Dynamic Modelling of a Two-link Flexible Manipulator System Incorporating Payload

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Abstract-This paper presents dynamic modelling of a two-link flexible manipulator based on closed-form equations of motion. The kinematic model is based on standard frame transformation matrices describing both rigid rotation and modal displacement, under small deflection assumption. The Lagrangian approach is used to derive the dynamic model of the structure. Links are modelled as Euler-Bernoulli beams with proper clamped-mass boundary conditions. A dynamic model of the system, incorporating structural damping, hub inertia and payload, is developed using finite assumed mode methods. Explicit equations of motions are detailed by assuming two modes of vibration for each link. Moreover, effects of payload on the response of the flexible manipulator are discussed. Extensive results that validate the theoretical derivation are presented in the time and frequency domains.

I. INTRODUCTION

Research on the dynamic modelling and control of flexible manipulators has received increased attention due to their several advantages over rigid robots: they require less material, are lighter in weight, consume less power, require smaller actuators, are more manoeuvrable and transportable, have less overall cost and higher payload to robot weight ratio. These types of robots are used in a wide spectrum of applications starting from simple pick and place operations of an industrial robot to micro-surgery, maintenance of nuclear plants and space robotics [1]. However, control of flexible manipulators to maintain accurate positioning is an extremely challenging. The complexity of the problem increases dramatically for a two-link flexible manipulator. Due to the flexible nature of the system, the dynamics are highly non-linear and complex [2,3]. In this respect, a control mechanism that accounts for both the rigid body and flexural motions of the system is required. Moreover, the complexity of this problem increases when a flexible manipulator carries a payload. If the advantages associated with lightness are not to be sacrificed, accurate models and efficient controllers have to be developed.

Modelling of a single-link flexible manipulator has been widely established. Various approaches have been developed which can mainly be divided into two categories: the numerical analysis approach and the assumed mode method (AMM). The numerical analysis methods that are utilised include finite difference (FD) and finite element (FE) methods. The FD and FE approaches have been used in obtaining the dynamic characterisation of single-link flexible

manipulator systems incorporating damping, hub inertia and payload [4,5,6]. Performance investigations of both techniques in modelling of flexible manipulators have shown that the FE method can be used to obtain a good representation of the system [4]. It has been reported that in using the FE method, a single element is sufficient to describe the dynamic behaviour of a flexible manipulator reasonable well [5].

AMM looks at obtaining approximate models by solving the partial differential equation characterising the dynamic behaviour of the system. Previous studies utilising this approach for modelling of a single-link flexible manipulator have been reported [2,7,8]. It has been shown that the first two modes are sufficient to identify the dynamic of flexible manipulators. A good agreement between theory and experiments has also been achieved utilising this approach [2].

Similar to the case of a single-link manipulator, the FE method and AMM have also been investigated for modelling of a two-link flexible robot manipulator. However, the complexity of the modelling increases dramatically as compared to the case of a single-link flexible manipulator. Yang and Sadler [9] have developed the FE model to describe the deflection of a planar two-link flexible robot manipulator. A dynamic model has been developed using the FE methods utilising a generalised inertia matrix [10]. De Luca and Siciliano [11] have utilised the AMM to derive a dynamic model of multilink flexible robot arms limiting to the case of planar manipulators with no torsional effects. The equations of motion which can be arranged in a computationally efficient closed form that is also linear with respect to a suitable set of constant mechanical parameters have been obtained. A systematic approach for deriving the dynamic equations for *n*-link manipulator has also been presented [12]. The utilisation of AMM to derive the dynamic model of the system has also been reported [13]. However, modelling of a two-link flexible robot manipulator using the AMM has not been adequately addressed in the literature. Moreover, the effects of other physical parameters such as payload on the dynamic characteristics of the system are not adequately discussed.

This paper presents a generalised modelling framework that provides a closed-form dynamic equation of motion of a two-link flexible manipulator system. Moreover, the works presents the effects of payload on the dynamic behaviour of

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the system. The Euler-Lagrange principle and assumed mode discretisation technique are used to derive the dynamic model of the system. The simulation algorithm thus developed is implemented in Matlab. Angular position responses of the system and the power spectral density (PSD) are obtained in both time and frequency domains. To study the effect of payloads, the results are evaluated with varying payloads of the flexible manipulator. Simulation results are analysed in both the time and frequency domains to assess the accuracy of the model in representing the actual system.

