FINDING THE ZEROS OF AHLFORS MAP USING INTEGRAL EQUATION METHOD ON BOUNDED MULTIPLY CONNECTED REGIONS

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To my mother, without her prayers, love and support I would never be able to complete this work, my wife whose presence with me helped me to overcome the most difficult period.

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ABSTRACT

The Ahlfors map of an n-connected region is a n-to-one map from the region onto the unit disk. The Ahlfors map being n-to-one map has n zeros. Previously, the exact zeros of the Ahlfors map are known only for the annulus region and a particular triply connected region. The zeros of the Ahlfors map for general bounded multiply connected regions has been unknown for many years. The purpose of this research is to find the zeros of the Ahlfors map for general bounded multiply connected regions using integral equation method. This work develops six new boundary integral equations for Ahlfors map of bounded multiply connected regions. The kernels of these integral equations are the generalized Neumann kernel, adjoint Neumann kernel, Neumann-type kernel and Kerzman-Stein type kernel. These integral equations are constructed from a non-homogeneous boundary relationship satisfied by an analytic function on a multiply connected region. The first four integral equations have kernels containing the zeros of the Ahlfors map which are unknown. The fifth integral equation has no zeros of the Ahlfors map in the kernel but involves derivative of the Ahlfors map at the unknown zeros. The sixth integral equation has unknown zeros appearing only at the right-hand side. The sixth integral equation proves to be useful for computing the zeros of the Ahlfors map. This work presents a numerical method for computing the zeros of Ahlfors map of any bounded multiply connected region with smooth boundaries. This work derives two formulas for the derivative of the boundary correspondence function of the Ahlfors map and the derivative of the Szegö kernel. The relation between the Ahlfors map and the Szegö kernel is classical. The Szegö kernel is a solution of a Fredholm integral equation of the second kind with the Kerzman-Stein kernel. These formulas are then used along with the sixth integral equation to compute all the zeros of the Ahlfors map for any bounded smooth multiply connected regions. Some examples are presented to demonstrate the efficiency of the presented method.

ABSTRAK

Pemetaan Ahlfors bagi rantau berkait berganda n adalah pemetaan n ke satu dari rantau tersebut ke atas cakera unit. Pemetaan Ahlfors yang merupakan pemetaan n-ke-satu mempunyai n sifar. Sebelum ini, pensifar yang tepat hanya diketahui untuk rantau anulus dan rantau berkait ganda tiga yang tertentu. Pensifar untuk pemetaan Ahlfors bagi rantau berkait berganda umum telah tidak diketahui bertahun-tahun lamanya. Kajian ini bertujuan untuk mencari pensifar pemetaan Ahlfors bagi rantu berkait berganda umum menggunakan kaedah persamaan kamiran. Penyelidikan ini telah membina enam persamaan kamiran sempadan yang baharu bagi pemetaan Ahlfors terhadap rantau berkait berganda terbatas. Inti untuk persamaan kamiran ini adalah inti Neumann teritlak, inti Neumann adjoin, inti jenis Neumann dan inti jenis Kerzman-Stein. Persamaan-persamaan kamiran ini dibina daripada hubungan sempadan tak homogen yang ditepati oleh fungsi analisis pada rantau berkait berganda. Empat persamaan kamiran yang pertama mempunyai inti yang mengandungi pensifar yang tidak diketahui bagi pemetaan Ahlfors. Persamaan kamiran kelima pula tiada pensifar di dalam inti tetapi persamaan kamiran ini melibatkan terbitan pemetaan Ahlfors pada pensifar yang tidak diketahui. Manakala persamaan kamiran keenam mengandungi pensifar yang tidak diketahui, tetapi berada di sebelah kanan persamaan sahaja. Persamaan kamiran keenam ini terbukti berguna dalam pengiraan pensifar bagi pemetaan Ahlfors. Penyelidikan ini memberikan suatu kaedah berangka untuk mengira pensifar bagi pemetaan Ahlfors ke atas sebarang rantau berkait berganda terbatas dengan sempadan yang licin. Penyelidikan ini menghasilkan dua rumus untuk terbitan fungsi hubungan sempadan bagi pemetaan Ahlfors dan terbitan inti Szego. Perhubungan antara pemetaan Ahlfors dan inti Szego adalah klasik. Inti Szego merupakan satu penyelesaian kepada persamaan kamiran Fredholm jenis kedua dengan inti Kerzman-Stein. Rumus yang terhasil ini kemudiannya digunakan bersama dengan persamaan kamiran keenam untuk mengira kesemua pensifar bagi pemetaan Ahlfors ke atas sebarang rantau berkait berganda yang licin. Beberapa contoh diberikan untuk menunjukkan keberkesanan kaedah yang dipersembahkan.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Conformal mapping is playing a vital role in complex analysis as a significant mathematical tool for science and engineering. Various regions with complicated boundaries can be transformed into simpler and a more manageable configurations by means of functions of complex variables. The magnitude as well as the directions of the angles between curves are preserved under conformal mapping function. Under conformal mapping functions, various problems in the applied sciences can be solved easily as complicated physical regions can be transformed into standard canonical regions in which calculations can be made easy. Then by the inverse mapping properties, the results can be transformed back into the original region to get required results. The process of conformally mapping of complicated regions onto simple regions has been applied in many applied problems like image processing, heat conduction and fluid mechanics.

