

NUMERICAL INVESTIGATION OF WAVES INTERACTIONS FROM  
FORCED KORTEWEG de VRIES EQUATION

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Dedicated to my beloved

father, mother,

brothers, sisters

&

my supervisors,

Assoc. Prof. Dr. Ong Chee Tiong,

and

Dr. Tiong Wei King.

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## ABSTRACT

Soliton generated by the Korteweg de Vries (KdV) equation forms a group of solitons ladder. During full interaction of multi-soliton solutions, three types of peaks were obtained, namely single, flat and double peak. Soliton generated by the forced Korteweg de Vries (fKdV) equation forms uniform solitons trains with equal amplitude. Various aspects of solitons interactions of the fKdV equation for free surface flow over uneven bottom topography have been investigated. Fluid flowing over uneven bottom topography can support wave propagation that generates upstream and downstream nonlinear wavetrains. Such forced nonlinear solitary waves occur naturally in the shallow water near the coastal region. The fKdV equation models the above phenomena in many cases, such as in the transcritical, weakly nonlinear and weakly dispersive region. Numerical method which involves the pseudo-spectral method is used to solve the fKdV equation as it is difficult to obtain the solution analytically, due to the presence of the forcing term and the broken symmetry. A group of uniform solitons having the same amplitude and speed will not collide when the bump size and bump speed are constant. A wave profile with time-dependent transcritical velocity was investigated with a variation of Froude number. As the Froude number changes, two sets of solitary waves travelling upstream were discovered. A set of these solitary waves have nearly uniform amplitude, while another set comprises of solitary waves with variable amplitude, which forms a pairwise and two pairwise interactions pattern in the transcritical region. In the case of multiple bumps, upstream-advancing nonlinear solitary waves which may be generated continuously and interact with each other when the distance between bumps, width and height of bumps were varied. Interesting interaction patterns of the collision between uniform solitons will provide a better understanding of the forcing caused by multiple bumps on water flow at the uneven bottom topography of a shallow water in a rectangular channel.

## ABSTRAK

Soliton yang dijana oleh persamaan Korteweg de Vries (KdV) membentuk sekumpulan soliton bertingkat. Ketika interaksi penuh penyelesaian soliton majmuk, tiga jenis puncak diperolehi iaitu satu, rata, dan dua puncak. Soliton yang dijana oleh persamaan Korteweg de Vries paksaan (fKdV) membentuk sekumpulan soliton seragam yang sama amplitud. Pelbagai aspek interaksi antara soliton bagi persamaan fKdV untuk aliran permukaan bebas yang merentasi topografi bahagian bawah yang tidak sekata telah dikaji. Bendalir mengalir merentasi topografi bahagian bawah yang tidak sekata boleh menjana perambatan gelombang tak linear huluan dan hiliran. Gelombang paksaan tak linear terjana secara semula jadi di kawasan air cetek berhampiran pantai. Persamaan fKdV adalah model yang menggambarkan fenomena dalam pelbagai kes, seperti dalam rantau transkritikal, tak linear lemah dan serakan lemah. Kaedah berangka yang melibatkan kaedah pseudospektral digunakan untuk menyelesaikan persamaan fKdV kerana penyelesaian analitikal sukar didapati dengan kewujudan unsur paksaan dan simetri yang termusnah. Sekumpulan soliton seragam yang mempunyai amplitud dan kelajuan yang sama tidak akan berlanggar apabila saiz bonggol dan kelajuan bonggol adalah tetap. Profil gelombang dengan halaju transkritikal yang bergantung kepada masa telah dikaji dengan variasi nombor Froude. Apabila nombor Froude diubah, dua set gelombang huluan ditemui. Salah satu set gelombang beramplitud hampir seragam, manakala satu set lagi terdiri daripada gelombang dengan amplitud yang berubah-ubah, membentuk sepasangan dan dua pasang perambatan gelombang huluan dalam rantau transkritikal. Dalam kes beberapa bonggol, penjanaan berterusan hulu-mara gelombang tak linear mungkin berinteraksi antara satu sama lain apabila jarak antara bonggol, lebar dan ketinggian bonggol diubah-ubah. Pola interaksi yang menarik untuk perlanggaran antara soliton seragam akan memberikan kefahaman yang lebih baik tentang halangan yang disebabkan oleh beberapa bonggol pada aliran air yang merentasi topografi bahagian bawah yang tidak sekata dalam saluran segi empat tepat.