II. THE FLEXIBLE MANIPULATOR SYSTEM

Fig. 1 shows a two-link flexible robot manipulator system considered in this study. The links are cascaded in a serial fashion and both links are actuated by individual motors at the hub of the flexible manipulator. X_0Y_0 is the inertial co-ordinate frame, X_iY_i is the rigid body coordinate frame associated with the ith link and $\hat{X}_i\hat{Y}_i$ is the moving coordinate frame. θ_i is the angular position of ith link while the transverse component of the displacement vector is designated as $v_i(x_i,t)$. An inertial payload of mass M_P with inertia J_P is attached at the end-point of link 2.
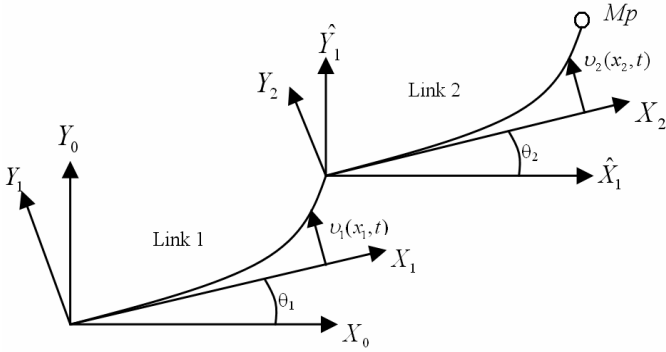

Fig. 1. Description of the two-link flexible manipulator system.

In this work, the following assumptions were considered in the development of a dynamic model of the flexible manipulator:

I. Each link is assumed to be long and slender. Therefore, transverse shear and the rotary inertia effects are negligible.
II. The motion of each link is assumed to be in the horizontal plane.
III. Links are considered to have constant cross-sectional area and uniform material properties, i.e. constant mass density and Young's modulus.
IV. Each link has a very small deflection.
V. Motion of the links can have deformations in the horizontal direction only.
VI. The kinetic energy of the rotor is mainly due to its rotation only, and the rotor inertia is symmetric about its axis of rotation.

VII. The backlash in the reduction gear and coulomb friction effects are neglected.

The physical parameters of the two-link flexible manipulator system are shown in Table I. l, ρ and EI represent the length, mass density and flexural rigidity of the links respectively. M_{h2} is the mass considered at the second motor which is located in between both links, J_{hi} is the inertia of the ith motor and hub. The input torque, $\tau_i(t)$ is applied at each motor and G_i is the gear ratio for the ith motor. Both links and motors are considered to have the same dimensions

TABLE I
PARAMETER OF THE MANIPULATOR

Symbol	Parameter	Value	Unit
ρ	Mass density	0.2	kgm^{-1}
EI	Flexural rigidity	1.0	Nm^2
l	Length	0.5	m
J_h	Motor and hub inertia	0.02	kgm^2
G	Gear ratio	1	-
M_{h2}	Mass of centre motor	1	kg

III. DYNAMIC MODELLING

A. Kinematic Formulation

Considering revolute joints and motion of the manipulator on a two-dimensional plane, the rigid transformation matrix, A_i, from $X_{i-1}Y_{i-1}$ to X_iY_i is written as

$$A_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix} \quad (1)$$

The elastic homogenous transformation matrix, $\mathbf{E}_i$, due to the deflection of the link i can be written as

$$E_i = \begin{bmatrix} 1 & -\dfrac{\partial v_i(x_i,t)}{\partial x_i}\bigg|_{x_i=l_i} \\ \dfrac{\partial v_i(x_i,t)}{\partial x_i}\bigg|_{x_i=l_i} & 1 \end{bmatrix} \quad (2)$$

where $v_i(x_i,t)$ is the bending deflection of the ith link at a spatial point x_i ($0 \leq x_i \leq l_i$) and l_i is the length of the ith link. The global transformation matrix $\mathbf{T}_i$ transforming co-ordinates from X_0Y_0 to X_iY_i follow a recursion as below

$$T_i = T_{i-1}E_{i-1}A_i \quad (3)$$

Let $^i r_i(x_i) = \begin{Bmatrix} x_i \\ v_i(x_i,t) \end{Bmatrix}$ be the position vector that describes an arbitrary point along the ith deflected link with respect to its local CF (X_iY_i) and $^0 r_i$ be the same point referring to X_0Y_0. The position of the origin of $X_{i+1}Y_{i+1}$ with