In complex analysis, the Riemann Mapping Theorem assures that for any simply connected region, there exists a unique conformal map onto a unit disk. The solution of the following extremal problem can be considered as a Riemann mapping function: For a simply connected region Ω and canonical region D in the complex plane $\mathbb{C}$ and fixed a in

Ω, construct an extremal analytic map

$$F : \Omega \to D, \text{ with } \quad F'(a) > 0.$$

The solution of this problem is the Riemann map. It is one-to-one, onto and unique conformal with $F(a) = 0$.

The conformal map from a multiply connected region onto the unit disk is known as the Ahlfors map. In above extremal problem if the region is multiply connected instead of simply connected, then the solution of this extremal problem becomes the Ahlfors map

$$f : \Omega \to D$$

that is onto, $f'(a) > 0$ and $f(a) = 0$ and a unique analytic map. Ahlfors map being mapping function from multiply connected region of connectivity n onto a unit disk, it maps each boundary of inner curves onto unit disk so it has $2n - 2$ branch points in the interior. Thus the Ahlfors map is not one-to-one and maps Ω onto D in an n-to-one manner, and maps each boundary curve one-to-one onto the unit circle [1,2]. Therefore in the multiply connected settings, the Ahlfors map can be considered as a generalization of the Riemann mapping function. If the region is simply connected then the Ahlfors map becomes the Riemann map. Many of the geometrical features of a Riemann map are shared with Ahlfors map. As Riemann map is one-to-one map, so it has only one zero which can be freely choosen. But Ahlfors map being n-to-one map have n zeros which are unknown in general where n refers to the connectivity of the multiply connected region. Ahlfors map can be proved to be useful in many applied problems. For example, in fluid mechanics the transonic flow computing problems passing through an obstacle in the planar region, by the conformal mapping of the exterior of the obstacle onto the unit disk, a grid is set up which is most favorable to make numerical computations [3]. Similarly in these sort of problems where more than one obstacles are involved, the Ahlfors map could be used.

1.2 Research Background

Conformal mapping of simply or multiply connected regions has limitations that exact conformal mapping functions are known only for some particular regions and for other general regions have to be computed numerically. As there is no theorem or result like Riemann Mapping Theorem in the setting of multiply connected regions, so multiply connected regions of same order of connectivity are not equivalent under conformal map. Nehari [4, p. 335], Bergmann [5] and Cohn [6] have specified five kinds of slit regions as important canonical regions for conformal mapping of multiply connected regions:

i- the disk with circular slits.
ii- an annulus with cicular slits.
iii- the circular slit region.
iv- the radial slit region.
v- the parallel slit region.

