# FAST NUMERICAL CONFORMAL MAPPING OF BOUNDED MULTIPLY CONNECTED REGIONS VIA INTEGRAL EQUATIONS

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# FAST NUMERICAL CONFORMAL MAPPING OF BOUNDED MULTIPLY CONNECTED REGIONS VIA INTEGRAL EQUATIONS

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*To my beloved family, lecturers and my friends, thank you for your great support in term of physically and mentally.*

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# ABSTRACT

This study presents a fast numerical conformal mapping of bounded multiply connected region onto a disk with circular slits, an annulus with circular slits, circular slits, parallel slits and radial slits regions and their inverses using integral equations with Neumann type kernel and adjoint generalized Neumann kernel. A graphical user interface is created to illustrate the effectiveness of the approach for computing the conformal maps of bounded multiply connected regions and image transformations via conformal mappings. Some image transformation results are shown via graphical user interface. This study also presents a fast numerical conformal mapping of bounded multiply connected region onto second, third and fourth categories of Koebe's canonical slits regions using integral equations with adjoint generalized Neumann kernel. The integral equations are discretized using Nyström method with trapezoidal rule. For regions with corners, the integral equations are discretized using Kress's graded mesh quadrature. All the linear systems that arised are solved using generalized minimal residual method (GMRES) or least square iterative method powered by fast multipole method (FMM). The interior values of the mapping functions and their inverses are determined by using Cauchy integral formula. Some numerical examples are presented to illustrate the effectiveness for computing the conformal maps of bounded multiply connected regions. This study also discussed a fast numerical conformal mapping of bounded multiply connected regions onto fifth category of Koebe's canonical regions using integral equations with the generalized Neumann kernel. An application of fast numerical conformal mapping to some coastal domains with many obstacles is also shown.

# ABSTRAK

Kajian ini mempersembahkan satu pemetaan konformal berangka bagi rantau terkait berganda terbatas yang pantas ke cakera dengan belahan membulat, annulus dengan belahan membulat, rantau belahan membulat, rantau belahan jejari dan rantau belahan selari dan songsangannya menggunakan persamaan kamiran dengan inti jenis Neumann dan inti Neumann dampingan terilak. Satu antara muka pengguna bergrafik dicipta untuk menunjukkan keberkesanan kaedah dalam pengiraan permetaan konformal berangka rantau terkait berganda terbatas dan transformasi imej melalui pemetaan konformal. Beberapa hasil transformasi imej telah ditunjukkan melalui antara muka pengguna bergrafik. Kajian ini juga mempersembahkan satu pemetaan konformal berangka rantau terkait berganda terbatas yang pantas ke kategori kedua, ketiga dan keempat rantau berkanun Koebe menggunakan persamaan kamiran bersama inti Neumann terilak. Persamaan kamiran tersebut didiskretkan menggunakan kaedah Nyström dengan petua trapezoid. Bagi rantau berpenjuru, persamaan kamiran tersebut telah didiskretkan menggunakan kuadratur Kress bergred. Semua sistem linear yang terhasil diselesaikan menggunakan kaedah reja minimum teritlak (GMRES) atau kaedah lelaran kuasa dua terkecil yang dikuasai oleh kaedah multikutub pantas (FMM). Nilai-nilai pedalaman bagi fungsi pemetaan dan pemetaan sonsangannya dikenal pasti dengan menggunakan formula kamiran Cauchy. Beberapa contoh berangka dipersembahkan bagi menggambarkan keberkesanan pengiraan permetaan konformal rantau terkait berganda terbatas. Kajian ini juga membincang pemetaan konformal berangka rantau terkait berganda terbatas yang pantas ke kategori kelima rantau berkanun Koebe menggunakan persamaan kamiran bersama inti Neumann terilak. Aplikasi pemetaan konformal berangka kepada domain pantai dengan banyak halangan juga ditunjukkan.

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# CHAPTER 1

# INTRODUCTION

## 1.1 Introduction

Conformal mapping is a one-to-one and onto function of a complex variable. It transforms a planar region to another planar region. The magnitudes as well as the angles between two curves are preserved under conformal mapping. As complicated regions can be transformed into simpler regions, various problems in the engineering and science fields can be solved easily.

