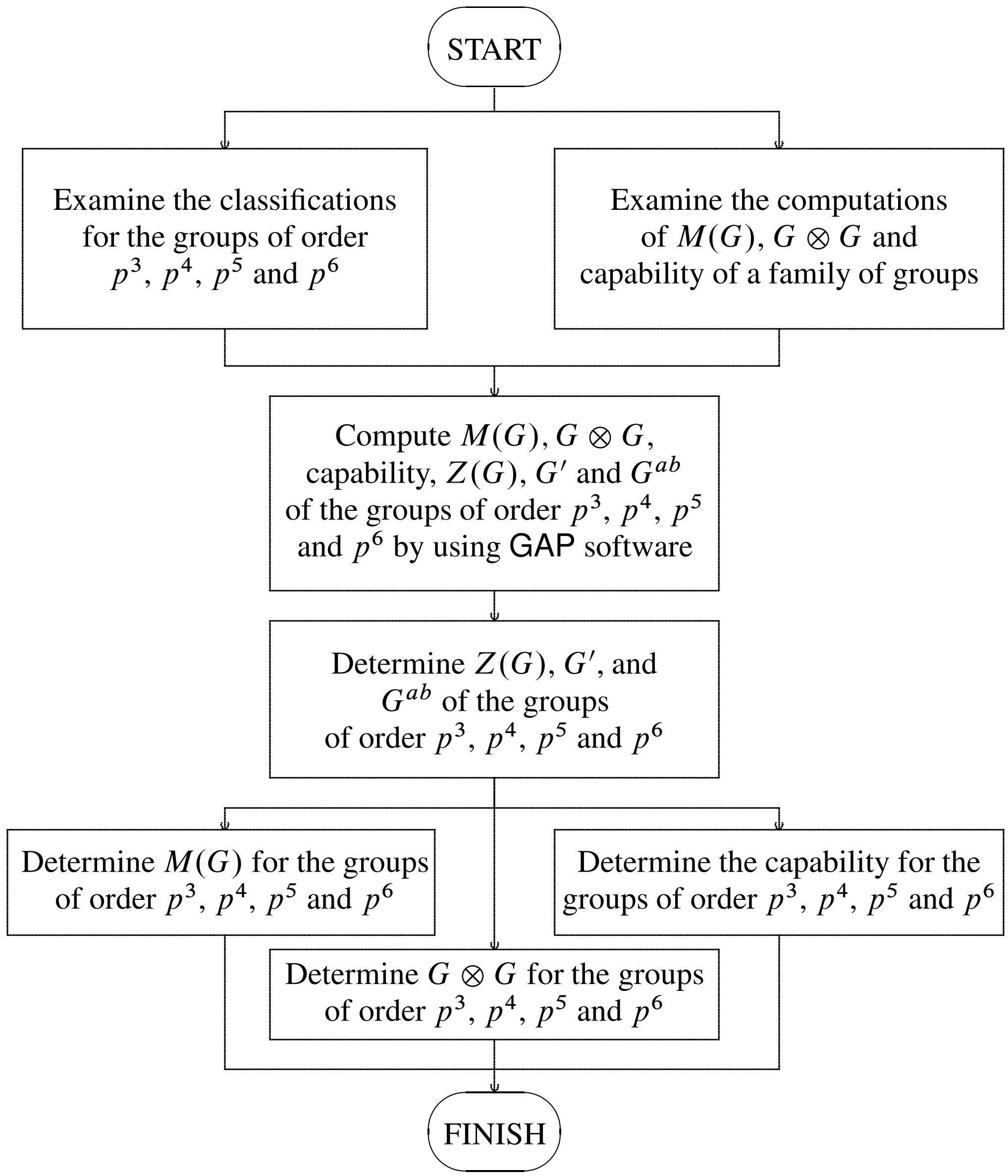

Figure 1.1 Research methodology

1.8 Thesis Organization

The first chapter gives the introduction to the whole thesis. The Schur multiplier, nonabelian tensor square and capability of groups are briefly discussed in this chapter. Chapter 1 includes research background, problem statements, research objectives, scope of the study, significance of the finding and research methodology.

The literature review of this research is presented in Chapter 2. Some concepts and established results from previous researchers on the Schur multipliers, nonabelian tensor squares and capability of groups are provided. Then, the classifications for groups of order p^3, p^4, p^5 and p^6 constructed by Burnside in [5] are discussed in this chapter. In addition, a brief introduction on Groups, Algorithms and Programming (GAP) software is also given.