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respect to X_iY_i is given by

$$^i p_{i+1} = {}^i r_i(l_i) \tag{4}$$

and $^0 p_i$ is its absolute position with respect to X_0Y_0. Using the global transformation matrix, $^0 r_i$ and $^0 p_i$ can be written as

$$^0 r_i = {}^0 p_i + T_i {}^i r_i, \qquad ^0 p_{i+1} = {}^0 p_i + T_i {}^i p_{i+1} \tag{5}$$

B. Dynamic Equations

To derive the dynamic equations of motion of the flexible manipulator, the total energy associated with the manipulator system needs to be computed using the kinematics formulations explained previously. The total kinetic energy of the manipulator (T) is given by

$$T = T_R + T_L + T_{PL} \tag{6}$$

where T_R, T_L and T_{PL} are the kinetic energies associated with the rotors, links and the hubs, respectively. By using a very small deflection assumption, the kinetic energy of the ith rotor is given by

$$T_{Ri} = \frac{1}{2} G_i^2 I_{ri} \dot\alpha_i^2 \tag{7}$$

where $\dot\alpha_i$ is the angular velocity of the rotor about the ith principal axis. The kinetic energy of a point $r_i(x_i)$ on the ith link can be written as

$$T_{Li} = \frac{1}{2} \rho_i \int_0^{l_i} {}^0\dot r_i{}^T(x_i) {}^0\dot r_i(x_i) dx_i \tag{8}$$

where ρ_i is the linear mass density for the ith link and $^0\dot r_i(x_i)$ is the velocity vector. The velocity vector can be computed by taking the time derivative of its position in (5):

$$^0\dot r_i(x_i) = {}^0\dot p_i + \dot T_i {}^i r_i(x_i) + T_i {}^i\dot r_i(x_i) \tag{9}$$

$^0\dot p_i$ in (9) can be determined using (4) and (5) along with

$$^i\dot p_{i+1} = {}^i\dot r_i(l_i) \tag{10}$$

The time derivative of the global transformation matrix $\dot T_i$ can be recursively calculated from

$$\dot T_i = \dot{\bar T}_{i-1} A_i + \dot{\bar T}_{i-1} \dot A_i, \quad \dot{\bar T}_i = T_i E_i + T_i \dot E_i \tag{11}$$

After determining the kinetic energy associated with the ith link, the kinetic energy of all the n links can be found as

$$T_L = \sum_{i=1}^{n} \frac{1}{2} \rho_i \int_0^{l_i} {}^0\dot r_i{}^T(x_i) {}^0\dot r_i(x_i) dx_i \tag{12}$$

Referring to Fig. 1 and the kinematics formulation described previously, the kinetic energy associated with the payload can be written as

$$T_{PL} = \frac{1}{2} M_p \dot p_{n+1}^T \dot p_{n+1} + \frac{1}{2} I_P (\dot\Omega_n + \dot v'_n(l_n))^2, \tag{13}$$

where $\dot\Omega_n = \sum_{j=1}^{n} \theta_j + \sum_{k=1}^{n-1} \dot v'_k(l_k)$; n being the link number, prime and dot represent the first derivatives with respect to spatial variable x and time, respectively. $\dot p_{n+1}$ can be determined using (4) and (5).

Next, neglecting the effects of the gravity, the potential energy of the system due to the deformation of the link i can be written as

$$U = \frac{1}{2} \int_0^{l_i} (EI)_i \left(\frac{d^2 v_i(x_i)}{dx_i^2} \right)^2 dx_i \tag{14}$$

C. Assumed Mode Shapes

Using the assumption of long and slender link, the dynamics of the link at an arbitrary spatial point x_i along the link at an instant of time t can be written using Euler-Beam theory [10] as

$$(EI)_i \frac{\partial^4 v_i(x_i,t)}{\partial x_i^4} + \rho_i \frac{\partial^2 v_i(x_i,t)}{\partial t^2} = 0 \tag{15}$$

where ρ_i is the linear mass density of the ith link. Equation (15) is solved by applying the boundary conditions of the manipulator. Considering a clamped-mass configuration of the manipulator, the boundary conditions can be written as

$$v_i(x_i,t)\big|_{x_i=0} = 0, \qquad v'_i(x_i,t)\big|_{x_i=0} = 0, \tag{16}$$

$$\begin{aligned} (EI)_i & \frac{\partial^2 v_i(x_i,t)}{\partial x_i^2}\bigg|_{x_i=l_i} \\ &= -I_{E_i} \frac{d^2}{dt^2}\left(\frac{\partial v_i(x_i,t)}{\partial t}\bigg|_{x_i=l_i} \right) \\ &\quad - M_{DE_i} \frac{d^2}{dt^2}\left(\frac{\partial v_i(x_i,t)}{\partial t}\bigg|_{x_i=l_i} \right) \end{aligned} \tag{17}$$

