THE PROBABILITY THAT AN ELEMENT OF A NON-ABELIAN GROUP FIXES A SET AND ITS APPLICATIONS IN GRAPH THEORY

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To my family with love.

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ABSTRACT

The commutativity degree, defined as the probability that two randomly selected elements of a group commute, plays a very important role in determining the abelianness of a group. In this research, the commutativity degree is extended by finding the probability that a group element fixes a set. This probability is computed under two group actions on the set namely, the conjugate action and the regular action. The set under study consists of all commuting elements of order two of metacyclic 2-groups and dihedral groups of even order. The probabilities found turned out to depend on the cardinality of the set. The results which were obtained from the probability are then linked to graph theory, more precisely to orbit graph and generalized conjugacy class graph. It is found that the orbit graph and the generalized conjugacy class graph consist of complete graphs, empty graphs or null graphs. Moreover, some graph properties including the chromatic number, clique number, dominating number and independent number are found. In addition, the necessary condition for the orbit graph and generalized conjugacy class graph to be a null graph is examined. Furthermore, two new graphs are introduced, namely the generalized commuting graph and the generalized non-commuting graph. The generalized commuting graph of all groups in the scope of this research turns out to be a union of complete graphs or null graphs, while the generalized non-commuting graph consists of regular graphs, empty graphs or null graphs.

ABSTRAK

Darjah kekalisan tukar tertib, ditakrifkan sebagai kebarangkalian bagi dua unsur yang dipilih secara rawak daripada suatu kumpulan adalah kalis tukar tertib, memainkan peranan yang amat penting dalam menentukan keabelanan bagi sesuatu kumpulan. Dalam penyelidikan ini, darjah kekalisan tukar tertib tersebut diperluaskan dengan mencari kebarangkalian bahawa suatu unsur kumpulan menetapkan suatu set. Kebarangkalian ini dikira berdasarkan dua tindakan kumpulan pada set yang dinamai tindakan konjugat dan tindakan biasa. Set yang dikaji terdiri daripada semua unsur kalis tukar tertib yang berperingkat dua bagi kumpulan 2-metakitaran dan kumpulan dwihedron berperingkat genap. Kebarangkalian yang diperolehi ternyata bergantung kepada kardinaliti bagi set tersebut. Keputusan yang diperoleh daripada kebarangkalian itu kemudian dihubungkan kepada teori graf, lebih tepatnya kepada graf orbit dan graf kelas konjugatnya yang teritlak. Didapati bahawa graf orbit dan graf kelas konjugatnya yang teritlak terdiri daripada graf sempurna, graf kosong atau graf nol. Selain itu, beberapa sifat graf termasuk nombor kromat, nombor klik, nombor berdominan dan nombor tak bersandar diperolehi. Tambahan lagi, syarat perlu bagi graf orbit dan graf kelas konjugat yang teritlak untuk menjadi graf nol ditentukan. Seterusnya, dua graf baharu diperkenalkan iaitu graf kalis tukar tertib yang teritlak dan graf bukan kalis tukar tertib yang teritlak. Graf kalis tukar tertib yang teritlak bagi semua kumpulan dalam skop kajian ini adalah kesatuan graf yang sempurna atau graf nol, sementara graf bukan kalis tukar tertib yang teritlak terdiri daripada graf biasa, graf kosong atau graf nol.

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LIST OF SYMBOLS

S	-	A non-empty set
$\in$	-	Belong to
$Z(G)$	-	Center of the group G
$\chi(\Gamma)$	-	The chromatic number
$\omega(\Gamma)$	-	The clique number of the graph Γ
$P(G)$	-	The commutativity degree of the group G
$[g,h]$	-	The commutator of g and h, that is, $[g,h]=g^{-1}h^{-1}gh$
K_n	-	Complete graph of n vertices
D_n	-	Dihedral groups of order $2n$
$\alpha(\Gamma)$	-	The independent number of the graph Γ
$\gamma(\Gamma)$	-	The dominating number of the graph Γ
K_e	-	Empty graph
$\forall$	-	For all
$\Gamma_G^{\Omega_c}$	-	Generalized conjugacy class
Γ	-	Graph
$\geq$	-	Greater than or equal
$H \leq G$	-	H is a subgroup of G
1	-	Identity element of the group G
$[G:H]$	-	The index of H in G
$\text{lcm}(a,b)$	-	The least common multiple of a and b
$\leq$	-	Less than or equal
$\mathbb{N}$	-	The set of natural numbers