In Chapter 3, the algebraic computations for some properties of the groups including the center, derived subgroups and abelianization, the Schur multipliers, nonabelian tensor squares and capability of groups of order p^3, p^4, p^5 and p^6 are computed using GAP software. The results are then used to make observations before a theoretical proof of the Schur multipliers, nonabelian tensor squares and capability of groups is provided. In addition, the GAP algorithms that generate the results are provided. Besides, the center, derived subgroups and abelianization of these groups are also determined in this chapter.

Chapter 4 until Chapter 6 focus on the determinations of the Schur multipliers, nonabelian tensor squares and capability of the groups, respectively. The Schur multipliers, nonabelian tensor squares and capability of groups of order p^3, p^4, p^5 and p^6 are considered in these chapters. The results are presented in several subsections.

Chapter 7 presents the summary and conclusion of this research. Some suggestions for future studies are also given in this chapter. Figure 1.2 illustrates the contents of this thesis.

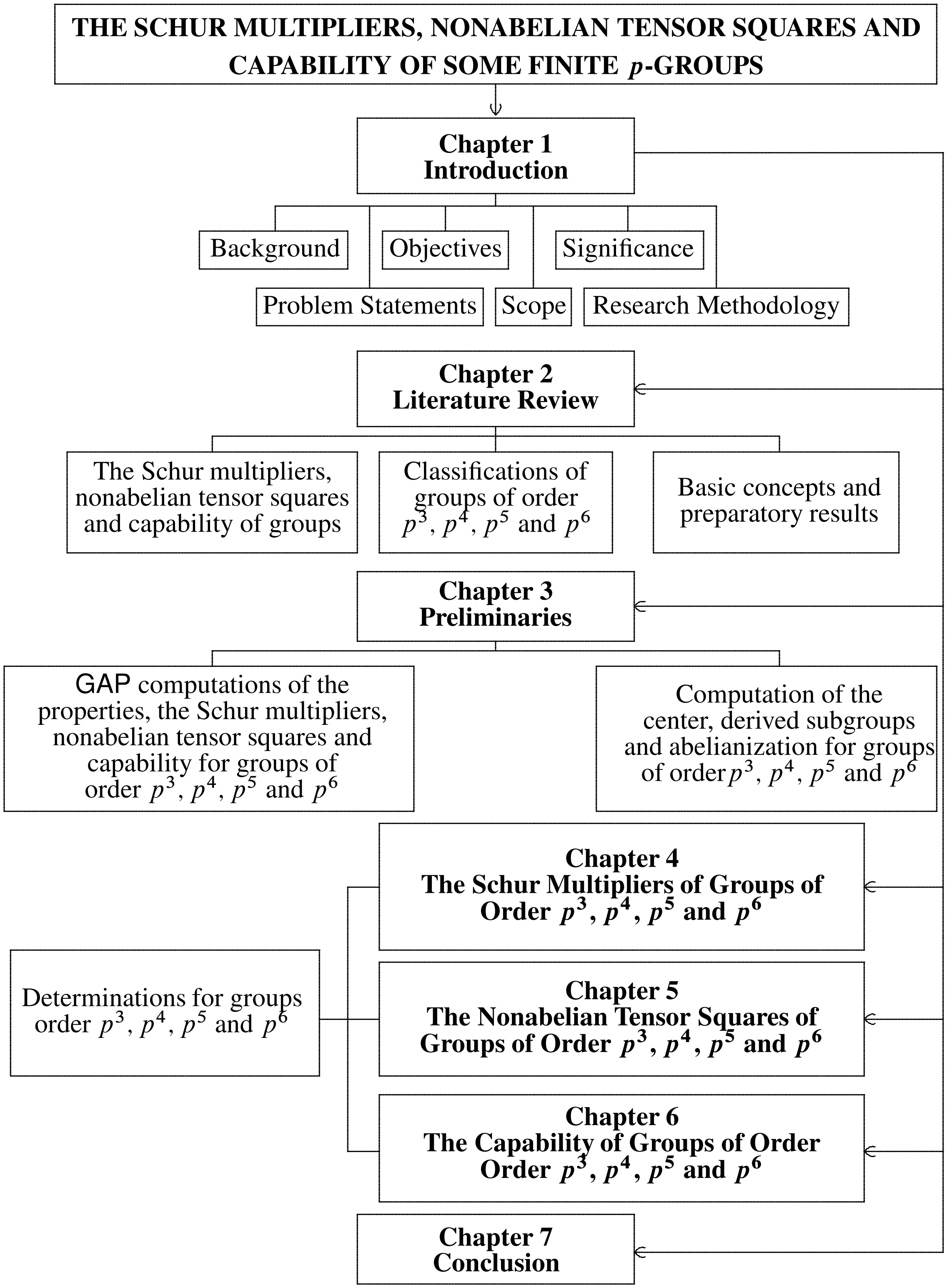

Figure 1.2 Thesis organization

REFERENCES

1. Schur, I. Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen. *J. F̈ur Math*, 1904. 127: 20–50.
2. Brown, R. and Loday, J. L. Van kampen theorems for diagrams of spaces. *Topology*, 1987. 26: 311–335.
3. Baer, R. Groups with preassigned central and central quotient group. *Trans. Amer. Math. Soc.*, 1938. 44: 387–412.
4. Hall, P. The classification of prime-power groups. *J. Reine Angew. Math.*, 1940. 182: 130–141.
5. Burnside, W. *The theory of groups of finite order*. 2th. ed. Cambridge: Cambridge University Press. 1911.
6. Green, J. A. On the number of automorphism of a finite group. *Proc. Royal Soc. A.*, 1956. 237: 574–581.
7. Berkovich, Y. On the order of the commutator subgroup and the Schur multiplier of a finite p-group. *J. Algebra*, 1991. 144: 269–272.
8. Brown, R., Johnson, D. L. and Robertson, E. F. Some computations of non-abelian tensor products of groups. *J. Algebra*, 1987. 111: 177–202.
9. Hall, M. and Senior, J. K. *The groups of order* 2^n*,* $n \leq 6$. New York: Macmillan. 1964.
10. Beyl, F. R., Felgner, U. and Schmid, P. On groups occuring as center factor groups. *J. Algebra*, 1979. 61: 161–177.
11. Ellis, G. Tensor products and q-crossed modules. *J. Lond. Math. Soc.*, 1995. 51: 243–258.