Conformal mapping of multiply connected regions onto five categories with 39 canonical slits regions had been introduced by Koebe (Koebe, 1916). Several numerical methods have been done to compute the conformal mapping of multiply connected regions onto canonical slits regions. A method has been presented to compute the conformal mapping onto all of 39 canonical slits regions in a unified ways based on uniquely solvable boundary integral equations with generalized Neumann kernel (Nasser, 2009a; Nasser, 2009b; Nasser, 2011; Nasser, 2013). Several other works have also been presented to compute conformal mapping of bounded and unbounded multiply connected regions onto several canonical slits regions using integral equation method (Amano, 1998; Okano et al., 2003; Yunus et al., 2012; Yunus et al., 2014a; Yunus et al., 2013; Yunus et al., 2014b; Sangawi et al., 2012a; Sangawi et al., 2012b; Sangawi et al., 2011; Sangawi et al., 2013; Sangawi, 2014a; Sangawi, 2014b; Sangawi and Murid, 2013).

The Nyström method with the trapezoidal rule has been used effectively to discretize the integral equations since the functions in the integral equations are all periodic (Davis and Rabinowitz, 1984). Normally the linear system obtained from discretization can be solved using the Gauss elimination method. However, the solution of the system is difficult to obtain when the number of nodes for discretization is large. A fast approach named as fast multipole method (FMM) has been proposed by Greengard and Rokhlin (1987) for faster particle simulations in plasma physics. Nasser and Al-Shihri (2013) proposed a fast method to solve integral equations with generalized Neumann kernel and compute conformal mapping using fast multipole method (FMM) and restarted version of generalized minimal residual (GMRES) method. This approach is fast, accurate and able to obtain the solution with very high number of nodes. However, the system obtained must be linear in order to be solved using GMRES method and also, this approach cannot be applied to overdetermined system.

## 1.2 Research Background

New integral equations have been derived to compute conformal mapping of bounded multiply connected regions onto first category of Koebe's canonical slits regions, namely disk with circular slits, annulus with circular slits, circular slits, parallel slits and radial slits regions. However, the approach requires three integral equations for each canonical region.

For some of these new integral equations, the systems obtained after discretization are overdetermined systems. An overdetermined system can been solved using QR decomposition and least square method (Hayami et al., 2010). But classical method like QR decomposition take high cost in time and memory, especially for large number of nodes.

Nasser and Al-Shihri (2013) and Nasser (2015) proposed a fast method to solve integral equations with generalized Neumann kernel and compute conformal mapping using fast multipole method (FMM) and restarted version of generalized minimal residual (GMRES) method. This approach can be applied to these new integral equations for numerical conformal mapping.

Integral equation methods are not only useful for computing conformal mapping of region with smooth boundaries, they are also applicable to compute conformal mapping of regions with non-smooth boundaries. Yunus (2013) had done numerical experiment for region with corners using Kress quadrature rule (Kress, 1990).

The integral equations derived in Sangawi (2014a), Sangawi (2014b), Sangawi and Murid (2013) and Sangawi et al. (2013) are solved numerically using MATLAB solver that requires explicitly defined coefficient matrix. The operation for this approach are $O((m+1)^2n^2)$, where $m+1$ is the number of connectivity of the region and $n$ is number of node for each boundary. It is not suitable for higher connectivity and more challenging region problems. Some improvements need to be done for reducing the operations and memory requirement for numerical conformal mapping in order to apply in real problem.

## 1.3 Research Statements

The research problem is to apply fast multipole method and LSQR method for computing conformal mapping of bounded multiply connected regions with smooth boundaries onto first category canonical regions including disk with circular slits region, annulus with circular slits region, circular slits region, parallel slits region and radial slits region using new integral equations developed by Ali Wahab Kareem Sangawi in Appendix B and Sangawi et al. (2013). This research problem is also to apply fast multipole method and generalized minimal residual (GMRES) method for computing conformal mapping of bounded multiply connected regions with smooth and non-smooth boundaries onto second, third and fourth categories of canonical regions developed by Ali Wahab Kareem Sangawi in Appendix C and Sangawi (2014a), Sangawi (2014b) and Sangawi and Murid (2013).

