THE RELATIVE COMMUTATIVITY DEGREE AND SUB-MULTIPLICATIVE DEGREE FOR NONCYCLIC SUBGROUPS OF SOME NONABELIAN METABELIAN GROUPS

FADHILAH BINTI ABU BAKAR

UNIVERSITI TEKNOLOGI MALAYSIA

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FADHILAH ABU BAKAR

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To my beloved Ummi and Abah

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ABSTRACT

A metabelian group is a group G that has at least an abelian normal subgroup N such that the quotient group ${}^{G}/_{N}$ is also abelian. The concept of commutativity degree plays an important role in determining the abelianness of the group. This concept has been extended to the relative commutativity degree of a subgroup H of a group G which is defined as the probability that an element of H commutes with an element of G. This notion is further extended to the notion of the multiplicative degree of a group G which is defined as the probability that the product of a pair of elements chosen randomly from a group G is in the given subgroup of H. By using those two definitions with an assistance from Groups, Algorithms and Programming and Maple software, the relative commutativity degree and sub-multiplicative degree for noncyclic subgroups of nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24 are determined in this dissertation.

ABSTRAK

Suatu kumpulan metabelan adalah kumpulan G yang mengandungi sekurang-kurangnya satu subkumpulan normal abelan N dengan syarat kumpulan pembahagi ${}^{G}/_{N}$ juga adalah abelan. Konsep darjah kekalisan tukar tertib memainkan peranan penting dalam menentukan keabelan bagi suatu kumpulan. Konsep ini telah dilanjutkan kepada darjah kekalisan tukar tertib relatif bagi suatu subkumpulan H bagi suatu kumpulan G yang ditakrifkan sebagai kebarangkalian bahawa suatu unsur H berkalis tukar tertib dengan suatu unsur dari G. Konsep ini diperlanjutkan kepada konsep darjah pendaraban yang ditakrifkan sebagai kebarangkalian bahawa hasil darab sepasang unsur yang dipilih secara rawak dari kumpulan G, berada dalam subkumpulan H. Dengan menggunakan kedua-dua takrifan tersebut beserta bantuan daripada perisian *Groups, Algorithms and Programming* dan *Maple*, darjah kekalisan tukar tertib relatif dan darjah sub-pendaraban bagi subkumpulan tidak kitaran bagi kumpulan metabelan yang tidak abelan bagi peringkat kurang daripada 24 dan kumpulan dwihedron bagi peringkat sekurang-kurangnya 24 telah ditentukan di dalam disertasi ini.

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LIST OF SYMBOLS

$Z(G)$	-	Center of a group G
$P(G)$	-	Commutativity degree of a group G
$[h^n, g]$	-	Commutator of two elements, h^n and g
X	-	Direct product
$\in$	-	Element of
$=$	-	Equal to
$\forall$	-	For all
$\geq$	-	Greater than or equal to
$\langle x \rangle$	-	Group generated by the element x
h^n	-	$n^{-1}hn$, conjugate element
$e, 1$	-	Identity element in a group
$\cong$	-	Isomorphic
$P_{mul}(G)$	-	Multiplicative degree of a group G
N	-	Normal subgroup
$\not\subseteq$	-	Not a subset of
$\notin$	-	Not an element of
$\neq$	-	Not equal to
$P_n(G)$	-	n-th commutativity degree of G
$\|G\|$	-	Order of the group G
$\|H\|$	-	Order of the subgroup H
${}^{G}/_{N}$	-	Quotient group
$P(H, G)$	-	Relative commutativity degree of G

$P_n(H, G)$	-	Relative n-th commutativity degree of G
$P_{subm}(H, G)$	-	Sub-Multiplicative degree of a group G
$\subset$	-	Subset
$x \in G \backslash H$	-	x is an element of G but not in H

CHAPTER 1

INTRODUCTION

1.1 Introduction

Probability theory is the mathematical study of uncertainty. Every day we face some situations where the result is uncertain, and perhaps without realizing it, we guess about the likelihood of one outcome or another. These days, probability theory plays an increasingly greater role in many fields. Algebra and analysis is one of the fields that take probabilistic theory importantly.

A metabelian group is a group G that has at least a normal subgroup N such that N and ${}^{G}/_{N}$ are both abelian. A metabelian group can be considered as a group that is closed to being abelian, in the sense that every abelian group is metabelian, but not all metabelian groups are abelian. By the definition of metabelian group, any dihedral group, D_n is metabelian since it has a cyclic normal subgroup, N and the quotient group of D_n/N is a cyclic group of order two, hence abelian.

Let a and b be elements in a group G. Then the commutativity degree of a group G is the probability of a random pair of elements in G commute. It is also equivalent as the number of pairs (a,b) such that $ab = ba$, divided by the total number of all possible pairs of (a,b).