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$$(EI)_i \frac{\partial^3 v_i(x_i,t)}{\partial x_i^3}\Big|_{x_i=l_i}$$

$$= M_{E_i} \frac{d^2}{dt^2}\left(v_i(x_i,t)\big|_{x_i=l_i}\right) \qquad (18)$$

$$+ M_{DE_i} \frac{d^2}{dt^2}\left(v_i(x_i,t)\big|_{x_i=l_i}\right)$$

where M_{E_i}, I_{E_i} are the effective mass and moment of inertias at the end of the ith link and M_{DE_i} is the contributions of masses of distal links.

Bending deflections $v_i(x_i,t)$ can be expressed as a superposition of mode-shapes and time dependent modal displacements:

$$v_i(x_i,t) = \sum_{j=1}^{n_m} \phi_{ij}(x_i) q_{ij}(t) \qquad (19)$$

where $\phi_{ij}(x_i)$ and $q_{ij}(t)$, respectively, are the jth mode shape function and jth modal displacement for the ith link. The solution of (19) and (15) gives

$$q_{ij}(t) = \exp(\omega_{ij} t) \qquad (20)$$

and

$$\phi_{ij}(x_i) = m_i[\cos(\beta_{ij} x_i) - \cosh(\beta_{ij} x_i) + \gamma_{ij}\sinh(\beta_{ij} x_i) - \cosh(\beta_{ij} x_i)] \qquad (21)$$

where m_i is the mass of the link i and γ_{ij} is given as

$$\gamma_{ij} = \frac{\sin\beta_{ij} - \sinh\beta_{ij} + \dfrac{M_{L_i}\beta_{ij}}{\rho_i}(\cos\beta_{ij} - \cosh\beta_{ij})}{\cos\beta_{ij} + \cosh\beta_{ij} - \dfrac{M_{L_i}\beta_{ij}}{\rho_i}(\sin\beta_{ij} - \sinh\beta_{ij})} \qquad (22)$$

and β_{ij} is the solution of the following equation:

$$1 + \cosh\beta_{ij}l_i \cos\beta_{ij}l_i$$

$$- \frac{M_{L_i}\beta_{ij}}{\rho_i}(\sin\beta_{ij}l_i \cosh\beta_{ij}l_i - \cos\beta_{ij}l_i \sinh\beta_{ij}l_i)$$

$$- \frac{J_{L_i}\beta_{ij}^3}{\rho_i}(\sin\beta_{ij}l_i \cosh\beta_{ij}l_i + \cos\beta_{ij}l_i \sinh\beta_{ij}l_i) \qquad (23)$$

$$+ \frac{M_{L_i}I_{L_i}\beta_{ij}^4}{\rho_i^2}(1 - \cos\beta_{ij}l_i \cosh\beta_{ij}l_i) = 0$$

B. Closed-form Equations

Using the generalised modelling scheme described in Sections 2, 3 and 4, equations with two flexible links are described in this section. In this case, the effective masses at the end of the individual links are set as

$$M_{E1} = m_2 + M_p,$$
$$I_{E1} = I_{b2} + J_2 + I_p + M_p l_2^2, \qquad (24)$$
$$M_{DE1} = (m_2 l_{c2} + M_p l_2 \cos\theta_2),$$

$$M_{E2} = M_p,$$
$$I_{E2} = I_p + M_p l_2^2, \qquad (25)$$
$$M_{DE2} = 0$$

In order to obtain a closed-form dynamic model of the manipulator, the energy expressions derived in section 3 are used to formulate the Lagrangian $L = T - U$. Using the Euler-Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{Q}_i}\right) - \frac{\partial L}{\partial Q_i} = F_i \qquad (26)$$

with the ith generalized co-ordinate of the system, Q_i ($Q=[\theta_1, \theta_2, q_{11}, q_{12}, q_{21}, q_{22}]^T$), and the corresponding generalized forces, F_i ($F=[\tau_1, \tau_2, 0, 0, 0, 0]^T$), a set of dynamic equations can be written in compact form as

