NUMERICAL SOLUTION OF ONE DIMENSIONAL BURGERS' EQUATION SOLVING WITH EXPLICIT FTCS METHOD AND IMPLCIT BTCS METHOD

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TO MY BELOVED PARENTS

Azmi Bin Zahari & Syarifah Fatimah Binti Syed Mahmood

for always loving and supporting me

TO MY RESPECTED SUPERVISOR

Tuan Haji Hamisan Bin Rahmat

for your patience and advice

AND TO ALL MY FIRENDS

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ABSTRACT

The turbulent flow in natural phenomena always exist everyday. This dissertation discusses on solving nonlinear of one dimensional Burgers' equation. The discussion begins with discretizing the equation using two different types of method which are Explicit Forward-Time-Central-Space (FTCS) method and Implicit Backward-Time-Central-Space (BTCS) method. The discretization from both method leads to form an algebraic system. For the case of using explicit FTCS, the system can be solve straightaway since it only consist of one unknown function for the next time step with the known function at previous time step. Apart from that, the discretization of implicit BTCS need to undergo an iterative technique since the system cannot be solve directly. Therefore Newton's method is used for system of nonlinear equations. Moreover, all numerical computation will be computed in MATLAB programming. The results achieve will be compared with from the journal. In conclusion, the results shows that solving with implicit BTCS are more preferable as it is unconditionally stable which can solve problem without the limitation of time step.

ABSTRAK

Aliran bergelora dalam fenomena alam semula sentiasa wujud dalam kehidupan seharian. Disertasi ini membincangkan tentang persamaan satu dimensi Burgers'. Perbincangan dimulakan dengan pendiskretan persamaan dengan menggunakan dua jenis kaedah iaitu kaedah Tak Tersirat masa beza depan – jarak beza tengah dan kaedah Tersirat masa beza belakang – jarak beza tengah. Pendiskretan daripada kedua-dua kaedah membawa kepada pembentukan sistem aljabar. Untuk kes yang menggunakan kaedah Tak Tersirat masa beza depan – jarak beza tengah, penyelesaian boleh diperoleh secara terus kerana ia melibatkan satu sebutan yang tidak diketahui untuk aras tertentu dengan sebutan yang diketahui pada aras sebelumnya. Selain daripada itu, pendiskretan tersirat masa beza belakang – jarak beza tengah perlu menjalani teknik lelaran kerana sistem tidak linear yang diperoleh tidak dapat diselesaikan secara langsung. Oleh itu, kaedah Newton digunakan untuk menyelesaikan sistem tidak linear. Tambahan itu, semua pengiraan berangka akan dikira dalam pengaturcaraan MATLAB. Hasil yang diperoleh daripada kedua-dua kaedah akan dibandingkan dengan jawapan yang boleh didapati dari journal. Kesimpulannya, hasil keputusan menunjukkan bahawa kaedah Tersirat masa beza belakang – jarak beza tengah adalah lebih baik kerana ia adalah tanpa syarat stabil yang boleh menyelesaikan masalah tanpa had langkah masa.

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v	-	Kinematic viscosity
r	-	Stability
ξ	-	Amplification factor
ϕ	-	phi, unknown function
φ	-	psi, unknown function

LIST OF ABBREVIATIONS

ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
FDM	Finite Difference Method
FTCS	Forward-Time Central-Space
BTCS	Backward-Time Central-Space

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CHAPTER 1

INTRODUCTION

1.1 Introduction

In this globalization era, the uses of applied mathematics in various fields are increasing year by year. Logan, (2013) stated that applied mathematics can be define as the abstract science of number, quantity and space which is applied to the other branch such as science, engineering, computer science and industry. Applied mathematics contains thousands of methods in order to solve ordinary differential equations (ODEs), Partial differential equations (PDEs), and integral equations. The mathematical models which are known as a set of equations that need to be formulate and analyze the model. Before solving the model, first must verify some important requirements such as initial and boundary conditions, variables used and other relevant quantities. After that, the model equations can be solved numerically or analytically. The important part is where we need to choose suitable method to avoid difficulties while solving the equations. Most of the mathematical models have a big problem, therefore it requires any relevant software such as MATLAB, C++, Maple and many more in order to achieve an accurate results. The significance of using software is that we can get result in a short time for calculating the equations and also avoid a lot of errors if we calculate manually.