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## LIST OF SYMBOLS

$d$	-	distance between two bumps
$F_r$	-	Froude number
$h_A$	-	height of bump A
$h_B$	-	height of bump B
$l$	-	the critical value for different types of peaks in multi-soliton solutions
$M$	-	number of loops
$N$	-	number of discrete points
$U$	-	free surface elevation
$V(\xi_j, t)$	-	discretise of $V(\xi, t)$ with respect to spatial variable $\xi$
$\hat{V}_{mt}$	-	short form of $\hat{V}(p, t - \Delta t)$
$\hat{V}(p, t)$	-	discrete Fourier transform of $V(\xi_j, t)$
$\hat{V}_{pt}$	-	short form of $\hat{V}(p, t + \Delta t)$
$\hat{V}_t$	-	short form of $\hat{V}(p, t)$
$\hat{W}(p, t)$	-	product of discrete Fourier transform for $W(\xi_j, t)$
$\hat{W}_t$	-	short form of $\hat{W}(p, t)$
$x_{0A}$	-	width of bump A
$x_{0B}$	-	width of bump B

### Greek Symbol

$\alpha$	-	nonlinear coefficient
$\beta$	-	dispersive coefficient
$\Delta t$	-	step size of distance
$\delta_x(x)$	-	Dirac-delta
$\gamma$	-	cross section area of the bump

$\lambda$	-	deviation of the bump speed from the shallow water velocity
$\lambda(t)$	-	trajectory of the deviatory transcritical velocity
$\lambda_s$	-	upper limit of the transcritical band for steady forcing
$\mu$	-	depth of penetration into the transcritical band
$\sigma_t$	-	residence time of the trajectory
$\xi$	-	spatial coordinate transformation of variable $x$ in pseudo-spectral method

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## CHAPTER 1

### INTRODUCTION

#### 1.1 Preface

In recent decades, many researchers have focused their research on linear and nonlinear waves, emphasizing on the importance of solitary waves and the theory of solitons. In mathematics and physics, a soliton is a class of nonlinear dispersive wave that maintains a balance between the effects of dispersion and nonlinearity. This balance admits localized solitary waves, where after a collision, the solitary waves reemerge, retain its shape and speed, which shows a similarity to the property of the elastic collision of a particle (Lakshmanan, 2011). When the solitons collide, the only result of the interaction of two solitons is a phase shift, whereby the faster soliton is further ahead, and the slower soliton further behind (Grimshaw, 2005).

Solitary water waves are long nonlinear waves that can propagate over long distances without dissipating. They consist of a single isolated wave elevation, or depression, whose speed is an increasing function of the amplitude (Grimshaw, 2002). The phenomenon of solitary waves was first observed by John Scott Russell in August 1834 whilst riding on a horseback alongside the Union Canal near Edinburgh in Scotland. The field of nonlinear dispersive waves has developed rapidly over the past 50 years. The origin of the water wave problems go back to the work of Stokes in 1847, Boussinesq in the 1870s and Korteweg and de Vries in

1895. The famous Korteweg de Vries (KdV) equation is then derived by Korteweg and de Vries and provides an explanation of the phenomenon observed by Scott Russell. Thus, the KdV equation was not just a notable integrable equation, but also a suitable model for solitary waves in a wide range of important physical fields. Indeed, the KdV equation is a “universal” model which arises whenever there are weak dispersive and nonlinear systems as the governing system.

There has been considerable interest in the generation of water surface flow due to uneven bottom topography. The problem of flow over a bump can be applied in many branches of fluid mechanics, especially in oceanographic applications. The fluid flow over a bump can be found generating upstream and/or downstream wavetrain depending on the system parameters (Grimshaw, 2010). The study of the generation of upstream propagating solitary waves forced by moving objects or bottom topographies has received much attention, for example, Huang *et al.* (1982) carried out laboratory experiments and Wu and Wu (1982) performed numerical simulation. Wu and Wu (1982) used a generalized Boussinesq (g-B) model, which is good for long waves (i.e.  $\frac{h_0}{\lambda}$  is small, where  $h_0$  is the mean water depth and  $\lambda$  is a typical wavelength.) generated by moving-surface pressure distribution or bottom topography. The remarkable findings by Wu and his colleagues is a transcritical water flow over bump generating a train of upstream-advancing uniform solitons, a depression zone behind the forcing site, followed by a zone of wake propagating downstream. Later, many theoretical studies (Wu, 1987) and objective experiments (Lee *et al.*, 1989) were conducted to better understand the phenomena of generation of forced nonlinear waves caused by forcing disturbance.