$$M(\theta,q)\begin{Bmatrix}\ddot{\theta}\\\ddot{q}\end{Bmatrix} + \begin{Bmatrix}f_1(\theta,\dot{\theta})\\f_2(\theta,\dot{\theta})\end{Bmatrix} + \begin{Bmatrix}g_1(\theta,\dot{\theta},q,\dot{q})\\g_2(\theta,\dot{\theta},q,\dot{q})\end{Bmatrix}$$
$$+ \begin{Bmatrix}0\\D\dot{q}\end{Bmatrix} + \begin{Bmatrix}0\\Kq\end{Bmatrix} = \begin{Bmatrix}\tau\\0\end{Bmatrix} \qquad (27)$$

where M is the mass matrix, f_1 and f_2 are the vectors containing terms due to coriolis and centrifugal forces, and g_1 and g_2 are the vectors containing terms due to the interactions of the link angles and their rates with the modal displacements. K is the diagonal stiffness matrix which takes on the values $\omega_{ij}^2 m_i$ and D is the passive structural damping ($D_i = 0.1\sqrt{K_i}, i = 3,...,6$).

IV. SIMULATION RESULTS

In this section, simulation results of the dynamic behaviour of the two-link flexible manipulator system are presented in the time and frequency domains. A bang-bang signal of amplitude 0.2 Nm and 1 s width is used as an input torque, applied at the rotors of the manipulator. A bang-bang torque has a positive (acceleration) and negative (deceleration) period allowing the manipulator to, initially, accelerate and then decelerate and eventually stop at a target location. System responses are verified by undertaking computer

simulation using the fourth-order Runge-Kutta integration method for duration of 5 s with a sampling time of 1ms. The angular position responses at both links of the system with the PSDs are obtained and evaluated.

To demonstrate the effects of payload on the dynamic behaviour of the system, various payloads of up to 100 grams weight were simulated. Figs. 2 and 3 show the angular position response with various payloads for link-1 and link-2 respectively. Moreover, Figs. 4 and 5 show the corresponding PSDs for both links. It is noted that the angular position for link-1 increases towards the negative direction while for link-2 the angular position decreases with increasing payloads. The time response specifications of angular positions have shown significant changes with the variations of payloads. Table II summarises the time response specifications of angular positions for the two-link flexible manipulator system. Moreover, the PSD of the system response shows that the resonance modes of vibration of the system shift to lower frequencies with increasing payloads. This implies that the manipulator oscillates at lower frequency rates than those without payload. Table III summarises the relation between payload and the resonance frequencies of the system.
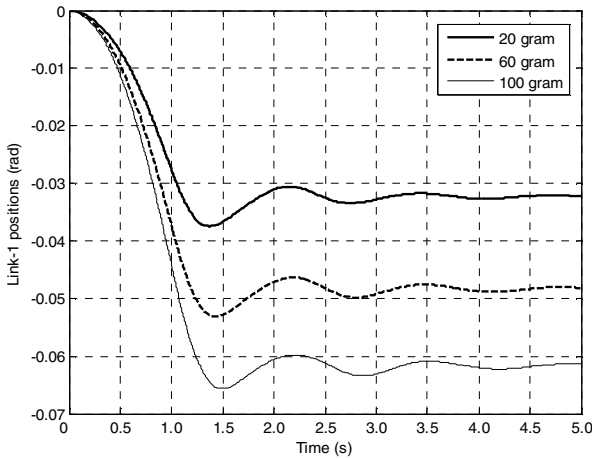

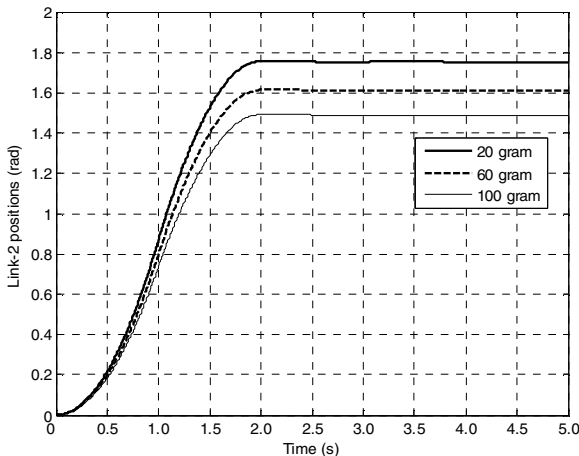Fig. 2. Angular position response of link-1

