SOLVING ONE DIMENSIONAL HEAT EQUATION AND GROUNDWATER FLOW MODELING USING FINITE DIFFERENCE METHOD

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To my beloved family and the person who loves me,

thanks for your love and support…

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Bismillahirrahmanirrahim. In the name of Allah, The Most Greatest and Most Merciful. Praise Upon the Beloved Prophet, His Family and Companion. There is no power except by the power of Allah and I humbly return my acknowledgement that all knowledge belongs to Allah. Alhamdulillah, I thank Allah for granting me this opportunity to broaden my knowledge in this field. Nothing is possible unless He made it possible.

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ABSTRACT

This research was conducted to solve one dimensional heat equation and groundwater flow equation using Finite Difference Method. Three Finite Difference methods were chosen to solve parabolic Partial Differential Equations which are Explicit, Implicit and Crank-Nicolson method. The algorithm for each method has been developed and the solution of the problem is simplified using *MATLAB* software. The result obtained by the explicit method is given the most accurate and the best results compared to the Crank-Nicolson method and implicit method.

ABSTRAK

Kajian ini telah dijalankan untuk menyelesaikan persamaan haba dan persamaan aliran air bawah tanah satu dimensi dengan menggunakan Kaedah Perbezaan Terhingga. Tiga kaedah Perbezaan Terhingga telah dipilih untuk menyelesaikan masalah Persamaan Terbitan Separa parabolik iaitu tidak tersirat, tersirat dan kaedah Crank-Nicolson. Algoritma bagi setiap kaedah telah dirumuskan dan penyelesaian masalah itu dipermudahkan dengan menggunakan perisian *MATLAB*. Keputusan yang diperolehi dengan kaedah tidak tersirat memberikan yang paling tepat dan hasil yang terbaik berbanding kaedah Crank-Nicolson dan kaedah tersirat.

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LIST OF ABBREVIATIONS

1-D	-	One-Dimensional
2-D	-	Two-Dimensional
3-D	-	Three-Dimensional
BEM	-	Boundary Element Method
FDM	-	Finite Difference Method
FEM	-	Finite Element Method
MQ-RBF	-	Multi-Quadric Radial Basis Function
PDE	-	Partial Differential Equation

LIST OF SYMBOLS

h	-	Mesh size (step size for x)
k	-	Iteration parameters (step size for t)
$u(x, t)$	-	Dependent variables
x, t	-	Independent variables
a, b, c, d, e, f, g	-	Constant
t	-	Time
$\Delta x, \Delta t$	-	Size of interval at x and t axis respectively
r	-	ratio
L	-	Length in x direction
T	-	Length in t direction

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CHAPTER 1

RESEARCH FRAMEWORK

1.1 Introduction

Mathematical problems can be solved analytically or numerically. However, analytic method scheme may be unsuccessful if the region of the problems is complex or the boundary conditions are time-dependent. In that case, numerical solution method will be very useful. The main goal in the field of numerical analysis is to provide compatible method in order to obtain the solutions to mathematical problems.

For scientific and engineering applications, it is often necessary to solve partial differential equations (PDE). Most partial differential equations for practical problems cannot be solved analytically. Therefore, numerical methods for partial differential equations are extremely important (Ya, Yan Lu).

In the 1920s, the finite difference method (FDM) was first developed by A. Thom. This method under the title "the method of square" was in order to solve nonlinear hydrodynamic equations. The basic of this finite difference techniques are, actually the approximations of solution to the differential equations by using finite difference equations. The approximations are algebraic in form and the solutions also related to the grid points (A. Thom and C. J. Apelt, 1961).

In this project, finite difference techniques; explicit method, implicit method and Crank-Nicolson method are applied for solving one dimensional heat equation and groundwater flow modeling.

1.2 Background of the Research

Groundwater is water located under the ground surface in soil pore spaces and in the fractures of rock formations (Adam Baharum et. al, 2009). A unit of rock or an unconsolidated deposit is called an aquifer when it can yield a usable quantity of water. The depth at which soil pore spaces or fractures and voids in rock become completely saturated with water is called the water table. Figure 1.1 shows the groundwater flow. Groundwater is recharged from, and eventually flows to the surface naturally; natural discharge often occurs at springs and seeps, and can form oases or wetlands. Groundwater is also often withdrawn for agricultural, municipal and industrial use by constructing and operating extraction wells. The study of the distribution and movement of groundwater is hydrogeology, also called groundwater hydrology.