K_0	-	Null graph
$K(G)$	-	The number of conjugacy classes in G
$K(\Omega)$	-	The number of orbits of Ω in G
Γ_G^{Ω}	-	The orbit graph
Γ_{Ω}^{GC}	-	The generalized commuting graph
Γ_{Ω}^{GN}	-	The generalized non-commuting graph
$\mathrm{cl}(x)$	-	Orbit of x in G
$\lvert G \rvert$	-	Order of the group G
$P_G(\Omega)$	-	The probability that a group element fixes a set
$E(\Gamma)$	-	Set of edges of Γ
$V(\Gamma)$	-	Set of vertices of Γ
Ω	-	The set of all subsets of commuting elements of size two
$C_G(s)$	-	The stabilizer of the element $s \in S$ under a group action of G on the set S
Γ_{sub}	-	Sub graph of the graph Γ

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Background of the Study

A group G acts on itself by conjugation if there is a function $\Phi: G \times G \to G$ such that $\Phi((g,x)) = xgx^{-1}$ for all $x \in G$, the conjugacy class of x in G is defined as $\text{cl}(x) = \{gxg^{-1} : g \in G\}$. If G acts on a set S, then there exists $\Phi: G \times S \to S$, such that the orbit of s under the group action is $O(s) = \{gs : g \in G\} \subseteq S$. Now if a group G acts regularly on S, then there exists $g \in G$, such that $gs_1 = s_2$, where $s_1, s_2 \in S$. These actions are used to find the commutativity degree and its generalizations, which are the focus in the first part of this research.

The commutativity degree is used to determine the abelianness of a group. This concept was first introduced by Miller in 1944 [1]. A few years later, the idea of commutativity degree was then investigated for symmetric groups by Erdos and Turan [2]. Their results encourage many researches to work on this topic, thus various generalizations have been done.

The second part of this research deals with graph theory. The idea of graphs came from a variety of real situations that can easily be described by drawing diagram consisting of a set of points and lines linking these points [3]. This concept was introduced in 1736 by Leonard Euler who considered the Kőnigsberg bridge problem. Euler solved this problem by drawing a graph with vertices and edges [3]. Years later, the usefulness of graph theory has been proven to a large number of diverse fields.

In 1990 Bertram *et al.* [4] introduced a new graph which relates to conjugacy classes. The vertices of this graph are non-central conjugacy classes in which two vertices are adjacent if they are not coprime. Many researches have been done using this algebraic graph theory. Recently, Erfanian and Tolue [5] introduced a new graph, namely conjugate graph in which the vertices are non-central elements of a group. Two vertices of this graph are adjacent if the elements represented by these vertices are conjugate with one another.

In 1975, the idea of non-commuting graphs comes from an old question of Erdős on the size of the cliques and answered in the affirmative by Neumann [6]. In [7], it is mentioned that the commuting graph is a graph whose vertices are non-central elements of a group, where two vertices are adjacent whenever they commute.

1.2 Objectives of the Research

The main objectives of this research are stated in the following:

(i) To extend the work on commutativity degree by defining the probability that a group element fixes a set.

(ii) To find the probability that a group element fixes a set for metacyclic 2-groups and dihedral groups.

(iii) To apply the results that are obtained from (ii) to graph theory, namely to the orbit graph, generalized conjugacy class graph, the generalized non-commuting graph and the generalized commuting graph.

(iv) To determine some graph properties such as the chromatic number, clique number, dominating number and independence number for the graphs in (iii).

1.3 Problem Statements

The commutativity degree has been generalized and extended by many authors, such as the probability that an automorphism of a finite group fixes an arbitrary element in the group, the probability that the commutator of two randomly chosen elements in a group is equal to a given element in the same group and the probability that an automorphism of a finite group fixes a subgroup element.

In this research, the following questions will be answered:

(i) What other extension can be made on the commutativity degree, based on the set of all commuting elements of $G \times G$ in the form of (a,b) where $|a| = |b| = 2$?

(ii) Can the group actions be applied to the set under investigation to find the probability?

(iii) How can the results obtained from the probability be applied to graph theory?

(iv) Can new graphs be introduced based on the obtained results from the probability?

(v) What are some graph properties that can be obtained?

1.4 Scope of the Research

This research consists of two parts. The first part focuses on the commutativity degree and its generalizations. This study covers some finite non-Abelian groups including metacyclic 2-groups and dihedral groups. One of the objectives in this research is to find the probability that a group element fixes a set under two group actions, namely conjugate action and regular action. The set under study is a non-empty subset of $G \times G$ in the form of (a, b), where a and b commute and $|a|=|b| = 2$.

