CHAOS AND STRANGE ATTRACTORS OF THE LORENZ EQUATIONS

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To my beloved parents,

Salim Bin Abdullah
Che Aini Binti Che Teh

To my supervisor,

Puan Halijah Binti Osman

Thank you for everything.

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ABSTRACT

This research introduces and analyzes the famous Lorenz equations which are a classical example of a dynamical continuous system exhibiting chaotic behavior. This system is a three-dimensional system of first order autonomous differential equations and their dynamics are quite complicated. Some basic dynamical properties, such as stability, bifurcations, chaos and attractor are studied, either qualitatively or quantitatively. The visualization of the strange attractor and chaotic orbit are displayed using phase portrait and also the time series graph. A way to detect the chaotic behavior of an orbit is by using the Lyapunov exponents which indicate chaoticity if there is at least one positive Lyapunov exponent. The Lyapunov dimension called Kaplan-Yorke dimension of the chaotic attractor of this system is calculated to prove the strangeness by non-integer number. Several visualization methods are applied to this system to help better understand the long time behavior of the system. This is achieved by varying the parameters and initial conditions to see the kind of behavior induced by the Lorenz equations. The mathematical algebra softwares, Matlab and Maple, are utilized to facilitate the study. Also, the compound structure of the butterfly-shaped attractor named Lorenz attractor is also explored.

ABSTRAK

Kajian ini memperkenalkan dan menganalisis persamaan Lorenz yang terkenal di mana ianya merupakan contoh klasik sistem selanjar dinamik yang mempamerkan tingkah laku kelam kabut. Sistem ini adalah sistem tiga dimensi peringkat pertama persamaan pembezaan autonomi yang dinamiknya agak rumit. Sesetengah sifat dinamik asas, seperti kestabilan, bifurkasi, kelam kabut dan attraktor dikaji, sama ada secara kualitatif atau kuantitatif. Visualisasi attraktor pelik dan orbit kelam kabut juga dipaparkan dengan menggunakan fasa potret dan juga graf siri masa. Cara untuk mengesan kelakuan kelam kabut edarannya adalah dengan menggunakan nilai Lyapunov yang menunjukkan keadaan kelam kabut jika terdapat sekurang-kurangnya satu nilai positif Lyapunov. Dimensi Lyapunov yang dipanggil juga sebagai dimensi Kaplan-Yorke dikira untuk membuktikan keanehan attraktor dengan wujudnya nombor bukan integer. Menggunakan kaedah visualisasi yang berbeza untuk bantuan sistem ini supaya lebih memahami tingkah-laku jangka masa panjang bergantung kepada nilai-nilai parameter yang berbeza dan syarat awal sistem Lorenz untuk menggambarkan tingkah-laku oleh persamaan Lorenz. Perisian matematik algebra, Matlab dan Maple, digunakan untuk memudahkan kajian ini. Selain itu, struktur sebatian attraktor berbentuk rama-rama yang dinamakan Lorenz attraktor juga diterokai.

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LIST OF SYMBOLS

x	-	Intensity of the convective motion
y	-	Temperature difference between the ascending and descending currents
z	-	Distortion of the vertical temperature profile from linearity
r	-	Rayleigh constant
r_c	-	Critical Value
b	-	Physical proportional of the cell
σ	-	Prandtl number
E	-	Eigenvalue
λ	-	Lyapunov exponent
D_{KY}	-	Kaplan-Yorke dimension
t	-	Dimensionless time

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CHAPTER 1

INTRODUCTION

1.1 Background of Study

Ordinary differential equations are divided into linear and nonlinear systems. Linear systems have only one equilibrium point at the origin. This is in contrast with nonlinear systems which may have many equilibrium points. According to Hirsch (2004), some nonlinear systems have no solutions to a given initial value problem and some systems have infinitely many different such solutions. Systems of nonlinear differential equations play an important part in subjects as diverse as meteorology, oceanography, optics, economics, biology, etc. Such dynamical systems are rarely have exact analytical solutions, and numerical modeling is frequently required to supplement experimental research. Nonlinear systems are found to exhibit some interesting behaviors, including chaos.

There are several difficulties in dealing with nonlinear dynamical system. Firstly, greater number of possible cases can occur, and the number increases with the dimension of the system. Secondly, it is not easy to plot accurate and understandable trajectories in a phase space of higher than two dimensional systems. The higher the number of dimensions, the harder it will be. Lastly, third order system or higher dimensions will sometimes produce different phenomena and behavior which one cannot find in one or two-dimensional system.

This research tries to explore some of these phenomena by investigating a three dimensional autonomous system involving Lorenz equations. We will see how these simple equations can give so many surprising behaviors such as chaos and strange attractor. We will be exposed on what is meant by chaos, attractors and how the attractor is known to be strange. Before we analyze Lorenz behavior, we start by studying the general properties of Lorenz system such as finding the equilibrium points, identifying the stability of the equilibrium points and how to linearize the system which involve a Jacobian matrix. The existence of bifurcations such as Pitchfork and Hopf which only exist at certain parameter r will be studied in the system.

The analysis of Lorenz system will be made by varying the parameter r within a certain range and we can observe that there is a change in behavior depending on the value of r chosen. Lorenz attractor exists at a certain state and this creates a beautiful butterfly which emerges when the solution is plotted in the phase portrait. The solution is very sensitive to the initial conditions and can result a big change with just a small change. The chaotic behavior of the Lorenz system will be measured by Lyapunov exponent and Kaplan-Yorke Dimension to detect the strangeness. This is done with the help of mathematical software tools such as Matlab and Maple.

1.2 Problem Statement

The behavior of nonlinear dynamical systems can differ completely from that of linear dynamical systems. The linear systems always exhibit straightforward, predictable behavior and eliminate any possibility of chaotic behavior. So, to find chaotic behavior, we need to look at nonlinear, higher dimensional systems which in this research is Lorenz equations. Lorenz as a vastly oversimplified model of atmospheric convection, produces what has come to be known as a strange attractor.

This research is done by focusing on research questions such as what are the properties of Lorenz system? What are the equilibria and how do they relate to stability? How does a change in value of a parameter r in the Lorenz system affect the behavior of the system in phase portrait? Does the chaotic behavior of the Lorenz system always exist? When do the bifurcation and attractor exist? How to measure chaos and how to detect strangeness in the attractor?

1.3 Objectives of study

1) To study the historical survey of Lorenz system, general properties, equilibria and their stability.
2) To explore the behavior of the Lorenz system by analyzing the phase portrait by changing the value of the parameter r in Lorenz equation.
3) To analyze the whole system of Lorenz including its stability, bifurcation, chaos and attractors.
4) To investigate the chaoticity of the system using Lyapunov exponent and Kaplan-Yorke dimension to detect strangeness.

1.4 Scope of Study

This research will focus on the behavior of Lorenz system based on qualitative tools as well as on some numerical experiment.

1.5 Significant of Study

This research can enhance our knowledge on dynamical systems especially on nonlinear autonomous of three-dimensional systems. This will expose us to some new environment of three-dimensional systems instead of two-dimensional systems which are always used in a particular problem. Furthermore, it can give us a better understanding on Lorenz system and the chaotic behavior of the system. By exploring their behavior in the phase portrait, we can see the presence of the Lorenz attractor producing the interesting butterfly effect under certain condition and this leads to chaos which involves in many of our real life applications. Last but not least, it also can widen our knowledge in the field of Mathematics and Engineering.

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