1.2 Background of the Study

A substance can exist as a solid, a liquid or a gas. These are the three common state of matter which exists around us. Besides, fluid is a substance that consists of liquid and gas. Whereas mechanics is define as a branch of science that studies the physical behaviour which dealing with motion and forces producing motion. By combining the words as "fluid mechanic", thus the term defines as the application of the laws of force and motion to fluids. Fluid mechanics are divided into two branches which are fluid statics and fluid dynamics. The application of fluid mechanics mostly related in engineering problems. Apart from that, fluid dynamics describes as study of the behaviour of the fluid either at rest or in motion. In other words it is the study about liquid and gases in motion. Fluid dynamics is the oldest branch in physics. Faber, (1995) stated that in eighteenth century, Euler and Daniel Bernoulli are among the earliest people who found out about the theory of physics. They study about fluid dynamics and carry out an experiment by applying the Newton principle which study about how the particles can be discretized into liquid continuously. However, this experiment is quite vague for some several reasons. It is because fluids dynamics nowadays can only be seen in engineering application such as in aerodynamics, hydrodynamics and in other branch of sciences. Mohanty, (2006) stated that fluid dynamics considered as an important application that are related to the engineering problems. For the past few decades, many scientists such as mathematician and physician have proved that the study of fluid dynamics brings a lot of benefits to the engineering problems. Some examples of popular applications in fluid dynamics which are rocket engines, wind turbines, and air conditioning systems.

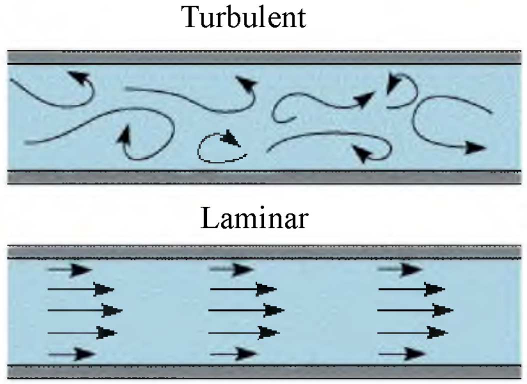

Figure 1.1: Type of fluid flows

Furthermore in fluid dynamics, flow can also be categorized as laminar or turbulent. Based on Figure 1.1 above shows that Laminar's flow are smoother than Turbulent's flow which are more chaotic. A steady flow refers when the flow is independent of time. For example, moving water in a pipe at a constant rate where it is also refers as laminar flow. Meanwhile, water moving on a surface is an example that shows an unsteady flow or called as turbulent flow. Mathematically, turbulent flow shows that it tends to be nonlinear problem since the flow are chaotic. Many researchers are more likely to study about turbulent flow compare to laminar flow. Because of fluid dynamics mostly referred as a nonlinear domain, thus there exist difference kinds of nonlinear equations.

Burgers' equation is known as nonlinear equations. In fields like science and engineering, are associated with PDEs either in the form of linear or nonlinear equations. It can be seen clearly through applications of turbulent. For example in fluid dynamics there are many applications use which involve PDEs such as differential formulations of the conservation laws apply Stokes' theorem to yield an expression which may be interpreted as the integral form of the law applied to an infinitesimal volume at a point within the flow. Apart from that, this study focuses on Burger's equations. In which Burger's equations is a fundamental part of PDEs.

Besides, the existence of Burger's equation can be seen in the area of applied mathematics such as in fluid mechanics, modeling dynamics, heat conduction and many more. It is named after Johannes Martinus Burgers (1895-1981). Burger's equation can be more than one dimension, hence for this study we only focus on one dimensional nonlinear Burger's equations. Solving nonlinear PDEs may having difficulties in the process to get the result compare to linear PDEs which are a lot more easier. Thus for this reason, many researchers found that numerical discretization techniques can be used to solve the nonlinear PDEs can make the solution less complicated.