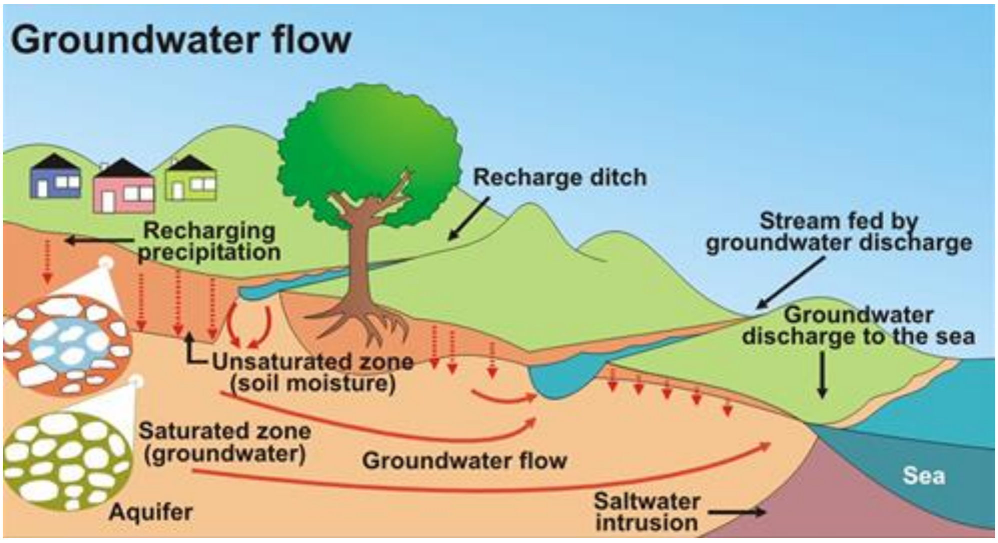


Figure 1.1: The hydrological cycle and groundwater flow

Some of the established solution techniques available for solving the governing equations of the model are Finite Difference Method and Finite Element Method approximation or a combination of both provided that model parameters also initial and boundary conditions are properly specified. The numerical solution applied in this research work is the finite difference method. This is an old method made more useful with the advent of high performance computer systems.

1.3 Problem Statement

In this research, we investigate numerical solution for solving one dimensional heat equation and groundwater flow modeling using finite difference method such as explicit, implicit and Crank-Nicolson method manually and using *MATLAB* software.

1.4 Objectives of the Research

The specific objectives of this research are:

1. To solve one dimensional heat equation by using explicit finite difference method, implicit finite difference method and Crank-Nicolson method manually and using *MATLAB* software;
2. To derive one dimensional groundwater flow modeling;
3. To solve one dimensional groundwater flow modeling by using explicit finite difference method, implicit finite difference method and Crank-Nicolson method using *MATLAB* software.

1.5 Scope of Research

In recent years, a number of numerical methods have been introduced. This project is limited to solve one dimensional groundwater flow by using finite difference method. We will focus on three methods namely, explicit, implicit and Crank-Nicolson method.

1.6 Significant of the Study

The result of this research will give alternative method to solve one dimensional groundwater flow equation. The research will lead to further study on the applying of finite difference method to solve the related problem to parabolic equation.

1.7 Dissertation Report Organization

This research has been organized into five chapters. In chapter one, we discuss about the information of the whole research. This chapter introduced the background of the research, problem statement and the objectives for this research. Besides that, we have state the scope of research and the significance of the research. Lastly, we discuss on the organization of the report.