The second part of this research focuses on graph theory. The probability under study in the first part is then applied to graph theory. The graphs under considerations in this research include orbit graph, graph related to conjugacy classes, non-commuting graph, commuting graph and conjugate graph. Some graph properties are obtained which include the chromatic number, clique number, independent number and dominating number.

1.5 Significance of the Research

The commutativity degree is one of the interesting topics in pure mathematics that has many applications on studying the structures of finite non-abelian groups.

The major contribution of this research is to provide new theoretical results on the commutativity degree by defining the probability that a group element fixes a set. This probability is computed for some finite non-abelian groups in the scope of this research. The results obtained from this probability are then linked to graph theory. Graph theory has many applications in chemistry, physics, computer science and engineering. In addition new graphs in this research are introduced. Besides, some new results on graph properties are obtained.

1.6 Research Methodology

In this research, commutativity degree is extended to the probability that a group element fixes a set, in which the set under consideration is a non-empty subset of $G \times G$ in the form of (a, b), where a and b commute and $|a| = |b| = 2$. The orbits, which are computed using two group actions on the set namely, conjugate action and regular action are used to determine the probability. This probability is the ratio of the orbits to the order of the set under examination. Beuerle [8] classified all non-Abelian metacyclic p-groups, where p is any prime. These classifications are used to obtain the probability that a group element fixes a set.

The obtained results on the probability that a group element fixes a set are applied to graph theory, more precisely the orbit graph and the generalized conjugacy class graph. The set under examination is used to find the number of vertices of the orbit graph and the generalized conjugacy class graph. Moreover the orbits and their sizes are used to determine the adjacency between the vertices of both graphs. The commuting elements of the set under the scope of this research are also used to obtain some new results in graph theory. In doing this, two new graphs are introduced, namely the generalized commuting graph and the generalized non-commuting graph. Lastly, the concept of orbits and their sizes are used to examine the relationship between the algebraic properties of graphs and groups.

1.7 Thesis Organization

There are six chapters in this thesis. In the first chapter the introduction to the whole thesis is provided. This chapter includes research background, problem statement, research objectives, scope and significance of findings of the thesis, research methodology and thesis organization.

In Chapter 2, the literature review of this research is presented. In this chapter, some earlier and recent works related to commutativity degree and its generalizations are provided. This chapter also provides some fundamental concepts of group theory and graph theory. A background about graph theory is stated, where different type of graphs and their algebraic properties are given.

In Chapter 3, the probability that an element of a group fixes a set is computed for some finite non-Abelian groups which include metacyclic 2-groups and dihedral groups. The probability is found under two group actions, namely conjugate and regular action.

In Chapter 4, the results that are found in Chapter 3 are then applied to graph theory, particularly to the orbit graph and generalized conjugacy class graph. These graphs are found for some finite groups which include metacyclic 2-groups and dihedral groups. Some graph properties are also obtained for the orbit graph and generalized conjugacy class graph of the groups.

In Chapter 5, new graphs are introduced, namely the generalized commuting graph and the generalized non-commuting graph. These two graphs are found for some finite groups which include metacyclic 2-groups and dihedral groups.

In the last chapter, namely Chapter 6, the summary of the whole thesis and suggestions for future research on the probability that a group element fixes a set, the orbit graph, generalized conjugacy class graph, generalized commuting graph and generalized non-commuting graph are provided. Figure 1.1 summarizes the content of this thesis.