12. Rashid, S. *The Schur multiplier, nonabelian tensor square and capability of some finite groups*. Department of Mathematical Sciences, Universiti Teknologi Malaysia: Ph.D. Thesis. 2012.

13. Karpilovsky, G. *The Schur multiplier*. Oxford: Clarendon Press. 1987.

14. Zhou, X. On the order of Schur multipliers of finite p-groups. *Comm. Algebra*. 1994. 22: 1–8.

15. Ellis, G. On the Schur multiplier of p-groups. *Commn. Algebra*. 1999. 27(9): 4173–4177.

16. Salemkar, A. R., Moghaddam, M. R. R., Davarpanah, M. and Saeedi, F. A remark on the Schur multiplier of p-groups. *Comm. Algebra*. 2007. 35(4): 1215–1221.

17. Niroomand, P. On the order of Schur multiplier of non-abelian p-groups. *Journal of Algebra*, 2009. 322(112): 4479–4482.

18. Rashid, S., Sarmin, N. H., Erfanian, A. and Mohd Ali, N. M. On the Schur multiplier of groups of order $8q$. *International Journal of Applied Mathematics & Statistic*, 2012. 28(4): 18–22.

19. Niroomand, P. Characterizing finite p-groups by their Schur multipliers. *Comptes Rendus Mathematique*, 2012. 350(19-20): 867–870.

20. Mat Hassim, H. I., Sarmin, N. H., Mohd Ali, N. M., Masri, R. and Mohd Idrus, N. The Schur Multipliers of certain bieberbach groups with abelian point groups. *Proceedings of the 20th National Symposium of Mathematical Sciences*, December 18–20, 2012. Melville, NY: AIP. 2013. 1522: 1069–1074.

21. Hannebaur, T. On nonabelian tensor square of linear groups. *Archiv der Mathematik*, 1990. 55(1): 30–34.

22. Bacon, M. R. and Kappe, L. C. The nonabelian of a 2-generator p-group of class 2. *Archiv der Mathematik*, 1993. 61(6): 508–516.

23. Bacon, M. R., Kappe, L. C. and Morse, R. F. On the nonabelian tensor square of a 2-Engel group. *Archiv der Mathematik*, 1997. 69(5): 353–364.

24. Kappe, L. C., Visscher, M. P., and Sarmin, N. H. Two-generator two-groups of class two and their nonabelian tensor square. *Glasgow Math. J.*, 1999. 41: 417–430.

25. Sarmin,. N. H. Infinite two-generator groups of class two and their nonabelian tensor square. *International Journal of Mathematics and Mathematical Sciences*, 2002. 32(10): 615–625.

26. Erfanian, A., Rezaei, R. and Jafari, H. Computing the nonabelian tensor square of general linear groups. *Italian J. of Pure and Applied Math.*, 2008. 24: 203–210.

27. Rashid, S., Sarmin, N. H., Zainal, R., Mohd Ali, N. M. and Erfanian, A. A note on the nonabelian tensor square. *Indian Journal of Science and Technology*, 2012. 5(6): 2877–2879.