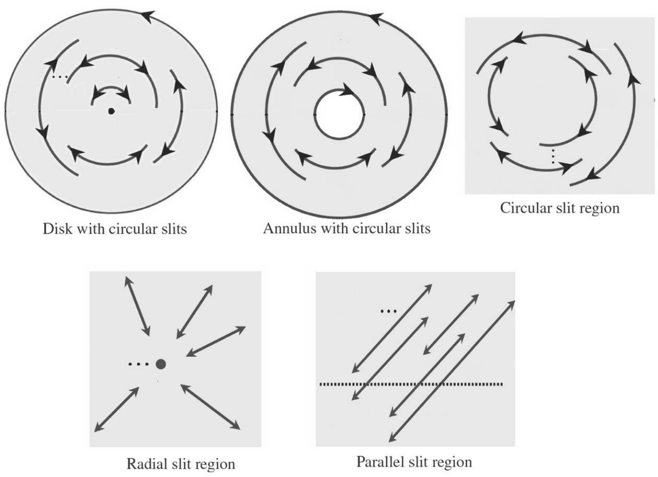

Figure 1.1 Five Canonical Regions

Several numerical methods for computing conformal mapping have been proposed in Henrici [7], Trefethen [8] and Wegmann [9]. The conformal mapping function for multiply connected regions can be computed efficiently using the integral equation method. The integral equation method has been used by many authors to compute the one-to-one conformal map from multiply connected regions onto some standard canonical regions in Kerzmann *et al*. [10, 11], Lee *et al.* [12], Nasser [13–16], Nasser *et al.* [17], O'Donnell *et al.* [18], Sangawi *et al*. [19–24], Yunus *et al.* [25–27]. Based on the relationship formed by an analytic fuction for the boundary of doubly connected region, Murid and Razali [28] obtained boundary integral equations for conformal map and Ahlfors map of doubly connected regions via Neumann, Kerzman-Stein and Szegö kernels. But no numerical experiments are reported and these are not the Fredholm integral equations. Based on Neuman kernel and Kerzmann-Stein kernel for conformal mapping of doubly connected region onto an annulus, Murid and Mohamed [29], Mohamed and Murid [30] and Mohamad [31] presented some numerical methods for solving the integral equations.

For solving the Riemann-Hilbert problems, Nasser [32] has also used the integral equation approach. Nasser [13, 14] discussed Riemann-Hilbert problems approach for numerical conformal mapping of bounded and unbounded multiply connected onto the canonical regions. Murid and Hu [33, 34] presented an integral equation method for conformal mapping of bounded and unbounded multiply connected regions onto a disk and annulus with circular slits respectively via the generalized Neumann kernel involving the circular radii which are assumed unknown.

Based on revised boundary relationship satisfied by an analytic function, Sangawi [19] formulated new integral equations for the conformal mapping of bounded multiply connected region with smooth boundaries via the classical Neumann kernel, the generalized Neumann kernel, the generalized Kerzman-Stein kernel and Neumann-type kernels onto the previous five canonical slit regions. Yunus [27] formulated new boundary integral equations for the conformal map of unbounded multiply connected region onto the previous five canonical slit regions. Also Al-Hatemi [35] made use of integral equation approach with the generalized Neumann kernel on multiply connected region for solving mixed boundary value problems.

Some integral equations for computing Ahlfors map have been given in [2, 28, 36–38]. To compute the Szegö kernel of a bounded region, Kerzman and Stein [10] have derived a uniquely solvable boundary integral equation and this method has been generalized in [36] to compute Ahlfors map of bounded multiply connected regions without depending upon on the zeros of Ahlfors map.

In [28] the integral equation for Ahlfors map of doubly connected regions requires knowledge of zeros of Ahlfors map, which are unknown in general. Computing the zeros of Ahlfors map for annulus region and a particular triply connected region are presented in [2] and [38]. Also for particular families of doubly connected regions in Bell domains, zeros of the Ahlfors map are known precisely in [39], but yet the problem of finding zeros of Ahlfors map for general doubly and higher connected regions is unsolved.

1.3 Problem Statement

The research problem is first to formulate new boundary integral equations for Ahlfors map of bounded multiply connected regions onto a unit disk via the generalized Neumann kernel, adjoint Neumann kernel, Neumann-type kernel and Kerzman-Stein kernel. Then use a suitable of these integral equations for computing the zeros of the Ahlfors map for multiply connected regions.

1.4 Research Objectives

The objectives of this research are:

(i) To derive new boundary integral equations for Ahlfors map of bounded multiply connected regions onto a unit disk based on the boundary relationship satisfied by an analytic function.

(ii) To verify numerically one of the derived boundary integral equations.

(iii) To determine a method for finding and to compute numerically the zeros of Ahlfors map for some selected multiply connected regions using the most suitable integral equation derived in (i).

(iv) To validate the results by means of numerical comparison of the proposed methods with the existing techniques for some selected regions.

1.5 Scope of the Study

This research focuses on the construction of new boundary integral equations for Ahlfors map of bounded and smooth multiply connected regions onto a unit disk. The theoretical development of the integral equations are based on the approach given by Sangawi [19]. Next the research focuses on finding the zeros of Ahlfors map for bounded multiply connected regions, which is now-a-days a main problem of interest.

In this research, some new boundary integral equations for Ahlfors map of bounded multiply connected regions via the classical Neumann kernel, adjoint Neumann kernel, Neumann type kernel, Kerzman-Stein kernel and Kerzman-Stein type kernel will be derived. These integral equations will be constructed from a non-homogeneous boundary relationship satisfied by an analytic function on bounded multiply connected regions. These integral equations will be applied to compute Ahlfors map of bounded multiply connected regions onto a unit disk, then the research will focus on finding some analytical or theoretical approach for finding the zeros of Ahlfors map for bounded multiply connected regions.

For numerical experiments, the integral equations will be discretized by the Nyström method with the trapezoidal rule which will lead to the system of equations.

This research also describes a numerical operation for computing the mapping of interior points based on Cauchy integral formula.

Finally, the research will present numerical examples to emphasize the advantage of using the proposed method.