By extending the notion $ab = ba$ to $a^n b = ba^n$, the n^{th} commutativity degree of a group G can be found. By considering the total number of pair (a,b) for which a^n and b commute i.e. $a^n b = ba^n$, divided by the total number of possible pairs of (a,b). Thus the n^{th} commutativity degree of a group is determined.

Next, the concept of relative commutativity degree of a subgroup H of a group G is defined as the extension of the concept of commutavity degree. The relative commutativity degree is defined as the probability for an element of H to commute with an element of G.

The notion of commutativity degree has lead to the concept of multiplicative degree of a group G which is defined as the probability that the product of a pair of elements chosen randomly from a group G is in the given subgroup H of G. There are four cases that can be considered in the concept of multiplicative degree which are $x, y \in H$, $x \in G \backslash H$ but $y \in H$, $x \in H$ but $y \in G \backslash H$ and lastly $x, y \in G \backslash H$. For the case $x, y \in G \backslash H$, we defined it as the sub-multiplicative degree where it is the probability that the product of a pair of elements chosen randomly from a group G but not in H is in the given subgroup H of G. In this research, the relative commutativity degree and sub-multiplicative degree of nonabelian metabelian groups of order less than 24 and dihedral group of order at most 24 are determined where H is their noncyclic subgroup.

1.2 Research Background

Metabelian group can be considered as a group close to being abelian, in the sense that every abelian group is metabelian, but not all metabelian groups are abelian. Metabelian can be described as Abelian-by-Abelian. Abdul Rahman *et al.* [1] has found all the metabelian groups of order at most 24 in 2012.

First, let a and b be elements in a group G. Then the commutativity degree of a group G is the total number of a random pairs of elements in G that commute divided by the total number of possible pairs of elements in G. Miller [2] was the first person who introduced the concept of commutativity degree of a finite group in his paper in 1944.

In 2006, Mohd Ali and Sarmin [3] extended the notion of commutativity degree of finite group G by defining the n^{th} commutativity degree of G denoted by $P_n(G)$. The n^{th} commutativity degree is the probability that the n^{th} power of a random element commutes with another random element from the same group.

In 2007, Erfanian *et al.* [4] introduced the concept of relative commutativity degree of a subgroup H of a group G, denoted as $P(H,G)$. The relative commutativity degree of a subgroup H, which is the probability for an element of H to commute with an element of G, is the extension of the concept of commutativity degree.

In early 2016, Abd Rhani *et al.* [5] introduced the multiplicative degree of a group G denoted as $P_{mul}(G)$. This is the further extension of the concept of relative commutativity degree. It is defined as the probability that the product of a pair of elements chosen randomly from a group G is in the given subgroup H of G. There are four cases that can be considered which are $x, y \in H$, $x \in G \backslash H$ but $y \in H$, $x \in H$ but $y \in G \backslash H$ and lastly $x, y \in G \backslash H$. For the case $x, y \in H$, then we have $P_{mul}(G) = \frac{|H|^2}{|G|^2}$. For the case $x, y \in G \backslash H$, it is defined as the sub-multiplicative degree which is the probability that the product of a pair of elements chosen randomly from a group G but not in H is in the given subgroup H of G.

Previously, the commutativity degree of nonabelian metabelian groups of order at most 24 has been determined by Che Mohd [6] in 2011. Two years later, Abd Halim and Mohd Ali [7] determined $P_n(G)$ for some nonabelian metabelian groups. The relative commutativity degree of a cyclic subgroup of metabelian group of order at most 24 has been determined by Hassan in [8] and recently, the sub-multiplicative degree for cyclic subgroups of nonabelian metabelian groups of order less than 24 is determined by Mustafa *et al.* in [9] and Jaafar *et al.* in [10].

In this research, the relative commutativity degree and multiplicative degree for noncyclic subgroups of nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24 are determined.

1.3 Problem Statement

Previous studies have been done in determining the relative commutativity degree and sub-multiplicative degree for various groups. However, the relative commutativity degree and sub-multiplicative degree for noncyclic subgroups of nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24 have not been done yet. Previous researchers only focus on cyclic subgroups of the groups. Thus problem that arises here is what is the relative commutativity degree and sub-multiplicative degree for noncyclic subgroups of nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24.

1.4 Research Objectives

The objectives of this research are:

1. To determine the noncyclic subgroups of nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24.
2. To find the relative commutativity degree for noncyclic subgroups of nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24.

3. To determine the sub-multiplicative degree for noncyclic subgroups of nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24.

1.5 Scope of the Study

This research consists of three parts. The first part focuses only on noncyclic subgroups of nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24. The second part of this research focuses on the relative commutativity degree for noncyclic subgroups of nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24. The last part covers on the sub-multiplicative degree for noncyclic subgroups of nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24.