28. Jafari, S. H., Saeedi, F., and Khamseh, E. Characterization of finite p-groups by their non-abelian tensor square. *Communications in Algebra*, 2013. 41(5): 1954–1963.

29. Basri, A. M., Sarmin, N. H., Mohd Ali, N. M. and Beuerle, J. R. Computing the nonabelian tensor square of metacyclic p-groups of nilpotency class two. *Jurnal Teknologi (Sciences & Engineering)*, 2013. 61(1): 31–34.

30. Wan Mohd Fauzi, W. N. F., Mohd Idrus, N., Masri, R. and Sarmin, N. H. The nonabelian tensor square of Bieberbach group of dimension five with dihedral point group of order eight. *Proceedings of the 21th National Symposium of Mathematical Sciences*, November 6–8, 2013. Melville, NY: AIP. 2014. 1605: 611–616.

31. Ellis, G. On the capability of groups *Proc. Edinburg Math. Soc.*, 1998. 41: 487–495.

32. Bacon, M. R. and Kappe, L. C. On capable p-group of nilpotency class two. *Illinois Journal of Math.*, 2003. 47: 49–62.

33. Magidin, A. Capable two-generator 2 -groups of class two. *Communications in Algebra*, 2006. 34: 2183–2193.

34. Kappe, L. C., Mohd Ali, N. M. and Sarmin, N. H. On the capability of finitely generated non-torsion groups of nilpotency class 2. *Glasgow Mathematical Journal*, 2011. 53(2): 411–417.

35. Pourmirzaei, A., Hokmabadi, A., and Kayvanfar, S. Capability of a pair of groups. *Bulletin of The Malaysian Mathematical Sciences Society*, 2012. 35(1): 205–213.

36. Rashid, S., Sarmin, N. H., Erfanian, A., Mohd Ali, N. M. and Zainal, R. On the nonabelian tensor square and capability of groups of order $8q$. *Indagationes Mathematicae*, 2013. 24(3): 581–588.

37. Magidin, A. On the capability of finite groups of class two and prime exponent. *Publicationes Mathematicae*, 2014. 85(3-4): 309–337.

38. Fraleigh, J. B. *A First Course in Abstract Algebra*. Pearson Education, Inc. 2003.

39. Robinson, D. J. S. *A Course in the Theory of Groups*. Berlin: Springer-Verlag. 1982.

40. Saeden S. G., Laine, J. and Behm, G. *On p-groups of low power order*. KTH: School of Engineering Sciences (SCI). 2010.

41. Oh, Y. J. *Group with irreducible representations of prime degree*. Danguk University: Ph.D Thesis. 1985.

42. GAP. The groups, algorithms, and programming, version 4.6.3; http://www.gap-system.org. 2013.

43. Johnson, D. L. *Topics in the Theory of Group Presentations. London Mathematical Society Lecture Note Series, 42*. Cambridge: Cambridge University Press. 2008.

44. Basri, A. M. *Capability and homological functors of finite metacyclic p groups*. Department of Mathematical Sciences, Universiti Teknologi Malaysia: Ph.D. Thesis. 2014.

45. Hungerford, T. W. *Algebra*. New York: Springer-Verlag. 1974.

46. Zomorodian, A. J. *Topology for Computing*. Cambridge: Cambridge University Press. 2005.

47. Visscher, M. P. *On the nonabelian tensor product of groups*. Department of Mathematics, State University of New York at Binghamton: Ph.D. Thesis. 1998.

48. Nakaoka, I. N. Nonabelian tensor products of solvable groups. *J. Group Theory*. 2000. 3: 157–167.

49. Blyth, R. D., Fumagalli, F. and Morigi, M. Some structural results on the nonabelian tensor square of groups. *Journal of Group Theory*, 2010. 13: 83–94.

50. Ellis, G. On the capability of groups. *Proceedings Edinburgh Math. Soc.*, 1998. 41: 487–495.

51. Niroomand, P. and Russo, F. A note on the exterior centralizer. *Arch. Math,*, 2009. 93: 505–512.

52. Kim, S. O. On p-groups of Order p^4. *Comm. Korean Math. Soc.*, 2001. 16(2): 204–210.

53. Niroomand, P. and Russo, F. G. On the size of the third homotopy group of the suspension of an EilenbergMacLane space. *Turk. J. Math.*, 2014. 38: 664–671.

54. Magidin, A. Capable two-generator 2-groups of class two. *Comm. Algebra*, 2006. 34: 2183–2193.

55. Behravesh, H. and Mousavi, H. A note on p-groups of order $\leq p^4$. *Proceedings of Indian Acd. Sci. (Math. Sci.)*, 2009. 119(2): 137–143.