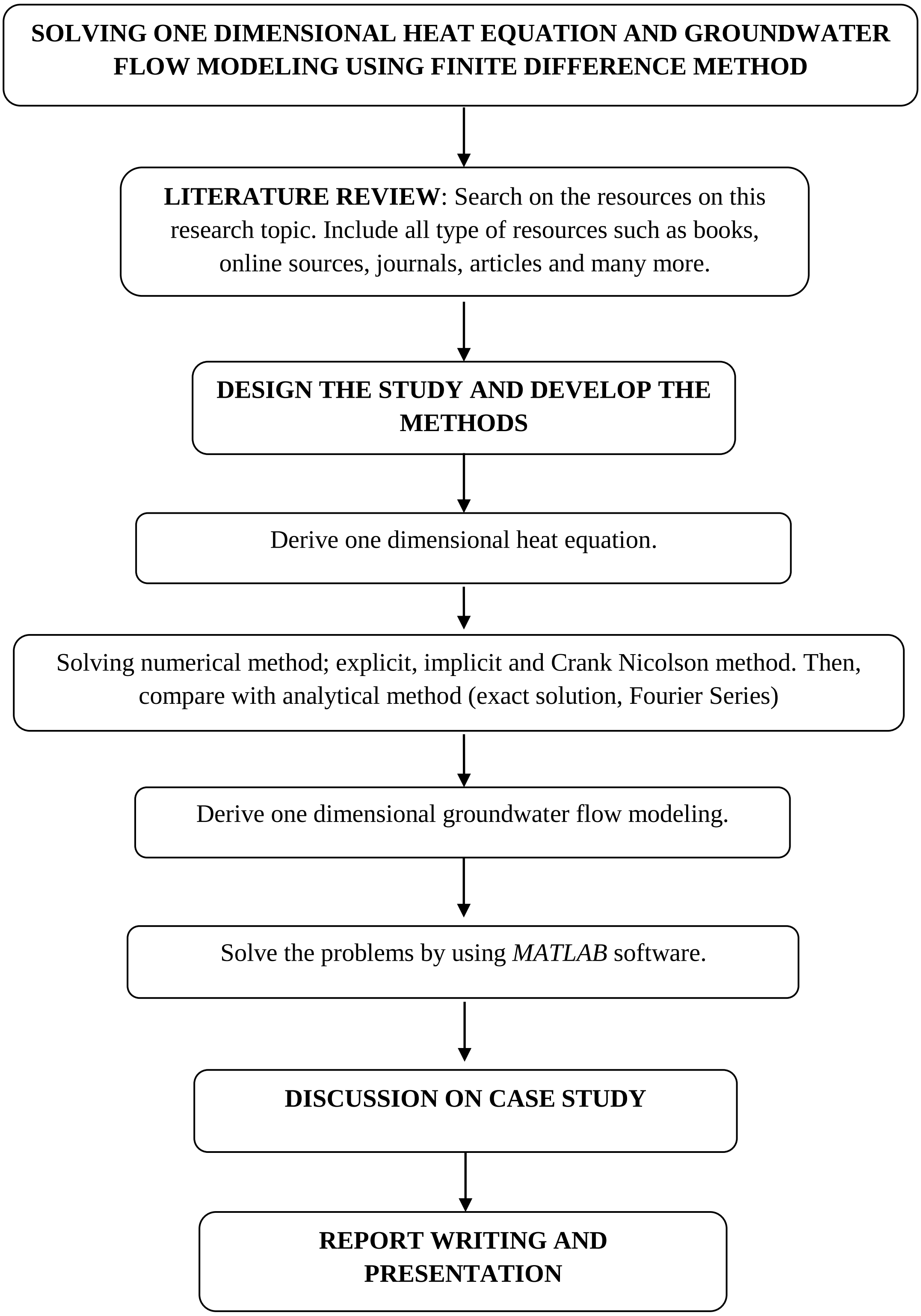
In chapter two, we discuss the literature review of this research. It give some review of previous work done by many researcher regarding for modeling of groundwater flow problem.

For chapter three, we explain about the algorithm of explicit, implicit and Crank-Nicolson method in order to solve the one dimensional of heat equation. We also give the example for all the methods. Additionally, we compare the results with exact solution.

In chapter four, we discuss the implementation of one dimensional groundwater flow modeling for explicit, implicit and Crank-Nicolson method. The problem we solve by using *MATLAB* software. Then, the results obtain are compared to each method and displayed in tables.

Finally, chapter five summarized the results obtain and some suggestions for the future research are also included.

1.8 Research Methodology



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