When the oncoming flow is closed to critical, i.e., the linear long wave speed is closed to zero in the reference frame of the obstacle. The linear solutions fail near criticality, thus energy cannot propagate away from the bump at the



linear group velocity and hence a strongly nonlinear response happens. This flow is termed resonant or transcritical (Smyth, 1990). Therefore, the forced Korteweg de Vries (fKdV) equation is an appropriate model and suitable theory due to its weak nonlinearity, under the assumptions that the upstream Froude number is near one. The bump base is relatively short compared to the length of the surface waves, and the bump height is relatively small compared to the wave amplitude (Shen, 1995). Grimshaw and Smyth (1986) and Smyth (1987) studied the flow of a stratified fluid over topography, where the upstream solution consists of a train of solitary waves when the flow is closed to a linear resonance. Due to the complexity of the generalized Boussinesq (g-B) model, a simpler and the most widely used nonlinear theoretical model for studying the phenomenon of upstream-running solitary waves is the fKdV equation. A comparison between experimental and numerical simulations of the fKdV and g-B models was carried out by Lee *et al.* (1989) for the generation of upstream wavetrains. According to Lee *et al.* (1989), for the fKdV model, the forcing provided by external surface pressure and the bottom bump is equal. For the g-B model, the external surface pressure acted as a stronger disturbance than the bottom bump. As a result, larger waves were produced in a shorter period.

The study of the transcritical flow over an obstacle using the fKdV equation has continued to attract the attention of many authors in more recent years. Grimshaw *et al.* (2009) considered the flow over a hole, and the results showed that the interaction over the obstacle occurs when two wavetrains are generated. Grimshaw (2010) studied on the effect of the obstacle width, which examined a local steady solution over the obstacle. Chardard *et al.* (2011) investigated the stability of the supercritical solitary-wave and hydraulic fall solutions over a single obstacle and table-top like solutions over a double bump. They derived exact stationary solutions of the fKdV equation using an inverse method. Grimshaw and Maleewong (2013) studied the stability of

both subcritical and supercritical steady waves generated by a moving localised pressure disturbance over water of finite depth. Yi and Lee (2014) proposed a new numerical method called Locally Conservative Eulerian-Lagrangian Finite Difference Method (fdLCELM) to solve the fKdV equation in the presence of one bump and two bumps.

The transcritical flow over a step has been studied using the fKdV equation. Zhang and Zhu (1997) integrated the time-dependent forced KdV equation numerically and found steady transcritical flows for negative forcing. Grimshaw *et al.* (2007) extended the asymptotic theory developed by Grimshaw and Smyth (1986) and compared it with numerical simulations of the full Euler equations for surface water waves (Zhang and Chwang, 2001). They found that a positive step generates an upstream propagating undular bore, and a negative step generates a downstream propagating undular bore.

The main interest in this research is the fKdV equation. However, transcritical flow over an obstacle can also be modelled by other forced nonlinear evolution equations depending on the physical circumstances. The forced extended KdV equation, also known as the forced Gardner equation, is an additional cubic nonlinear term which has been used to study the transcritical flow over an obstacle by Grimshaw *et al.* (2002) and transcritical flow over a hole by Ee *et al.* (2011). Besides, the forced Su-Gardner equation has been used to study the transcritical flow of a stratified fluid over a broad localised topographic obstacle by El *et al.* (2009) and Kamchatnov *et al.* (2013).

## 1.2 Background of the Problem

Transcritical flows over uneven bottom topographies have received considerable attention from many applied mathematicians in the last three

decades due to its physically rich and mathematically tractable phenomena of a nonlinear dispersive system, which extends a linear resonance condition (Redekopp and You, 1995). These phenomena can happen in atmospheric and oceanic waves. It is necessary to include a forcing term especially in oceanographic applications. Typical examples are when waves are generated by ships moving along canals or flowing over uneven bottom topography (Grimshaw *et al.*, 2002).

The flow over a bump in a horizontal channel will reach criticality when the upstream flow velocity equals a linear long wave velocity. The linear solution fails when the flow nears criticality, thus the energy of the waves cannot propagate away from the bump. Indeed it is this feature which leads to the necessity for invoking weak nonlinearity to obtain a suitable theory. The fKdV equation is then used as a successful mathematical model equation to describe the nonlinear surface waves forced by an uneven bottom topography. In the transcritical regime, the flow over an obstacle generates upstream and downstream nonlinear wavetrains, connected by a locally steady solution over the obstacle which is elevated on the upstream and depressed on the downstream (Grimshaw *et al.*, 2009).