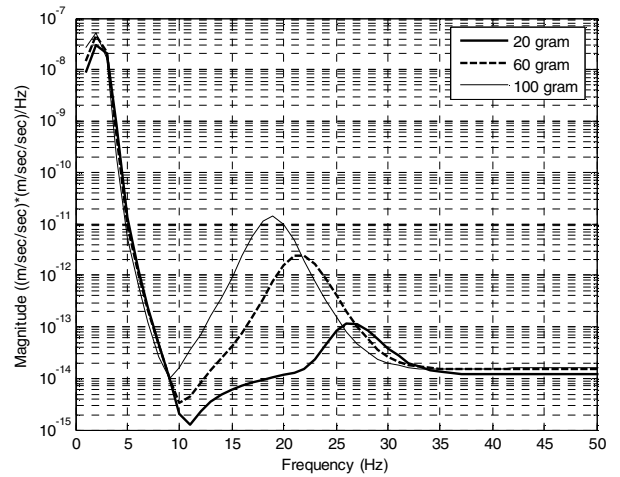Fig. 3. Angular position response of link-2

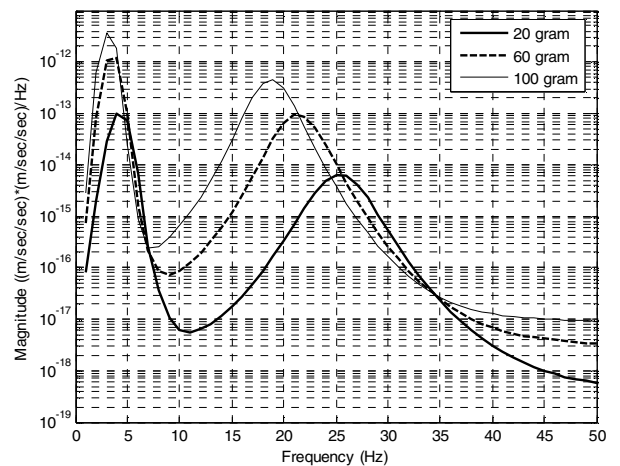Fig.4. PSD of end-point acceleration of link-1

Fig.4. PSD of end-point acceleration of link-2

By comparing the results presented in Table II, it is noted that the settling time of the manipulator response was affected by variations in the payload. It is also evidenced that the settling time response for both links decreases with increasing payloads. It shows that, by incorporating more weight at distal link resulted in a faster response. However, the percentage overshoot results produce a contrast pattern between link-1 and link-2 with variation of payloads. With increasing payload, the overshoots of link-1 slightly decrease whereas for link-2 the overshoots gradually increase. The changes of percentage of overshoot for link-1 are significant as link-1 exhibits higher magnitude of vibration compared to link-2. Besides, link-1 has to cater a higher mass of load from the second rotor, link-2 and payload which produced a smaller angular position as compared to link-2.

TABLE II
RELATION BETWEEN PAYLOAD AND SPECIFICATIONS OF ANGULAR POSITION
RESPONSES OF THE FLEXIBLE MANIPULATOR

Payloads (grams)	Time responses specifications of angular positions		
	Link-1		
	Rise time (s)	Settling time (s)	Overshoot (%)
0	0.652	1.769	26.20
20	0.677	1.765	21.18
40	0.700	1.760	17.35
60	0.721	1.754	14.41
80	0.742	1.752	12.25
100	0.761	1.750	10.57
	Link-2		
0	1.112	1.803	0.22
20	1.113	1.802	0.34
40	1.115	1.799	0.42
60	1.116	1.800	0.50
80	1.119	1.796	0.58
100	1.121	1.798	0.61

TABLE III
RELATION BETWEEN PAYLOAD AND RESONANCE FREQUENCIES OF THE FLEXIBLE
MANIPULATOR

Payloads (grams)	Resonance frequencies (Hz)			
	Link-1		Link-2	
	Mode 1 (Hz)	Mode 2 (Hz)	Mode 1 (Hz)	Mode 2 (Hz)
0	2	35	6	32
20	2	27	4	25
40	2	23	4	23
60	2	22	3	21
80	2	20	3	20
100	2	19	3	19

V. CONCLUSION

Investigations into the development of a dynamic model of a two-link flexible manipulator incorporating structural damping, hub inertia and payload have been presented. A closed-form finite dimensional dynamic model of a planar two-link flexible manipulator has been developed using the Euler-Lagrange approach combined with the AMM. The derived dynamic model has been simulated with bang-bang torque inputs and angular position responses of both links of the system have been obtained and analysed in time and frequency domains. Moreover, the effects of payload on the dynamic characteristic of the system have been studied and discussed. These results are very helpful and important in the development of effective control algorithms for a two-link flexible robot manipulator with varying payload

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