## 1.4 Research Objectives

The objectives of this research are:

(i) To construct fast numerical technique to compute conformal mapping of bounded multiply connected regions with smooth boundaries onto first category of Koebe's canonical slits regions based on new integral equations developed by Ali Wahab Kareem Sangawi in Appendix B and Sangawi et al. (2013) by using LSQR and fast multipole method and illustrate the results by constructed a graphical user interface.

(ii) To construct fast numerical technique to compute conformal mapping of bounded multiply connected region with smooth and non-smooth boundaries onto second, third and fourth categories of canonical regions based on new integral equations developed by Ali Wahab Kareem Sangawi in Appendix C and Sangawi (2014a), Sangawi (2014b) and Sangawi and Murid (2013) based on GMRES method, FMM, and Kress quadrature rule.

(iii) To develop graphical user interface to illustrate the image transformation based on objective (i).

(iv) To illustrate an application of fast numerical conformal mapping to a coastal domain with many obstacles.

## 1.5 Scope of the Study

This research focuses on boundary integral equation methods to compute the conformal mapping function of bounded multiply connected region onto canonical slits regions. The fast multipole method is considered to compute matrix-vector multiplication and the iterative methods including GMRES method and LSQR method are considered to solve the linear system. Normally the integral equation will be discretized using Nyström's method with trapezoidal rule when the function is periodic. Kress quadrature rule is used to discretize the integral equation when conformal mappings of non-smooth regions are computed.

## 1.6 Significance of the Study

Conformal mapping is important in solving several problems that arises from science, engineering and medical fields. Many real problems involve smooth and non-smooth regions. Conformal mapping play the role in transforming the complicated region to a standard region, where the problem can be solved easily. The solution is obtained after transforming back to the original region.

The usual algorithm without fast multipole method takes a longer time and requires much memories in the computation by using computer. When fast multipole method is applied into the computation, the time taken to compute shorten from $O(n^2(m+1)^2)$ to $O(n(m+1))$ with less memory needed. The fast algorithm is capable to compute the mapping of more complex regions. Many hospitals have also been using medical imaging extensively for rapid diagnosis with visualization and quantitative assessment. Conformal mapping algorithm can be applied to medical image processing, which help the medical experts in diagnosis (Gu et al., 2004).

## 1.7 Organization of the Thesis

The thesis is organized into five chapters. Chapter 1 contains six sections which are introduction, problem statements, objectives of the study, scope of the study, significance of the study and organization of the thesis.

Chapter 2 begins with some explanations on concepts of conformal mapping of bounded multiply connected regions onto some Koebe's canonical regions followed by auxiliary materials. Furthermore, Cauchy's integral formula, FMM and Iterative Methods for solving linear system are also briefly discussed.

Chapter 3 discusses integral equations developed in Sangawi et al. (2013) for computing conformal mapping of bounded multiply connected regions onto first category of Koebe's canonical slits regions. Chapter 3 also presents the numerical computation for the conformal mapping of any bounded multiply connected region with smooth boundaries onto first category of canonical regions using FMM, GMRES and LSQR. Some numerical examples including graphical user interface are also given.

Chapter 4 discusses new integral equations for computing conformal mapping of bounded multiply connected regions onto second, third and fourth Koebe's canonical slits regions. The numerical computation for the conformal mapping of any bounded multiply connected region with smooth and non-smooth boundaries onto second, third and fourth Koebe's canonical slits regions using FMM and GMRES method is also discussed in Chapter 4. Some numerical examples are also given.

An example on application of conformal mapping to compute potential flow with a coastal domain with many obstacles is given in Chapter 5. It is shown how to construct uniform flow, uniform flow with a point vortex, multiple vortices, uniform flow with source at boundary with a coastal domain by means of conformal mapping.

The concluding chapter, Chapter 6 contains summary of the research and some recommendations for future research.

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