The study of the flow over a bump has been quite well-known for the past decades but much less is known about the flow over multiple bumps. Such forced nonlinear solitary waves with multiple bumps occur naturally in many physical phenomena, especially in oceanographic applications. Fluid flow phenomena over the bumps generate various interaction patterns depending on the upcoming stream velocity and bump shape. The most interesting feature of this phenomenon is that the fluid interaction of multiple bumps where the disturbance profiles local to each bump can interact. Obtaining this remarkable result which differ drastically from the flow over a bump at the bottom topography of the earlier work will be more fascinating.

Mathematically, the classical KdV equation without the forcing term is completely integrable, which means that it is easier to obtain analytical solutions. However, analytical solutions are not found for forced nonlinear evolution equations including various forms of perturbed KdV equation. Therefore, numerical simulations of perturbed or forced KdV class equations play an important role for modelling a wave evolution on a free surface and simulations of these wave patterns seem to be an important contribution in forced nonlinear evolution systems.

### 1.3 Statement of the Problem

One and two soliton solution of KdV equation are well-known solutions. There are three types of peaks during full interaction for two solitons, which are single peak, plateau and double peak. The two-soliton solutions can be solved analytically and numerically easily. However, the construction of multi-soliton solutions become complicated and lengthy as the number of soliton increases. Moreover, it is not simple to obtain the critical value that determines the different types of peaks for multi-soliton solutions. Therefore, three and four-soliton solutions or even more soliton solutions are no longer easy to solve. Hence, an iterative method will be useful to obtain the critical value which will become more complex as the number of soliton increases.

It is still a challenge to understand the flow over obstacles, especially in the atmosphere or oceans because of various ambient conditions and responses. Typical factors that control the phenomena are the uneven bottom topography and the time-varying upstream flow speed. The effects of transcritical flow over uneven bottom topography are important, due to the circumstances of the physical effect which can cause a forcing effect. These obstacles cause hindrance and resistance to the smooth flow of wind and water waves and the forcing effects

create interesting physical happening with nonlinearity phenomenon, such as the flow over uneven bottom topography creates various interaction patterns. Interactions of trains of multiple solitons arise from the fKdV equation that models surface water wave flow over a bump is interesting. Past researchers concentrated on the studies of periodical generation of upstream solitary waves of uniform amplitude, equally spaced and speed, with parameters of the bump size and speed set to constants. Thus these parameters do not resemble the real physical problem. With interaction patterns that change due to the varying upstream velocity, more complicated problems will arise. The KdV equation no longer models this phenomenon well and a more suitable mathematical model will be needed. Moreover, this type of forced nonlinear evolution equation become more difficult as no analytical solution has ever been found. With the presence of a forcing term, the translation-invariant type of group symmetry is broken, hence the traditional analytical method such as inverse scattering method and Bäcklund transformations can no longer generate analytic solutions of solitons. When the above method fails, numerical solution will be a useful tool to solve a system of forced nonlinear evolution equation.

However, due to the complex geometries (localized multiple bumps, holes, or platform, etc.), more interaction patterns of surface dynamics can be observed. Most studies for multiple bumps focused on the hydraulic falls solution. Nevertheless, the study of forced solitary waves generated by multiple bumps has not been carried out yet. With the focus on numerical methods in solving fKdV equations, more work on physical phenomenon involving two or more bumps at the bottom topography will be investigated. A more suitable mathematical model is sought to fit in two bumps or more since it can represent a real physical phenomenon.

## 1.4 Objectives of the Study

Objectives of this study are to:

- (a) investigate and obtain simulation for types of peaks of full interaction in multi-soliton solutions of Korteweg de Vries (KdV) equation.
- (b) investigate and obtain simulation of interaction patterns of trains of multiple forced solitons using the forced Korteweg de Vries (fKdV) equation that models surface water waves in a flow over uneven bottom topography.
- (c) generate and simulate 3D forced solitary waves of interaction patterns created by multiple bumps.

## 1.5 Scope of the Study

In this research, an incompressible, inviscid fluid in a two-dimensional channel forced by a distributed pressure on a free surface and a small bump on the flat bottom topography, which uses the forced Korteweg de Vries (fKdV) equation

$$U_t + \lambda U_x + 2\alpha U U_x + \beta U_{xxx} = \frac{\gamma}{2} f'(x), \quad (1.1)$$

where  $\alpha < 0$ ,  $\beta < 0$  and  $\gamma$  are constants,  $f(x)$  is the forcing term.

## 1.6 Significance of the Study

In mathematics, this research will provide an advanced application of modelling of nonlinear wave. In real life, this research will provide advanced knowledge on the concept of solitons in water wave problem to provide and

simulate various properties of nonlinear waves.