1.6 Significance of Findings

In complex analysis, a conformal mapping uses functions to transform a complicated region into a simpler region. The conformal transformation of a simply connected region in the complex plane to the unit disk is known as Riemann map. As Riemann map is one-to-one map, so it has only one zero which can be freely choosen. But Ahlfors map has n zeros being an n-to-one map. Tegtmeyer and Thomas [2, 38] presented analytical methods for computing the exact zeros of the Ahlfors map only for the annulus region and a particular triply connected region, also for particular families of doubly connected regions in Bell domains, zeros of the Ahlfors map are known precisely in [39].

The major contribution of this research will be the presentation of a boundary integral equation method for finding the Ahlfors map and its zeros both graphically and numerically for bounded multiply connected regions. The problem of finding the zeros of Ahlfors map for arbitrary doubly, triply and the regions with higher connectivity is the first time presented in this research.

Furthermore, computer programming codes using Mathematica and MATLAB software will be constructed for the numerical examples of the Ahlfors mapping of bounded multiply connected regions and its zeros. Some of the results have been presented or published in national and international conference or journals. These will contribute to new findings in the field of complex analysis.

1.7 Research Methodology

This research wish to obtain new integral equations for Ahlfors map of multiply connected regions on the unit disk. To achieve these, the theorems on the integral equations based on the boundary relationship satisfied by the analytic function presented by Sangawi [19] need to be used. This research consists of four steps. The first step is to construct some integral equations involving Neumann kernel, Neumann Type kernels, Kerzman-Stein kernel and Kerzman-Stein type kernel related to the Ahlfors map of multiply connected region on the unit disk. The second step is to study the suitability of these integral equations for computing the zeros of Ahlfors map. The third step is to solve the integral equation numerically by using Nyström method with the trapezoidal rule [40]. The fourth step is to compute the zeros of Ahlfors map and to compare the numerical results with the exact solutions .

1.8 Thesis Organization

This thesis consists of six chapters and is organized as follows:

Chapter 1 is essentially an introduction, which consists of introduction, some research background of the problem, the problem statement, research objectives, scope of the research, significance of the findings of this research, research methodology and thesis organization.

Chapter 2 presents some literature review on conformal mapping of multiply connected regions and also states some theorems on conformal mappings, which will be proved to useful in this study. After explaining the idea of the conformal mapping in general, the theory of Riemann mapping function with some of its related theorems, the conformal mapping of multiply connected regions and canonical regions will be discussed. Ahlfors mapping function, some previous studies on Ahlfors map on multiply connected regions and its zeros will also be discussed. The definitions of classical and generalized Neumann kernels are also given. Finally the integral equations related to non-homogeneous boundary relationship satisfied by an analytical function derived in Sangawi [19] are also presented.

Chapter 3 contains the formulations of some new boundary integral equations for Ahlfors map of bounded multiply connected regions. The kernels of these boundary integral equations are the generalized Neumann kernel, adjoint Neumann kernel, Neumann type kernel, Kerzman-Stein kernel and Kerzman-Stein type kernel. These integral equations are constructed from a non-homogeneous boundary relationship satisfied by an analytic function on multiply connected regions. Also verified numerically one of the integral equations derived.

Chapter 4 consists of some modification of the integral equation which derived in Chapter 3, and also determination of some new formulas, to be used with the integral equation to find Ahlfors map and its zeros for smooth bounded multiply connected regions.

In Chapter 5, some numerical examples are presented for computing the zeros of the Ahlfors map for several multiply connected regions. The values of zeros of Ahlfors map for particular regions by other methods in literature has been compared with the values of zeros of Ahlfors map obtained by the proposed method. The examples demonstrate that the proposed method can be applied to any multiply connected regions for finding the zeros of Ahlfors map.

Finally, Chapter 6 contains a summary of this thesis with final conclusion and some recommendations for future research. There are three appendices in this thesis. Appendix A presents the list of the papers that have been published, submitted and presented during the authors candidature. Appendix B presents the Hölder condition and Nyström method. Appendix C displays some samples of computer programs coded in MATHMATICA 10.0 and MATLAB.

an obstacle in the plane, a conformal map of the outside of the obstacle onto the unit disk is used to set up a grid which is most convenient for making numerical computations. The Ahlfors map could be used in the similar way in problems of this sort in which more than one obstacle is involved". Thus it can conclude that much work can be done using this work in different applications.

This thesis is mainly on the Ahlfors mapping of bounded smooth multiply connected regions onto the unit disk. Extending the work of this thesis to mapping of bounded multiply connected regions with corners onto the unit disk canonical region constitute a good problem for future research.

With the above summary, conclusions and future recommendations, we conclude this thesis.

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