Fan and Li (2014) stated that although Burger's equation and the momentum of well-known Navier-Stokes equation are similar, but the Burger's equations are easier to be solved than the governing equations of Navier-Stokes equation. Many studies have been done which was to compare between these two systems. Basically Burger's is a system with time-dependent and space-dependent partial differential equations. Meanwhile there are many meshfree numerical schemes were proposed for discretization. After all each method have its own advantages and privileges when solving the problem. In fact, many researches are interested in studying the solution of Burger's equation and trying to solve the problems related with many type of numerical techniques.

For this study we will use finite difference method (FDM) in order to solve one dimensional Burgers' equation. Two types of method from FDM will be proposed. First method namely Explicit Forward-Time Central-Space or (FTCS) method and second method is Implicit Backward-Time Central-Space or called as (BTCS) method. Result from both methods is then can be compared with exact solution to determine which method is more stable and have nearest value to the exact solution.

1.3 Problem Statement

In this area of study, will solve the one dimensional of Burgers' equations with two types of methods which are Explicit FTCS method and Implicit BTCS method. Besides, solving with Explicit FTCS can produce the solution directly. Meanwhile when solving with Implicit BTCS cannot be solved directly and need to undergo an iterative technique to get the solutions. The iterative techniques that will be used is Newton's method. Since Burgers' equation are known as a nonlinear equation, usually the number of unknown variables is typically very large. In spite of that, there are some problems that will appear while solving the solutions which are the stability issue and the accuracy issue. Numerical stability has to do with the behaviour of the solution as the time step increase. Indicating that there must be some limit on the size of the time step for there to be a solution. Apart from that, for the accuracy issue can be seen clearly when comparing the chosen solution from these two methods with the exact solution.

1.4 Objectives of the Study

The main purpose on doing this project is to solve the nonlinear of one-dimensional Burgers' equations. Here are the three objectives which are:

- To formulate Burgers' equation into discretization form.
- To investigate the consequences of using an Explicit FTCS versus an Implicit BTCS method.
- To determine which method are more stable for solving the problem.

1.5 Scope of the Study

The study focused on Burgers' equation. This study only concerns on nonlinear one-dimensional Burgers' equation. Since there are many methods that can be used to solve this problem. Thus for this study will be restricted to use Explicit FTCS method, Implicit BTCS method and Newton's method. Therefore, for this case of study MATLAB software is used for the numerical computation.

1.6 Significant of the Study

The importance of this study are:

- To provide numerical solution about Burgers' equation by using Explicit FTCS and Implicit BTCS.
- To adopt the method that can solve system of nonlinear algebraic equations in a very effective way and has great potential for large scale of engineering problems.

1.7 Dissertation Organization

The dissertation organizations are divided into five chapters. These chapter discussed about one dimensional of Burgers' equation which apply two types of methods such as explicit FTCS method. Then another methods used is an implicit BTCS method which form system of nonlinear equations then solving the systems by using Newton's method.

Chapter 1 discussed about short introduction regarding this study. Chapter 1 also includes background of study, problem statement, objectives of study, scope of study and significant of study.

Chapter 2 are about literature reviews for this study. Where lots of information mostly in books, research paper, journals is collected. The topics that have been discussed are nonlinear equations of PDEs, one dimensional of Burgers' equation, explicit FTCS method, implicit BTCS method, von Neumann stability analysis, and lastly is Newton's method.

Chapter 3 illustrates the numerical discretization to solve the one dimensional Burgers' equation. The first derivations is for explicit FTCS method. Second is implicit BTCS method. Lastly is about Newton's methods.

Next, in Chapter 4 shows the implementation of explicit FTCS method and implicit BTCS method on the problem for one dimensional Burgers' equation. Then problem will be compute in MATLAB software to get the result. Comparison between explicit FTCS, implicit BTCS, and exact solution will be provided in this section. Lastly in chapter 5, summary and recommendation for future research is included.

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