1.6 Significance of Findings

The results of this research can be used for further research in related areas such as to find other generalized commutativity degree involving noncyclic subgroups of nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24. In addition, this research will lead to the production of technical reports and papers published in an open literature.

1.7 Research Methodology

This research begins by studying the various concepts and properties of the nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24. To find the Cayley table of respective groups, Maple Software [29]is use.Next, the noncyclic subgroups of those groups are determined by the assistance of Groups, Algorithms and Programming (GAP) Software [11]. Then, the concept of commutativity degree is studied to have more understanding on the relative commutativity degree and sub-multiplicative degree. Next, the relative commutativity degree are computed for noncyclic subgroups of nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24. Lastly, the sub-multiplicative degree are computed for the same groups. Figure 1.1 shows the methodology of this dissertation

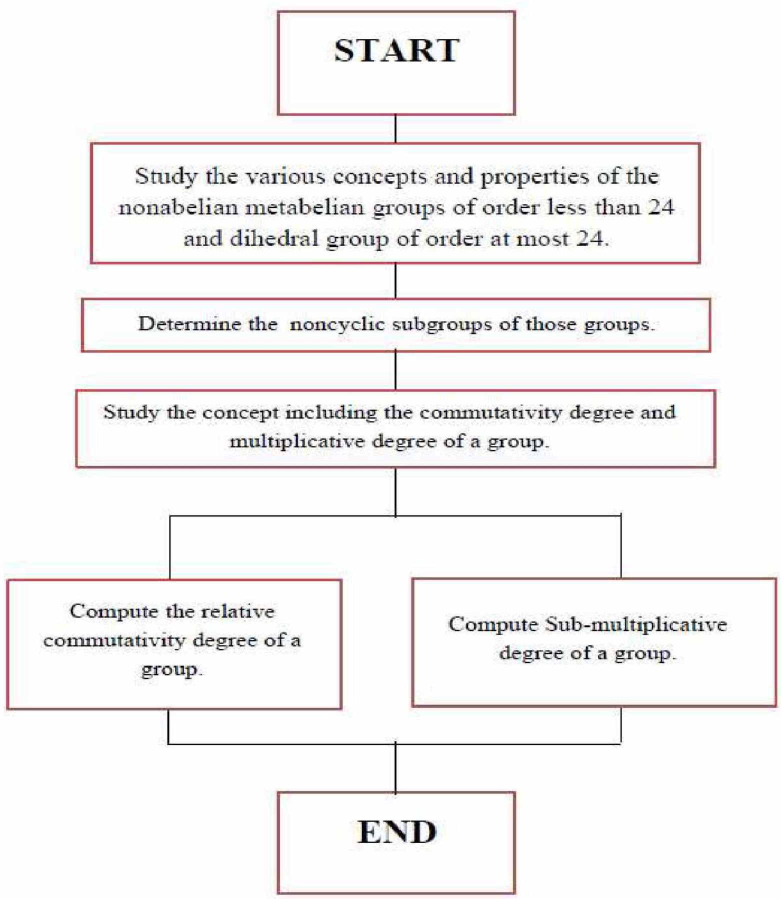

Figure 1.1 Research Methodology

1.8 Dissertation Organization

This thesis is divided into six chapters which includes the introduction, literature review, the noncyclic subgroups of some nonabelian metabelian groups, the relative commutativity degree of some nonabelian metabelian groups, the sub-multiplicative degree of some nonabelian metabelian groups and lastly the conclusion.

The first chapter serves as an introduction to the whole dissertation including the research background, problem statements, research objectives, scope of the study, significance of finding, research methodology and dissertation organization.

In Chapter 2, the literature review of this research is presented. In this chapter, a brief history about commutativity degree is stated in the background of the study, where some earlier and recent works related to commutativity degree and its generalizations namely the relative commutativity degree, n^{th} commutativity degree and the sub-multiplicative degree are provided. Some definitions and basic concepts that are used throughout this research are also included.

All noncyclic subgroups of all nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24 are presented in Chapter 3. Groups, Algorithms and Programming (GAP) Software [11] is used to assist in the finding of those noncyclic subgroups.

Chapter 4 and 5 discuss on the findings of this dissertation. Chapter 4 includes the results on the relative commutativity degree for noncyclic subgroups of all nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24. Chapter 5 discusses on the findings and the results on the sub-multiplicative degree for noncyclic subgroups of all nonabelian metabelian groups of order less than 24 and dihedral groups of order at most 24.

Lastly, Chapter 6 is the conclusion chapter where the summary and some useful suggestions for future research are included. Published papers in conferences are listed in Appendix A. The Cayley tables and 0-1 Table for all nonabelian metabelian groups of order less than 24 with the exception of dihedral groups are included in Appendix B, and the Cayley tables and 0-1 Table for all dihedral groups of order at most 24 are included in Appendix C.

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