Single bump or multiple bumps can be applied to reduce damages caused by water surface waves. Although one or multiple bumps do not completely control water surface waves, it is important to lower the risk of damage to seacoast residences. When a bump or multiple bumps appear along the path of the fluid, it can reduce the severity of the effects of the wave and the fluid's velocity. Moreover, multiple bumps can act as a barrier in order to reduce the depth of runup and velocity of the onshore watermass flow. In other words, multiple bumps are function like protective walls separating harbors from residences.

Sometimes, it is costly and difficult to build and maintain a monitoring lab, especially in deep waters. Therefore, a mathematical model plays an important role and it usually describes important relationships between the variables with a system by a set of variables. Oceanographers, engineers and mathematicians can use a mathematical equation to build a model of the internal soliton system and also to estimate how an unforeseeable event could affect the system with virtual simulation.

## 1.7 Methodology

Research on nonlinear evolution equation will begin with the study of non-forced system governed by the KdV equation. To derive a forced nonlinear evolution equation, the study of fluid flows induced by a moving pressure over a flat bottom is needed. Later, the fKdV equation derived for flow over one bump is established. The literature review on the fKdV equation is needed to obtain a suitable model for the study of flow over single and multiple bumps.

Due to the forcing term in the nonlinear evolution equation, the

translation-invariant type of group symmetry is broken. Therefore, the traditional analytical method such as inverse scattering method and Bäcklund transformations cannot be used to solve the equation with forcing terms. Numerical solutions seem to be the method to solve the fKdV equation. The forced nonlinear evolution equation will be solved using pseudo-spectral method, which is based on the semi-implicit scheme (Chan and Kerhoven, 1985). By integrating the fKdV equation in time in Fourier space and using the dispersive term,  $U_{xxx}$  by Crank-Nicolson method and the nonlinear term,  $U U_x$  with leap-frog method, a numerical solver will be developed. In order to carry out the Chan and Kerhoven scheme effectively, an FFT and inverse FFT are set up in computer program. FFT is an algorithm that effectively computes the discrete Fourier Transform (DFT) and MATLAB SimBiology is a computer software used to discover how the solitary wave solutions of various forces evolve.

## 1.8 Outline of Thesis

This thesis focuses on three major parts. Part 1 studies the shapes of peaks during the full interaction of multi-soliton solutions of KdV equation. Part 2 discusses the phenomenon of forced nonlinear solitary waves in the presence of a single bump with particular emphasis on the varying upstream velocity. Part 3 deals with on the problem of free surface flow over multiple bumps on the bottom topography and the effects caused by the parameters of forcing term. Numerical solutions of the fKdV equation using pseudo-spectral method will be solved and various graphical outputs will be simulated.

In Chapter 2, literature review starts with the background of solitary waves and the KdV equation. Later, the problem of fluid flow over one bump and multiple bumps will be discussed.



Chapter 3 discusses the peaks pattern of full interactions solitons of the KdV equation. The Hirota's bilinear method will be used to obtain the exact solutions of KdV equation. After that, a computer program to obtain explicit multi-soliton solutions of KdV equation is built. Besides that, the shapes of three different types of peaks during full interaction of two solitons discovered by Ong (1993), which are single, double and flat peak is discussed. Ong's (1993) work is extended and use Hirota's bilinear method to obtain multi-soliton solutions.

Chapter 4 describes the numerical solutions of the fKdV equation using the pseudo-spectral method. The initialization equation and forward scheme equation will be used to develop a program to obtain numerical solutions for fKdV equation. MATLAB SimBiology will be used to simulate the graphical output based on the solutions obtained. Various numerical simulations were generated easily by using MATLAB Simbiology with different types of forcing.

Chapter 5 focuses on a wave profile where time-dependent transcritical velocity was discovered by changing the parameter  $\lambda$  with respect to time. Two interesting phenomena can be obtained, which are soliton pair and soliton triad. In this research, the less studied case is emphasized when the value of  $\sigma_t$  is small for specific  $t_0$ . Later, four regimes from wave profile of  $\lambda(t)$  will be identified.

Chapter 6 discusses about the problem of free surface flow over multiple bumps on the bottom topography of the channel. The effects caused by the parameters of forcing term, such as distances between obstacles, width of bumps and height of bumps were discussed. Besides, the effects of the dip on the bottom of channel are also investigated. The values of  $\lambda$  are then varied to generate different flow of regimes.

Chapter 7 is the final chapter of the thesis. The chapter is introduced to conclude the overall research. Some suggestions and recommendations for future

research in forced system that models both generation and propagation of solitary waves will be discussed.

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