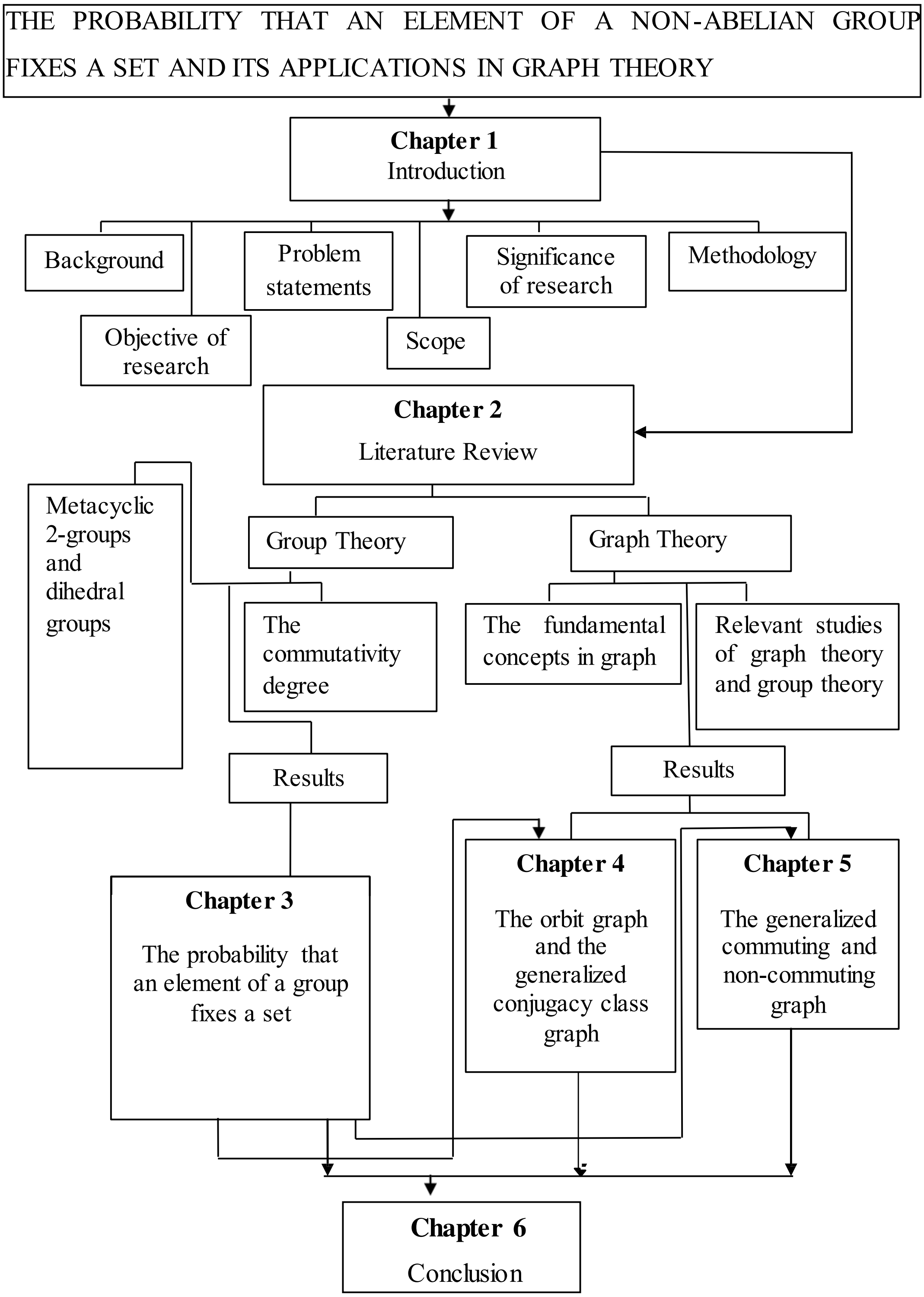

Figure 1.1 Thesis Outline

independent number are found. Moreover, a result that linked between the probability under conjugate action and the null graph is provided.

In this study, a new graph called the generalized commuting graph is introduced. The generalized commuting graph is null whenever the group is Abelian. The generalized commuting graph is found for all groups in the scope of this research. In addition, a new graph called the generalized non-commuting graph is also introduced. It is proven that the generalized non-commuting graph is null for all abelian groups. The generalized non-commuting graph is also obtained for all groups in the scope of this research.

6.2 Suggestions for Further Research

Based on this research, further studies on the probability that an element of a group fixes a set can be done. The probability can be obtained for other groups such as 2-generated 2-groups, metacyclic p-groups where p is odd prime, metabelian groups and nilpotent groups. This probability can be obtained under some group action on the set including, regular action, conjugate action and faithful action. Moreover, the order of the set can be changed to higher order and the probability can be obtained under group action on the set.

The results obtained from the probability, which obtained under some group actions on the set under consideration can be applied to graph theory. Where the relation between the elements of the set Ω and it is size are applied to some graphs including, the orbit graph, generalized conjugacy class graph, the generalized commuting graph and the generalized non-commuting graph. Moreover, some work in the characterization problems can be done such as, to characterize the groups which have complete graphs as their generalized commuting graphs. In addition some graph properties can be obtained to the mention graphs. Also, new graphs can be introduced based on the size and the relation between the elements of the set Ω, with proper clear definitions.

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